Analysis of a Variational Multiscale Method for Large-Eddy Simulation and its Application to Homogeneous Isotropic Turbulence

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Abstract

In this paper a variational multiscale method based on local projection and grad-div stabilization for Large-Eddy Simulation for the incompressible Navier-Stokes problem is considered. An a priori error estimate is given for a case with rather general nonlinear (piecewise constant) coefficients of the subgrid models for the unresolved scales of velocity and pressure. Then the design of the subgrid scale models is specified for the case of homogeneous isotropic turbulence and studied for the standard benchmark problem of decaying homogeneous isotropic turbulence.

Keywords: Navier-Stokes problem, large-eddy simulation, variational multiscale method, local projection stabilization, grad-div stabilization, homogeneous isotropic turbulence

1. Introduction

Incompressible viscous flows of a Newtonian fluid are modeled by the Navier-Stokes equations which read: Given a bounded domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$, the simulation time T, and a force field $\mathbf{f} : (0,T] \times \Omega \to \mathbb{R}^3$, find a velocity field $\mathbf{u} : (0,T] \times \Omega \to \mathbb{R}^3$ and a pressure field $p : (0,T] \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{u} - 2\nu \nabla \cdot \mathbb{D}\mathbf{u} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla p &= \mathbf{f} \quad && \text{in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad && \text{in } [0,T] \times \Omega, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 \quad && \text{in } \Omega, \end{aligned} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity coefficient and $\mathbb{D}\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ denotes the velocity deformation tensor. Some boundary conditions have to be imposed on $\partial\Omega$ to obtain a closed set of equations. In the analysis below, we impose homogeneous Dirichlet conditions for simplicity, but see Remark 2.1.

In many industrial applications, simulations of turbulent flows are of major interest. Such flows are characterized by large Reynolds numbers $Re = UL/\nu$ with given characteristic length L und velocity scale U. For the numerical approximation, the finite-element (FE) method is one of the most popular and mathematically sound variants. The standard Galerkin method aims to simulate all persistent scales in the range of order $\mathcal{O}(\mathrm{diam}(\Omega))$ down to $\mathcal{O}(Re^{-3/4})$ which is not feasible even in next futures for the case of large Re. Residual-based stabilization techniques, like the streamline-upwind (or SUPG) method and/or the pressure stabilization (or PSPG) technique, add numerical viscosity acting at all scales. For a representative overview, we refer to Ref. [29]. The natural approach to simulate only the behavior of large scales accurately has been considered for a long time in the classical Large-Eddy simulation (LES). Several drawbacks like commutation errors and the unsolved question of appropriate boundary conditions for the large scales have been critically discussed in recent times. For a review, see Ref. [4].

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Based on ideas in Refs. [12, 11], the class of Variational Multiscale (VMS) methods provides an alternative approach to the simulation of large scales. For a first application to turbulent flow problems, we refer to Ref. [13]. The basic idea of VMS-methods is to define the large scales by projections into appropriate function spaces. Within a three-scale decomposition of the flow field into large, resolved small and unresolved scales, the influence of the unresolved small scales is described by a subgrid model acting directly only on the resolved small scales. A series of numerical studies reports good experience with VMS methods for standard benchmark problems. Meanwhile, different variants of VMS methods have been considered, for a review and comparison of different variants see Refs. [10, 17].

The numerical analysis of VMS methods for turbulent flows is still in its infancy. Let us remark that the analysis depends on the choice of the discrete velocity-pressure approximation. For the case of equal-order interpolation, we refer to the contributions of Codina and his co-workers, see, e.g., Ref. [26]. The analysis may differ in certain aspects from the equal-order case if inf-sup stable FE pairs are applied (as in the present paper). Some progress has been made by V. John and co-workers for projection-based variants of the VMS method with inf-sup stable FE pairs. A globally constant turbulent viscosity ν_T together with an elliptic projection for the definition of the large scales had been used in Ref. [14] and analyzed in Ref. [15]. The recent paper Ref. [16] analyzes a Smagorinsky-type subgrid model applied together with an L^2-projection. The latter approach avoids some open problems of the elliptic projection in Ref. [14]. For a discussion on subgrid modelling of the unresolved pressure scales based on grad-div stabilization, we refer to Ref. [25].

In the present paper, we consider a modified projection-based FE VMS method which had been presented in Refs. [20, 14]. The subgrid model for the unresolved velocity scales is based on the L^2-projection Π_H for the definition of the large scales of the velocity deformation tensor. One difference to the approach in Ref. [16] is that the so-called fluctuation operator $I - \Pi_H$ is applied to the velocity deformation tensor whereas the velocity deformation tensor is applied to the fluctuation operator in Ref. [16] first. Please notice that these operators do not commute in the general case Ref. [24]. Another difference to Refs. [15, 16] is the application of the so-called grad-div stabilization as a subgrid model for the pressure. In particular, we address implementation issues and the relation of the method to stabilization techniques based on local projection. We derive an a priori error estimate for the semidiscrete problem where the definition of the subgrid models for the unresolved velocity and pressure scales remains rather general. For the case of homogeneous isotropic turbulence, we specify the velocity subgrid model to be of Smagorinsky type. In particular, Lilly's argument Refs. [22, 13] is modified for this model. The subgrid parametrization includes the dependence on the polynomial degree of the velocity approximation. Finally, the parametrization of the two subgrid models is checked for the case of decaying homogeneous isotropic turbulence.

The paper is organized as follows: In Section 2, we introduce the projection-based VMS method under consideration. Then, in Section 3, we provide the error analysis for the model after spatial semidiscretization based on inf-sup stable finite element pairs for velocity and pressure. In Section 4 we specify the subgrid model for the case of homogeneous isotropic turbulence. Then, Section 5 is devoted to the application of the approach to the standard benchmark of decaying homogeneous isotropic turbulence. Finally, we summarize the results in Section 6 and give some conclusions.

2. A modified projection-based finite-element variational multiscale method

2.1. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Standard notations are used for Lebesgues spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega), m \in \mathbb{N}, 1 \le p \le \infty$, together with the corresponding norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$. The inner product in $[L^2(\Omega)]^3$ will be denoted by $(\cdot,\cdot)$. A similar notation will be used on subdomains $D \subseteq \Omega$. For clarity we write $\|\cdot\|_0$ for the L^2 norm $\|\cdot\|_{L^2(\Omega)}$ of the whole domain Ω.

For a normed space X with functions defined on Ω, let $L^p(0,t;X)$ be the space of all functions defined on $(0,t) \times X$ with finite norm

$$\|\mathbf{u}\|_{L^p(0,t;X)} := \Big(\int_0^t \|\mathbf{u}\|_X^p \, ds \Big)^{1/p}, \quad 1 \le p < \infty$$

and with the obvious modification for $p = \infty$.
Setting $V = [H_0^1(\Omega)]^3$ and $Q = L_*^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}$, we consider the variational formulation of the Navier-Stokes equations: find $\mathbf{u}\colon [0,T] \to V$ and $p\colon (0,T] \to Q$ satisfying

$$
\begin{aligned}
(\partial_t \mathbf{u}, \mathbf{v}) + (2\nu \mathbb{D}\mathbf{u}, \mathbb{D}\mathbf{v}) + b_S(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad & \forall \mathbf{v} \in V, \\
(q, \nabla \cdot \mathbf{u}) = 0 \quad & \forall q \in Q.
\end{aligned} \tag{2}
$$

Here, the skew-symmetric trilinear form $b_S(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}[((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - ((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v})]$ has the important property $b_S(\mathbf{u}.\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{u}, \mathbf{v} \in V$.
For the present analysis, we will use Korn's inequality with constant C_{Ko} and the Poincaré-Friedrichs inequality with constant C_F such that

$$
\|\nabla \mathbf{v}\|_0 \leq C_{\mathrm{Ko}} \|\mathbb{D}\mathbf{v}\|_0 \quad \text{and} \quad \|\mathbf{v}\|_0 \leq C_F \|\nabla \mathbf{v}\|_0 \qquad \forall \mathbf{v} \in V. \tag{3}
$$

Remark 2.1. *The analysis of this paper can be applied in the case of periodic boundary conditions for the velocity as well. The proof for Korn's inequality under such conditions is very similar to the case of no-slip boundary conditions, see Ref. [28]. Please notice that the application of the VMS approach to the case of decaying homogeneous isotropic turbulence requires such periodic boundary conditions. It seems also possible to extend the analysis to cases where no-slip boundary conditions and periodic boundary conditions appear simultaneously, e.g. in channel flows.*

2.2. Variational multiscale method

Let $\mathcal{T}_h$ be an admissible triangulation of Ω in the usual sense, see Ref. [8], with maximal diameter $h > 0$ of the mesh cells $K \in \mathcal{T}_h$. The FE spaces $V_h \times Q_h \subset V \times Q$ of the basic Galerkin FE method will be standard inf-sup stable velocity-pressure spaces, i.e. with

$$
\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in V_h \setminus \{0\}} \frac{(q_h, \nabla \cdot \mathbf{v_h})}{\|q_h\|_0 \|\nabla \mathbf{v}_h\|_0} \geq \beta > 0 \tag{4}
$$

where β is h-independent. The Galerkin FE method reads: find $\mathbf{u}_h\colon [0,T] \to V_h$, $p_h\colon (0,T] \to Q_h$ such that

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad & \forall \mathbf{v}_h \in V_h, \\
(q_h, \nabla \cdot \mathbf{u}_h) = 0 \qquad & \forall q_h \in Q_h.
\end{aligned}
$$

For turbulent flows, let a three-scale decomposition of the flow and pressure fields be given by

$$
\mathbf{v} = \overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h + \hat{\mathbf{v}} \quad \forall \mathbf{v} \in V; \qquad q = \overline{q}_h + \tilde{q}_h + \hat{q} \quad \forall q \in Q.
$$

We search for the resolved scales $(\mathbf{v}_h, q_h) := (\overline{\mathbf{v}}_h + \tilde{\mathbf{v}}_h, q_h + \tilde{q}_h) \in V_h \times Q_h \subset V \times Q$. The influence of the unresolved small velocity scales $(\hat{\mathbf{v}}_h, \hat{q}_h)$ on the resolved small scales will be modelled using a variant of the variational multiscale approach of Ref. [20], Section 3. To this goal, we define the following.

Definition 2.2. *Let $\mathcal{T}_H$ be the triangulation of a coarser grid, i.e. $H \geq h$. Then the finite-element space L_H of coarse scales of the deformation tensor is*

$$
\{0\} \subseteq L_H \subseteq \mathbb{D}V_h \subseteq L := \left\{ \mathbf{L} = (l_{ij}) \mid l_{ij} = l_{ji} \in L^2(\Omega) \ \forall i,j \in \{1,2,3\} \right\}.
$$

Within this article we assume that $\mathcal{T}_h$ is a conforming refinement of $\mathcal{T}_H$. A possible choice of the space L_H with $H = h$ will be discussed in Section 4. In particular, an adaptive choice of L_H, which varies on $\mathcal{T}_H$, is not excluded, see Ref. [17].
A first version of the VMS model reads: find $\mathbf{u}_h\colon [0,T] \to V_h$, $p_h\colon (0,T] \to Q_h$, $\mathbf{G}_H\colon (0,T] \to L_H$ such that

$$
\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) & \\
-(p_h, \nabla \cdot \mathbf{v}_h) + (\nu_T(\mathbf{u}_h)(\mathbb{D}\mathbf{u}_h - \mathbf{G}_H), \mathbb{D}\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad & \forall \mathbf{v}_h \in V_h, \\
(q_h, \nabla \cdot \mathbf{u}_h) = 0 \quad & \forall q_h \in Q_h, \\
(\mathbf{G}_H - \mathbb{D}\mathbf{u}_h, \mathbf{L}_H) = 0 \quad & \forall \mathbf{L}_H \in L_H.
\end{aligned}
$$

The third equation implies $\mathbf{G}_H = \Pi_H(\mathbb{D}\mathbf{u}_h)$, where $\Pi_H \colon L \to L_H \subset L$ is the L^2-orthogonal projection. Having observed this we give the following Definition.

Definition 2.3. *Let $\Pi_H \colon L \to L_H \subset L$ be the L^2-orthogonal projection. The fluctuation operator is*

$$\kappa \colon = Id - \Pi_H.$$

Then one can write

$$(\nu_T(\mathbf{u}_h)(\mathbb{D}\mathbf{u}_h - \Pi_H \mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h) = (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h)$$

for the turbulence model and skip the third equation. The operator κ takes the resolved small-scale fluctuations of the deformation. If we now think of a FE approximation and take a cellwise constant $\nu_T(\mathbf{u}_h)$, which will be denoted by $\nu_T^K(\mathbf{u}_h)$ per cell $K \subset \Omega$, the additional term becomes symmetric with

$$\begin{aligned} (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h) &= \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)(\kappa(\mathbb{D}\mathbf{u}_h), \mathbb{D}\mathbf{v}_h)_K \\ &= \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)(\kappa(\mathbb{D}\mathbf{u}_h), \kappa(\mathbb{D}\mathbf{v}_h))_K = (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \kappa(\mathbb{D}\mathbf{v}_h)). \end{aligned} \tag{5}$$

In this case, one can write the problem as: find $\mathbf{u}_h \colon [0,T] \to V_h$ and $p_h \colon (0,T] \to Q_h$ satisfying

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) + (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \kappa(\mathbb{D}\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h, \\ (q_h, \nabla \cdot \mathbf{u}_h) &= 0 \qquad\quad \forall q_h \in Q_h. \end{aligned}$$

In the context of stabilization techniques based on local projection, this approach is strongly related to the one denoted as *gradient-based* local projection stabilization, e.g. in Ref. [18]. In the first part of the paper, we consider a rather general subgrid model for $\nu_T(\mathbf{u}_h)$. Later on, we consider a Smagorinsky-type model. In a second step, we incorporate a pressure subgrid scale model using the so-called grad-div stabilization. For simplicity, we introduce no coarse grid space for the pressure. Then the modified VMS method reads:

$$\begin{aligned} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h) & \\ +(\gamma(\mathbf{u}_h)(\nabla \cdot \mathbf{u}_h), \nabla \cdot \mathbf{v}_h) + (\nu_T(\mathbf{u}_h)\kappa(\mathbb{D}\mathbf{u}_h), \kappa(\mathbb{D}\mathbf{v}_h)) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h, \\ (q_h, \nabla \cdot \mathbf{u}_h) &= 0 \qquad\quad \forall q_h \in Q_h \end{aligned} \tag{6}$$

with

$$(\gamma(\mathbf{u}_h)(\nabla \cdot \mathbf{u}_h), \nabla \cdot \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_K$$

where $\gamma(\mathbf{u}_h)$ denotes a non-negative user-chosen parameter function being cellwise constant on $K \in \mathcal{T}_h$.

2.3. Implementation Aspects

Let us consider some implementation aspects related to the VMS model for the velocity in the context of Section 2.2. The approach presented above is strongly related to the stabilization via local projection. That is why one can find hints for the implementation in papers on that topic. The following is based especially on Refs. [14, 23].

One has to implement a term $\sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)(\kappa \mathbb{D}\mathbf{u}_h, \kappa \mathbb{D}\mathbf{v}_h)_K$, where $\nu_T^K(\mathbf{u}_h)$ is constant per cell K due to (5). Let $\{\Psi_1, \ldots, \Psi_{N_V}\} \subset V_h$ be a basis of V_h and $\{\Phi_1, \ldots, \Phi_{N_L}\} \subset L_H$ be a basis of the coarse space L_H. In the following, we only consider the local numbering of the basis functions on a fixed cell K. It holds

$$\begin{aligned} (\kappa \mathbb{D}\Psi_j, \kappa \mathbb{D}\Psi_i)_K &= (\mathbb{D}\Psi_j - \Pi_H \mathbb{D}\Psi_j, \mathbb{D}\Psi_i - \Pi_H \mathbb{D}\Psi_i)_K = (\mathbb{D}\Psi_j - \Pi_H \mathbb{D}\Psi_j, \mathbb{D}\Psi_i)_K \\ &= (\mathbb{D}\Psi_j, \mathbb{D}\Psi_i)_K - (\Pi_H \mathbb{D}\Psi_j, \mathbb{D}\Psi_i)_K, \end{aligned}$$

where the local projection Π_H is defined by

$$(\Pi_H \mathbb{D}\Psi_j, \Phi_k)_K = (\mathbb{D}\Psi_j, \Phi_k)_K$$

for all $k = 1, \dots N_L$. We can also write this operator in the form

$$\Pi_H \mathbb{D}\Psi_j = \sum_{l=1}^{N_L} P_{K,l,j} \Phi_l.$$

For clarity, let us define the local mass matrix, the local stiffness matrix and the transfer matrix which does the transfer between the deformation tensor and the coarse space on a cell K

$$M_K := ((\Phi_l, \Phi_k)_K)_{k,l}, \qquad K_K := ((\mathbb{D}\Psi_j, \mathbb{D}\Psi_i)_K)_{i,j}, \qquad \text{and} \quad T_K := ((\mathbb{D}\Psi_j, \Phi_k)_K)_{k,j}. \tag{7}$$

With these matrices we get $M_K P_K = T_K$, where M_K is invertible and

$$(\Pi_H \mathbb{D}\Psi_j, \mathbb{D}\Psi_i)_K = \sum_{l=1}^{N_L} P_{K,l,j} (\Phi_l, \mathbb{D}\Psi_i)_K = \sum_{l=1}^{N_L} (T_K)_{l,i} (M_K^{-1} T_K)_{l,j} = (T_K^T M_K^{-1} T_K)_{i,j}.$$

Together with the stiffness matrix, we are now able to write the turbulence model with matrices we can implement

$$\sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)(\kappa \mathbb{D}\mathbf{u}_h, \kappa \mathbb{D}\mathbf{v}_h)_K = \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)(K_K - T_K^T M_K^{-1} T_K).$$

A concrete implementation of a Smagorinsky-type $\nu_T^K(\mathbf{u}_h)$ is given later in (20), where the matrices of this subsection are used.

3. A Priori Error Analysis

In this section, we will consider the a priori analysis of the VMS-scheme (6) where the (nonlinear) coefficients of the subgrid models are not specified yet. Thus it is an extension to the work of Refs. [14, 15, 16] where the authors only considered a constant turbulent viscosity. Moreover, in Ref. [16] the order of the fluctuation operator $\kappa = I - \Pi_H$ in Definition 2.3 and of the velocity deformation tensor are interchanged, and a subgrid pressure term is missing. There were also other publications on similar approaches based on the framework of local projection stabilization like Ref. [5]. For the theory presented here, we are using the space of discretely divergence-free functions

$$V_{h,\mathrm{div}} := \{ \mathbf{v}_h \in V_h \mid (\nabla \cdot \mathbf{v}_h, q_h) = 0 \ \forall q_h \in Q_h \}.$$

Since we will use only finite element spaces V_h and Q_h which fulfill the discrete Ladyžhenskaya-Babuška-Brezzi condition (4), the space $V_{h,\mathrm{div}}$ is not empty. Later on we take advantage of the existence of a divergence-preserving interpolation operator I_h onto $V_{h,div}$ which had been shown in Ref. [9].
The problem (6) is equivalent to: find $\mathbf{u}_h : [0,T] \longrightarrow V_{h,\mathrm{div}}$ satisfying

$$\begin{aligned}(\partial_t \mathbf{u}_h, \mathbf{v}_h) + (2\nu \mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \sum_{K \in \mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}_h)_K& \\ + \sum_{K \in \mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,(\kappa \mathbb{D}\mathbf{u}_h, \kappa \mathbb{D}\mathbf{v}_h)_K = (\mathbf{f}, \mathbf{v}_h)&\end{aligned} \tag{8}$$

for all $\mathbf{v}_h \in V_{h,\mathrm{div}}$.

3.1. Stability

We derive a semidiscrete a priori error estimate for the problem (8). Therefore one has to prove the stability of the continuous and the discrete solutions $\mathbf{u}$ and $\mathbf{u}_h$.

Lemma 3.1. *Let $\mathbf{u}_h$ be the solution of (8). Assume $\mathbf{f} \in L^1(0,T;L^2(\Omega))$ and $\mathbf{u}_0 \in [L^2(\Omega)]^3$, then it holds $\mathbf{u}_h \in L^\infty(0,T;L^2(\Omega))$, $\mathbb{D}\mathbf{u}_h \in L^2(0,T;L^2(\Omega))$ such that*

$$\|\mathbf{u}_h(t)\|_0 \leq \|\mathbf{u}_0\|_0 + \|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))},$$

and

$$\|\mathbf{u}_h(t)\|_0^2 + 4\nu \|\mathbb{D}\mathbf{u}_h\|^2_{L^2(0,t;L^2(\Omega))} + 2\int_0^t \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h) \|\nabla\cdot\mathbf{u}_h\|^2_{L^2(K)}\,dt$$
$$+2\int_0^t \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\|\kappa\mathbb{D}\mathbf{u}_h\|^2_{L^2(K)}\,dt \leq 2\|\mathbf{u}_0\|_0^2 + 3\|\mathbf{f}\|^2_{L^1(0,t;L^2(\Omega))}$$

for all $t \in (0,T]$.

The proof is a variant of a similar proof in Ref. [19], Section 1. One only has to take care of the different turbulence models.

Remark 3.2. *One can prove a similar stability lemma for the solution $\mathbf{u}$ of the continuous problem (2). For this purpose it is easy to also adapt the proof in Ref. [19].*

3.2. A Priori Error Estimate

In this subsection, we prove a semidiscrete a priori error estimate for the problem (8). On this account we have to consider the regularity of the continuous and discrete solutions. For the proof of Theorem 3.5, we need that for the continuous solution $\mathbf{u}$ of (2) and the discrete solution $\mathbf{u}_h$ of (8)

$$\partial_t \mathbf{u}_{(h)} \in L^2(0,t;H^{-1}(\Omega)) \qquad \text{and} \qquad \nabla\mathbf{u} \in L^4(0,t;L^2(\Omega)) \tag{9}$$

holds true. Together with the regularity assumptions on the data from Lemma 3.1 we obtain the validity of Serrin's uniqueness condition from Ref. [31], Section V.1.5, up to $n = 3$, since from the Sobolev imbedding Theorem we get $H^1(\Omega) \hookrightarrow L^6(\Omega)$. The main point to check for Serrin's condition is whether the continuous solution fulfills $\mathbf{u} \in L^s(0,t;L^q(\Omega))$ with $\Omega \subset \mathbb{R}^n$ and $\frac{n}{q} + \frac{2}{s} \leq 1$. This is fulfilled here and hence, the solution of (2) is unique.
For clarity, let us define some abbreviations, where we make use of (5) and the Definitions 2.2 and 2.3.

Definition 3.3. *Let $\Pi_H \colon L \to L_H$ be the L^2-projection of Definition 2.3 and $K \in \mathcal{T}_h$.*

(i) The multiscale viscosity $\nu^K_{VMS}(\mathbf{u}_h,\mathbf{v}_h)$ is defined by

$$\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\|\kappa\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)} = \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\left(\|\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)} - \|\Pi_H\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)}\right)$$
$$= \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\left(1 - \frac{\|\Pi_H\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)}}{\|\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)}}\right)\|\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)} =: \sum_{K\in\mathcal{T}_h} \nu^K_{VMS}(\mathbf{u}_h,\mathbf{v}_h)\|\mathbb{D}\mathbf{v}_h\|^2_{L^2(K)}. \tag{10}$$

(ii) The local modified viscosity $\nu^K_{mod}(\mathbf{u}_h,\mathbf{v}_h)$ is defined by

$$\nu^K_{mod}(\mathbf{u}_h,\mathbf{v}_h) := 2\nu + \nu^K_{VMS}(\mathbf{u}_h,\mathbf{v}_h).$$

(iii) We will also use global variants of this viscosity function within this article. An elementwise constant viscosity function will be denoted by $\tilde{\nu}_{mod}(\mathbf{u}_h,\mathbf{v}_h)$ and the global minimum will be denoted by $\nu^{min}_{mod}(\mathbf{u}_h,\mathbf{v}_h)$, where

$$\tilde{\nu}_{mod}(\mathbf{u}_h,\mathbf{v}_h)|_K := \nu^K_{mod}(\mathbf{u}_h,\mathbf{v}_h) \qquad \textit{and} \qquad \nu^{min}_{mod}(\mathbf{u}_h,\mathbf{v}_h) := \min_{K\in\mathcal{T}_h} \nu^K_{mod}(\mathbf{u}_h,\mathbf{v}_h).$$

(iv) *Together with (ii) we will write*

$$|||\mathbf{u}(t)|||^2 := \|\mathbf{u}(t)\|_0^2 + \sum_{K\in\mathcal{T}_h} \int_0^t \frac{1}{2}\nu_{mod}^K(\mathbf{u}_h,\mathbf{u})\,\|\mathbb{D}(\mathbf{u})\|_{L^2(K)}^2\,dt + \sum_{K\in\mathcal{T}_h}\int_0^t \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{u}\|_{L^2(K)}^2\,dt.$$

Remark 3.4. *(i) In case of $\|\mathbb{D}\mathbf{v}_h\|_{L^2(K)}^2 = 0$, we set $\nu_{VMS}^K(\mathbf{u}_h,\mathbf{v}_h) = 0$. For this reason one could demand $\nu_T^K(\mathbf{u}_h) = 0$ if $\|\mathbb{D}\mathbf{u}_h\|_{L^2(K)}^2 = 0$, which is the case for the Smagorinsky-type model considered in Section 4 and 5.*

(ii) Note that $\nu_{VMS}^K(\mathbf{u}_h,\mathbf{v}_h) \geq 0$, because of $0 \leq \|\Pi_H\mathbb{D}\mathbf{v}_h\|_{L^2(K)} \leq \|\mathbb{D}\mathbf{v}_h\|_{L^2(K)}$. Hence, we also get $\tilde{\nu}_{mod}(\mathbf{u}_h,\mathbf{v}_h) \geq 2\nu > 0$ on the whole domain Ω.

(iii) The local modified viscosity $\nu_{mod}^K(\mathbf{u}_h,\mathbf{v}_h)$ contains the sum of the model viscosity and the viscosity stemming from the turbulence model. Even if formally the turbulence model only acts on the small resolved scales the term $\nu_{VMS}^K(\mathbf{u}_h,\mathbf{v}_h)$ measures the influence on all resolved scales, where we used the properties of the L^2-projection Π_H. One can also think of a reduced Reynolds number $Re_{red} = 1/\inf_{t\in[0,T]}\nu_{mod}^{min}(\mathbf{u}_h,\mathbf{u})$, where $Re_{red} \leq 2Re = 2/\nu$ holds always true. See also Ref. [15] for similar considerations.

Theorem 3.5. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of (2) and of (8), respectively. Let $\mathbf{f} \in L^1(0,T;L^2(\Omega))$, $\mathbf{u}_0 \in [L^2(\Omega)]^3$, let I_h be an interpolation operator onto $L^4(0,T;V_{h,div})$ and J_h an interpolation operator onto $L^2(0,T;Q_h)$. Suppose that (9) is true and let*

$$\sum_{K\in\mathcal{T}_h}\nu_{VMS}^K(\mathbf{u}_h,\mathbf{u}-I_h\mathbf{u})\,\|\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^2 \in L^1(0,T), \qquad \sum_{K\in\mathcal{T}_h}\nu_{VMS}^K(\mathbf{u}_h,\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2 \in L^1(0,T); \tag{11}$$

then

$$\begin{aligned}
&|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 \leq 2\,|||(\mathbf{u}-I_h\mathbf{u})(t)|||^2\\
&+\exp\left(\frac{27C_{LT}^4}{2\inf_{\tau\in[0,t]}(\nu_{mod}^{min}(\mathbf{u}_h,\mathbf{u}_h-I_h\mathbf{u})^3)}\,\|\mathbb{D}\mathbf{u}\|_{L^4(0,t;L^2(\Omega))}^4\right)\Bigg\{\\
&\quad 2\,\|\mathbf{u}_0-I_h\mathbf{u}(0)\|_0^2 + 24\nu\,\|\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(0,t;L^2(\Omega))}^2 + \frac{12C_{Ko}^2}{\nu_{mod}^{min}(\mathbf{u}_h,\mathbf{u}_h-I_h\mathbf{u})}\,\|\partial_t(\mathbf{u}-I_h\mathbf{u})\|_{H^{-1}(\Omega)}^2\\
&\quad + \frac{12C_{LT}^2}{\inf_{\tau\in[0,t]}(\nu_{mod}^{min}(\mathbf{u}_h,\mathbf{u}_h-I_h\mathbf{u}))}\Big(\|\mathbf{u}_h\|_{L^\infty(0,t;L^2(\Omega))}\,\|\mathbb{D}\mathbf{u}_h\|_{L^2(0,t;L^2(\Omega))}\,\|\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^4(0,t;L^2(\Omega))}^2\\
&\qquad + C_FC_{Ko}\,\|\mathbb{D}\mathbf{u}\|_{L^4(0,t;L^2(\Omega))}^2\,\|\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^4(0,t;L^2(\Omega))}^2\Big)\\
&\quad + 4\sum_{K\in\mathcal{T}_h}\Bigg[\int_0^t \min\left(\frac{9C_{Ko}^2}{\inf_{\tau\in[0,t]}(\nu_{mod}^{min}(\mathbf{u}_h,\mathbf{u}_h-I_h\mathbf{u}))},\frac{1}{\gamma_K(\mathbf{u}_h)}\right)\Big(\|p-J_hp\|_{L^2(K)}^2\\
&\qquad + \gamma_K^2(\mathbf{u}_h)\,\|\nabla\cdot(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^2\Big)\,dt\\
&\quad + 6\int_0^t \nu_T^K(\mathbf{u}_h)\left(\|\kappa\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^2 + \|\kappa\mathbb{D}\mathbf{u}\|_{L^2(K)}^2\right)dt\Bigg]\Bigg\}
\end{aligned} \tag{12}$$

holds for every $t \in [0,T]$.

Proof. We split the error into model error $\mathbf{e}_h$ and approximation error ϵ

$$\mathbf{u}_h - \mathbf{u} = (\mathbf{u}_h - I_h\mathbf{u}) - (\mathbf{u} - I_h\mathbf{u}) =: \mathbf{e}_h - \epsilon.$$

Now one can subtract (2) from (8), use $\mathbf{e}_h \in V_{h,\text{div}}$ as test function and obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t \|\mathbf{e}_h\|_0^2 + \sum_{K\in\mathcal{T}_h} \nu_{\text{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)}^2 \\
&\qquad = (\partial_t\epsilon,\mathbf{e}_h) + (2\nu\mathbb{D}\epsilon,\mathbb{D}\mathbf{e}_h) + b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h) - b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}_h) - (p-\lambda_h,\nabla\cdot\mathbf{e}_h) \\
&\qquad + \sum_{K\in\mathcal{T}_h} \gamma_K\,(\nabla\cdot\epsilon,\nabla\cdot\mathbf{e}_h)_K + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,(\kappa\mathbb{D}\epsilon,\kappa\mathbb{D}\mathbf{e}_h)_K - \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,(\kappa\mathbb{D}\mathbf{u},\kappa\mathbb{D}\mathbf{e}_h)_K
\end{aligned} \tag{13}
$$

for all $\lambda_h \in Q_h$. Next, one has to estimate all the terms on the right hand side of (13). Many of them are already included in Ref. [15], nevertheless we do all the estimations here since there are some differences. The first term is estimated in detail

$$
\begin{aligned}
(\partial_t\epsilon,\mathbf{e}_h) &\le \|\partial_t\epsilon\|_{H^{-1}(\Omega)}\,\|\nabla\mathbf{e}_h\|_0 \le C_{\text{Ko}}\,\|\partial_t\epsilon\|_{H^{-1}(\Omega)}\,\|\mathbb{D}\mathbf{e}_h\|_0 = C_{\text{Ko}}\,\|\partial_t\epsilon\|_{H^{-1}(\Omega)} \left\|\sqrt{\frac{\tilde\nu_{\text{mod}}(\mathbf{u}_h,\mathbf{e}_h)}{\tilde\nu_{\text{mod}}(\mathbf{u}_h,\mathbf{e}_h)}}\mathbb{D}\mathbf{e}_h\right\|_0 \\
&\le \frac{C_{\text{Ko}}}{\sqrt{\nu_{\text{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}}\,\|\partial_t\epsilon\|_{H^{-1}(\Omega)} \left\|\sqrt{\tilde\nu_{\text{mod}}(\mathbf{u}_h,\mathbf{e}_h)}\mathbb{D}\mathbf{e}_h\right\|_0 \\
&\le \frac{3C_{\text{Ko}}^2}{\nu_{\text{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\,\|\partial_t\epsilon\|_{H^{-1}(\Omega)}^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\text{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 .
\end{aligned}
$$

For the others it holds

$$
\begin{aligned}
(2\nu\mathbb{D}\epsilon,\mathbb{D}\mathbf{e}_h) &\le 2\nu\,\|\mathbb{D}\epsilon\|_0\,\|\mathbb{D}\mathbf{e}_h\|_0 \le 6\nu\,\|\mathbb{D}\epsilon\|_0^2 + \frac{\nu}{6}\,\|\mathbb{D}\mathbf{e}_h\|_0^2 \\
(p-\lambda_h,\nabla\cdot\mathbf{e}_h) &\le \|p-\lambda_h\|_0\,\|\nabla\cdot\mathbf{e}_h\|_0 \le C_{\text{Ko}}\sqrt{3}\,\|p-\lambda_h\|_0\,\|\mathbb{D}\mathbf{e}_h\|_0 \\
&\le \frac{9C_{\text{Ko}}^2}{\nu_{\text{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\,\|p-\lambda_h\|_0^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\text{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 \\
(p-\lambda_h,\nabla\cdot\mathbf{e}_h) &\le \sum_{K\in\mathcal{T}_h} \frac{1}{\gamma_K(\mathbf{u}_h)}\,\|p-\lambda_h\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\gamma_K(\mathbf{u}_h)}{4}\,\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)}^2 \\
\sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,(\nabla\cdot\epsilon,\nabla\cdot\mathbf{e}_h)_K &\le \|\tilde\gamma(\mathbf{u}_h)\nabla\cdot\epsilon\|_0\,C_{\text{Ko}}\sqrt{3}\,\|\mathbb{D}\mathbf{e}_h\|_0 \\
&\le \frac{C_{\text{Ko}}\sqrt{3}}{\sqrt{\nu_{\text{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}}\,\|\tilde\gamma(\mathbf{u}_h)\nabla\cdot\epsilon\|_0 \left\|\sqrt{\tilde\nu_{\text{mod}}(\mathbf{u}_h,\mathbf{e}_h)}\mathbb{D}\mathbf{e}_h\right\|_0 \\
&\le \sum_{K\in\mathcal{T}_h} \frac{9C_{\text{Ko}}^2\gamma_K^2(\mathbf{u}_h)}{\nu_{\text{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\,\|\nabla\cdot\epsilon\|_{L^2(K)}^2 + \frac{1}{12}\sum_{K\in\mathcal{T}_h} \nu_{\text{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 \\
\sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,(\nabla\cdot\epsilon,\nabla\cdot\mathbf{e}_h)_K &\le \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\epsilon\|_{L^2(K)}\,\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)} \\
&\le \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\epsilon\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\gamma_K(\mathbf{u}_h)}{4}\,\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)}^2 \\
\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,(\kappa\mathbb{D}\epsilon,\kappa\mathbb{D}\mathbf{e}_h)_K &\le \sum_{K\in\mathcal{T}_h} 6\nu_T^K(\mathbf{u}_h)\,\|\kappa\mathbb{D}\epsilon\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\nu_T^K(\mathbf{u}_h)}{24}\,\|\kappa\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 \\
&= \sum_{K\in\mathcal{T}_h} 6\nu_{\text{VMS}}^K(\mathbf{u}_h,\epsilon)\,\|\mathbb{D}\epsilon\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\nu_{\text{VMS}}^K(\mathbf{u}_h,\mathbf{e}_h)}{24}\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 \\
\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,(\kappa\mathbb{D}\mathbf{u},\kappa\mathbb{D}\mathbf{e}_h)_K &\le \sum_{K\in\mathcal{T}_h} 6\nu_{\text{VMS}}^K(\mathbf{u}_h,\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\nu_{\text{VMS}}^K(\mathbf{u}_h,\mathbf{e}_h)}{24}\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2 .
\end{aligned}
$$

For these estimates we used Cauchy-Schwarz inequality, Young's inequality and Korn's inequality, see (3). When the divergence came into play we used $\|\nabla\cdot\mathbf{v}\|_0^2 \leq 3\,\|\nabla\mathbf{v}\|_0^2$ as well. In the 5th term we used a global variant $\tilde{\gamma}$ of the cellwise parameter γ_K, which is defined analogously to $\tilde{\nu}_{\mathrm{mod}}$ in Definition 3.3. There are two possibilities to estimate the pressure term and the grad-div term leading to

$$
\begin{aligned}
(p-\lambda_h,\nabla\cdot\mathbf{e}_h) \leq{}& \sum_{K\in\mathcal{T}_h} \min\left(\frac{9C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)},\frac{1}{\gamma_K(\mathbf{u}_h)}\right)\|p-\lambda_h\|_{L^2(K)}^2\\
&+\frac{1}{12}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2+\sum_{K\in\mathcal{T}_h}\frac{\gamma_K(\mathbf{u}_h)}{4}\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)}^2\,,\\
\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\,(\nabla\cdot\epsilon,\nabla\cdot\mathbf{e}_h)_K \leq{}& \sum_{K\in\mathcal{T}_h} \min\left(\frac{9C_{\mathrm{Ko}}^2\gamma_K^2(\mathbf{u}_h)}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)},\gamma_K(\mathbf{u}_h)\right)\|\nabla\cdot\epsilon\|_{L^2(K)}^2\\
&+\frac{1}{12}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2+\sum_{K\in\mathcal{T}_h}\frac{\gamma_K(\mathbf{u}_h)}{4}\|\nabla\cdot\mathbf{e}_h\|_{L^2(K)}^2\,.
\end{aligned}
$$

We did not discuss the terms related to the nonlinearity in (13) yet. To deal with them we can use that

$$
\begin{aligned}
&2\left[b_S(\epsilon,\mathbf{u},\mathbf{e}_h)-b_S(\mathbf{e}_h,\mathbf{u},\mathbf{e}_h)+b_S(\mathbf{u}_h,\epsilon,\mathbf{e}_h)\right]\\
=&[((\epsilon\cdot\nabla)\mathbf{u},\mathbf{e}_h)-((\epsilon\cdot\nabla)\mathbf{e}_h,\mathbf{u})]-[((\mathbf{e}_h\cdot\nabla)\mathbf{u},\mathbf{e}_h)-((\mathbf{e}_h\cdot\nabla)\mathbf{e}_h,\mathbf{u})]+[((\mathbf{u}_h\cdot\nabla)\epsilon,\mathbf{e}_h)-((\mathbf{u}_h\cdot\nabla)\mathbf{e}_h,\epsilon)]\\
=&[((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{e}_h)-((\mathbf{u}\cdot\nabla)\mathbf{e}_h,\mathbf{u})]-((I_h\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{e}_h)+((I_h\mathbf{u}\cdot\nabla)\mathbf{e}_h,\mathbf{u})\\
&-((\mathbf{u}_h\cdot\nabla)\mathbf{u},\mathbf{e}_h)+((I_h\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{e}_h)-((\mathbf{u}_h\cdot\nabla)\mathbf{e}_h,\mathbf{u})-((I_h\mathbf{u}\cdot\nabla)\mathbf{e}_h,\mathbf{u})\\
&+((\mathbf{u}_h\cdot\nabla)\mathbf{u},\mathbf{e}_h)-((\mathbf{u}_h\cdot\nabla)I_h\mathbf{u},\mathbf{e}_h)-((\mathbf{u}_h\cdot\nabla)\mathbf{e}_h,\mathbf{u})+((\mathbf{u}_h\cdot\nabla)\mathbf{e}_h,I_h\mathbf{u})\\
=&2b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h)-((\mathbf{u}_h\cdot\nabla)(\mathbf{u}_h-\mathbf{e}_h),\mathbf{e}_h)+((\mathbf{u}_h\cdot\nabla)\mathbf{e}_h,I_h\mathbf{u})\\
=&2\left[b_S(\mathbf{u},\mathbf{u},\mathbf{e}_h)-b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{e}_h)\right].
\end{aligned}
$$

The terms arising from the skew-symmetric trilinear form are estimated by the inequality

$$
b_S(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C_{LT}\,\|\mathbf{u}\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{u}\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{v}\|_0\,\|\mathbb{D}\mathbf{w}\|_0 \tag{14}
$$

which is proved in Ref. [21], Lemma 2.2 (f). The term $b_S(\mathbf{e}_h,\mathbf{u}_h,\mathbf{e}_h)$ is the most difficult one. It holds

$$
\begin{aligned}
b_S(\mathbf{e}_h,\mathbf{u},\mathbf{e}_h) &\leq C_{LT}\,\|\mathbf{e}_h\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{u}\|_0\,\|\mathbb{D}\mathbf{e}_h\|_0^{\frac{3}{2}} \leq \frac{C_{LT}}{(\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h))^{3/4}}\,\|\mathbf{e}_h\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{u}\|_0\left\|\sqrt{\tilde{\nu}_{\mathrm{mod}}(\mathbf{u}_h,\mathbf{e}_h)}\mathbb{D}\mathbf{e}_h\right\|_0^{\frac{3}{2}}\\
&\leq \frac{27C_{LT}^4}{4\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)^3}\,\|\mathbb{D}\mathbf{u}\|_0^4\,\|\mathbf{e}_h\|_0^2+\frac{1}{4}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2\,,
\end{aligned}
$$

where Young's inequality

$$
ab \leq \frac{\varepsilon}{p}a^p+\frac{\varepsilon^{-q/p}}{q}b^q
$$

with $p=\frac{4}{3}$, $q=4$ and $\varepsilon=\frac{1}{3}$ was applied. For the remaining two terms we obtain

$$
\begin{aligned}
b_S(\epsilon,\mathbf{u},\mathbf{e}_h) &\leq C_{LT}\,\|\epsilon\|_0^{\frac{1}{2}}\,\|\mathbb{D}\epsilon\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{u}\|_0\,\|\mathbb{D}\mathbf{e}_h\|_0\\
&\leq \frac{3C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\,\|\epsilon\|_0\,\|\mathbb{D}\epsilon\|_0\,\|\mathbb{D}\mathbf{u}\|_0^2+\frac{1}{12}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2\\
b_S(\mathbf{u}_h,\epsilon,\mathbf{e}_h) &\leq C_{LT}\,\|\mathbf{u}_h\|_0^{\frac{1}{2}}\,\|\mathbb{D}\mathbf{u}_h\|_0^{\frac{1}{2}}\,\|\mathbb{D}\epsilon\|_0\,\|\mathbb{D}\mathbf{e}_h\|_0\\
&\leq \frac{3C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\,\|\mathbb{D}\epsilon\|_0^2\,\|\mathbf{u}_h\|_0\,\|\mathbb{D}\mathbf{u}_h\|_0+\frac{1}{12}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\,\|\mathbb{D}\mathbf{e}_h\|_{L^2(K)}^2\,.
\end{aligned}
$$

Now each term in (13) is estimated and we can summarize

$$
\begin{aligned}
&\frac{1}{2}\partial_t \left\|\mathbf{e}_h\right\|_0^2+\frac{1}{4}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\left\|\mathbb{D}\mathbf{e}_h\right\|_{L^2(K)}^2+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\left\|\nabla\cdot\mathbf{e}_h\right\|_{L^2(K)}^2\\
\le&\frac{3C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\left\|\partial_t\epsilon\right\|_{H^{-1}(\Omega)}^2+6\nu\left\|\mathbb{D}\epsilon\right\|_0^2+\frac{3C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\left(\left\|\mathbb{D}\mathbf{u}\right\|_0^2\left\|\epsilon\right\|_0\left\|\mathbb{D}\epsilon\right\|_0+\left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\mathbb{D}\epsilon\right\|_0^2\right)\\
&+\sum_{K\in\mathcal{T}_h}\Bigg[\min\left(\frac{9C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)},\frac{1}{\gamma_K(\mathbf{u}_h)}\right)\left(\left\|p-\lambda_h\right\|_{L^2(K)}^2+\gamma_K^2(\mathbf{u}_h)\left\|\nabla\cdot\epsilon\right\|_{L^2(K)}^2\right)\\
&+6\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\epsilon)\left\|\mathbb{D}\epsilon\right\|_{L^2(K)}^2+6\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{u})\left\|\mathbb{D}\mathbf{u}\right\|_{L^2(K)}^2\Bigg]\\
&+\frac{27C_{LT}^4}{4\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)^3}\left\|\mathbb{D}\mathbf{u}\right\|_0^4\left\|\mathbf{e}_h\right\|_0^2.
\end{aligned}
\tag{15}
$$

The next step in the proof will be the application of Gronwall's Lemma, see Ref. [27], Lemma 1.4.1. It states that for $g_i(t)\in L^1(0,T)$ $(i\in\{1,2,3\})$, $\int_0^t g_1(s)ds$, $\int_0^t g_2(s)ds$ continuous and non-decreasing on $[0,T]$, $g_3(t)$ non-negative and

$$
\partial_t\left\|\mathbf{e}_h\right\|_0^2+g_1(t)\le g_2(t)+g_3(t)\left\|\mathbf{e}_h\right\|_0^2 \tag{16}
$$

there holds

$$
\left\|\mathbf{e}_h(t)\right\|_0^2+\int_0^t g_1(t)dt\le\exp\left(\int_0^t g_3(s)ds\right)\left(\left\|\mathbf{e}_h(0)\right\|_0^2+\int_0^t g_2(s)ds\right) \tag{17}
$$

for all $t\in[0,T]$.
At this point we have to find functions g_i which are contained in $L^1(0,T)$. For this reason we set

$$
\begin{aligned}
g_1(t):=&\frac{1}{2}\sum_{K\in\mathcal{T}_h}\nu_{\mathrm{mod}}^K(\mathbf{u}_h,\mathbf{e}_h)\left\|\mathbb{D}\mathbf{e}_h\right\|_{L^2(K)}^2+\sum_{K\in\mathcal{T}_h}\gamma_K(\mathbf{u}_h)\left\|\nabla\cdot\mathbf{e}_h\right\|_{L^2(K)}^2,\\
g_2(t):=&\frac{6C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\left\|\partial_t\epsilon\right\|_{H^{-1}(\Omega)}^2+12\nu\left\|\mathbb{D}\epsilon\right\|_0^2+\frac{6C_{LT}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)}\left(\left\|\mathbb{D}\mathbf{u}\right\|_0^2\left\|\epsilon\right\|_0\left\|\mathbb{D}\epsilon\right\|_0+\left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\mathbb{D}\epsilon\right\|_0^2\right)\\
&+2\sum_{K\in\mathcal{T}_h}\Bigg[\min\left(\frac{9C_{\mathrm{Ko}}^2}{\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)},\frac{1}{\gamma_K(\mathbf{u}_h)}\right)\left(\left\|p-\lambda_h\right\|_{L^2(K)}^2+\gamma_K^2(\mathbf{u}_h)\left\|\nabla\cdot\epsilon\right\|_{L^2(K)}^2\right)\\
&+6\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\epsilon)\left\|\mathbb{D}\epsilon\right\|_{L^2(K)}^2+6\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{u})\left\|\mathbb{D}\mathbf{u}\right\|_{L^2(K)}^2\Bigg],\\
g_3(t):=&\frac{27C_{LT}^4}{2\inf_{t\in[0,T]}\left(\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{e}_h)^3\right)}\left\|\mathbb{D}\mathbf{u}\right\|_0^4.
\end{aligned}
$$

With these definitions we have (16). By using (15) and if the $L^1(0,T)$ regularity is checked the setting will comply the requirements, because the other conditions are straightforward. To prove this we will use the stability result of Lemma 3.1 for the terms stemming from the nonlinearities in g_2, all the other terms are directly clear. We obtain

$$
\begin{aligned}
\int_0^t\left\|\mathbf{u}_h\right\|_0\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\mathbb{D}\epsilon\right\|_0^2dt\le&\left\|\mathbf{u}_h\right\|_{L^\infty(0,t;L^2(\Omega))}\int_0^t\left\|\mathbb{D}\mathbf{u}_h\right\|_0\left\|\mathbb{D}\epsilon\right\|_0^2dt\\
\le&\left\|\mathbf{u}_h\right\|_{L^\infty(0,t;L^2(\Omega))}\left\|\mathbb{D}\mathbf{u}_h\right\|_{L^2(0,t;L^2(\Omega))}\left\|\mathbb{D}\epsilon\right\|_{L^4(0,t;L^2(\Omega))}^2\\
\le&\sqrt{\frac{2}{\inf_{t\in[0,T]}\nu_{\mathrm{mod}}^{\min}(\mathbf{u}_h,\mathbf{u}_h)}}\left(\left\|\mathbf{u}_0\right\|_0^2+\frac{3}{2}\left\|\mathbf{f}\right\|_{L^1(0,t;L^2(\Omega))}^2\right)\left\|\mathbb{D}\epsilon\right\|_{L^4(0,t;L^2(\Omega))}^2<\infty
\end{aligned}
$$

and

$$\int_0^t \|\mathbb{D}\mathbf{u}\|_0^2 \|\epsilon\|_0 \|\mathbb{D}\epsilon\|_0 \, dt \leq C_F C_{\mathrm{Ko}} \int_0^t \|\mathbb{D}\mathbf{u}\|_0^2 \|\mathbb{D}\epsilon\|_0^2 \, dt$$
$$\leq C_F C_{\mathrm{Ko}} \|\mathbb{D}\mathbf{u}\|^2_{L^4(0,t;L^2(\Omega))} \|\mathbb{D}\epsilon\|^2_{L^4(0,t;L^2(\Omega))} < \infty$$

via Poincaré-Friedrichs inequality from (3) and Hölder inequality. With the regularity assumptions of Theorem 3.5, in particular assumptions (9) and (11), we are now able to apply Gronwall's Lemma. This gives the estimate for the discretization error according to (17).
Finally a careful consideration of Definition 3.3 and $\nu^K_{\mathrm{VMS}}(\mathbf{u}_h, \mathbf{e}_h)$ shows $|||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 \leq 2\,|||\epsilon(t)|||^2 + 2\,|||\mathbf{e}_h(t)|||^2$. This concludes the proof. □

Finally we are interested in an L^2-error estimate for the pressure within the framework of the Definitions 2.2, 2.3 and 3.3.

Corollary 3.6. *Let $(\mathbf{u}, p) \in V \times Q$ and $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ be solutions of (2) and (6), respectively. Suppose that the regularity assumptions of Theorem 3.5 are valid, then*

$$\|p - p_h\|^2_{L^2(0,t;L^2(\Omega))} \leq 2\Big(1 + \frac{\sqrt{3}}{\beta}\Big)^2 \|p - J_h p\|^2_{L^2(0,t;L^2(\Omega))} + \frac{12}{\beta^2} \|\partial_t(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(0,t;H^{-1}(\Omega))}$$
$$+ C_u\, |||(\mathbf{u}-\mathbf{u}_h)(t)|||^2 + \frac{12}{\beta^2} \int_0^t \sum_{K \in \mathcal{T}_h} \nu^K_T(\mathbf{u}_h)^2 \|\kappa \mathbb{D}\mathbf{u}\|^2_{L^2(K)} \, dt, \quad (18)$$

where

$$C_u \;=\; \frac{12}{\beta^2}\left(4\nu + \max_{\tilde t \in (0,t), K \in \mathcal{T}_h} (3\gamma_K(\mathbf{u}_h)) + \max_{\tilde t \in (0,t), K \in \mathcal{T}_h} (2\nu^K_T(\mathbf{u}_h)) + \max_{\tilde t \in (0,t)} \frac{2C^2_{LT2}(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0)^2}{\nu^{min}_{mod}(\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h)}\right).$$

Proof. With the definition $\mathbf{e}_u := \mathbf{u}_h - \mathbf{u}$ we see that

$$b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b_S(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) = b_S(\mathbf{e}_u, \mathbf{u}_h, \mathbf{v}_h) + b_S(\mathbf{u}, \mathbf{e}_u, \mathbf{v}_h) \leq C_{\mathrm{LT2}} \left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right) \|\mathbb{D}\mathbf{e}_u\|_0 \|\nabla \mathbf{v}_h\|_0$$

for all $\mathbf{v}_h \in V_h$, where the constant C_{LT2} comes from Ref. [21], Lemma 2.2 (e) together with Korn's inequality.
For the proof we subtract (2) from (6) with an arbitrary function $\mathbf{v}_h \in V_h$ and obtain

$$(\partial_t \mathbf{e}_u, \mathbf{v}_h) + 2\nu\,(\mathbb{D}\mathbf{e}_u, \mathbb{D}\mathbf{v}_h) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b_S(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) + \sum_{K \in \mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla \cdot \mathbf{e}_u, \nabla \cdot \mathbf{v}_h)_K$$
$$+ \sum_{K \in \mathcal{T}_h} \nu^K_T(\mathbf{u}_h)(\kappa(\mathbb{D}\mathbf{e}_u), \kappa(\mathbb{D}\mathbf{v}_h)) + \sum_{K \in \mathcal{T}_h} \nu^K_T(\mathbf{u}_h)(\kappa(\mathbb{D}\mathbf{u}), \kappa(\mathbb{D}\mathbf{v}_h)) - (\nabla \cdot \mathbf{v}_h, (p_h - p)) = 0. \quad (19)$$

We split the pressure error as $p - p_h = (p - J_h p) + (J_h p - p_h)$. The discrete inf-sup condition (4) yields

$$\beta \|p_h - p\|_0 \leq \beta \|p_h - J_h p\|_0 + \beta \|p - J_h p\|_0 \leq \frac{(p_h - J_h p, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_0} + \beta \|p - J_h p\|_0$$
$$\leq \frac{(p_h - p, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_0} + \frac{(p - J_h p, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_0} + \beta \|p - J_h p\|_0 \leq \frac{(p_h - p, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_0} + (\beta + \sqrt{3}) \|p - J_h p\|_0 .$$

Furthermore, the remaining fraction can be estimated via (19), the Cauchy-Schwarz inequality, the Hölder inequality, Definition 3.3 and (5) by

$$
\begin{aligned}
\frac{(p_h - p, \nabla\cdot\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} \leq & \frac{1}{\|\nabla\mathbf{v}_h\|_0}\Bigg((\partial_t\mathbf{e}_u, \mathbf{v}_h) + 2\nu\,\|\mathbb{D}\mathbf{e}_u\|_0\,\|\mathbb{D}\mathbf{v}_h\|_0 \\
& + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,\|\kappa\mathbb{D}\mathbf{e}_u\|_{L^2(K)}\,\|\mathbb{D}\mathbf{v}_h\|_{L^2(K)} + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\,\|\kappa\mathbb{D}\mathbf{u}\|_{L^2(K)}\,\|\mathbb{D}\mathbf{v}_h\|_{L^2(K)} \\
& + \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_u\|_{L^2(K)}\,\|\nabla\cdot\mathbf{v}_h\|_{L^2(K)} + [b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) - b_S(\mathbf{u},\mathbf{u},\mathbf{v}_h)]\Bigg) \\
\leq & \frac{(\partial_t\mathbf{e}_u, \mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|_0} + \sqrt{\sum_{K\in\mathcal{T}_h} 3\gamma_K^2(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_u\|_{L^2(K)}^2} + C_{\mathrm{LT2}}\left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right)\|\mathbb{D}\mathbf{e}_u\|_0 \\
& + 2\nu\,\|\mathbb{D}\mathbf{e}_u\|_0 + \sqrt{\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{e}_u)\,\|\mathbb{D}\mathbf{e}_u\|_{L^2(K)}^2} + \sqrt{\sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)^2\,\|\kappa\mathbb{D}\mathbf{u}\|_{L^2(K)}^2}.
\end{aligned}
$$

That is why

$$
\begin{aligned}
\frac{(p_h - p, \nabla\cdot\mathbf{v}_h)^2}{6\,\|\nabla\mathbf{v}_h\|_0^2} \leq & \frac{(\partial_t\mathbf{e}_u, \mathbf{v}_h)^2}{\|\nabla\mathbf{v}_h\|_0^2} + \sum_{K\in\mathcal{T}_h} 3\gamma_K^2(\mathbf{u}_h)\,\|\nabla\cdot\mathbf{e}_u\|_{L^2(K)}^2 + C_{\mathrm{LT2}}^2\left(\|\mathbb{D}\mathbf{u}_h\|_0 + \|\mathbb{D}\mathbf{u}\|_0\right)^2\|\mathbb{D}\mathbf{e}_u\|_0^2 \\
& + 4\nu^2\,\|\mathbb{D}\mathbf{e}_u\|_0^2 + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)\nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{e}_u)\,\|\mathbb{D}\mathbf{e}_u\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h} \nu_T^K(\mathbf{u}_h)^2\,\|\kappa\mathbb{D}\mathbf{u}\|_{L^2(K)}^2 .
\end{aligned}
$$

Integration over $(0,t)$ concludes the proof. In particular, the term stemming from the time derivative can be estimated by

$$
\int_0^t \frac{(\partial_t\mathbf{e}_u, \mathbf{v}_h)^2}{\|\nabla\mathbf{v}_h\|_0^2}dt \leq \int_0^t \|\partial_t(\mathbf{u}_h - \mathbf{u})\|_{H^{-1}(\Omega))}^2\,dt = \|\partial_t(\mathbf{u}_h - \mathbf{u})\|_{L^2(0,t;H^{-1}(\Omega))}^2 .
$$

$\square$

Let us discuss the results of Theorem 3.5 and of Corollary 3.6 and compare it to Theorem 5.1 in Ref. [15]. All right hand side terms, with the exception of the last term in (12) and the last term in (18), depend on the approximation properties of a standard (quasi)-interpolation operator in the discrete pressure space Q_h and of the divergence-preserving interpolation operator I_h in the space $V_{h,div}$.
Let $(\mathbf{u}, p) \in [W^{k+1,2}(\Omega)]^3 \times W^{k,2}(\Omega)$ for $t \in (0,T]$ with $k \in \mathbb{N}$, and let the FE-spaces $V_h \times Q_h$ of velocity/pressure be of piecewise order k and $k-1$, respectively. Then, the optimal convergence order of the corresponding left hand side terms is $\mathcal{O}(h^k)$. Moreover, the results of Theorem 3.5 and of Corollary 3.6 provide control of the divergence error and of the L^2-error of the pressure.
The last two terms of the right hand side of (12) and the last term of (18) correspond to a model error. Please note that different turbulence models can be applied in different parts of the domain. The influence of this model error depends on the approximation properties of the fluctuation operator $\kappa = \mathrm{I}d - \Pi_h$, see Definition 2.3. Such approximation results for κ with the L^2-orthogonal projector Π_h can be found, e.g., in the paper Ref. [24] for different variants of elements.

4. Specification of the subgrid models

Up to now we did not specify the subgrid models for the unresolved velocity and pressure scales. In general, the former may differ from the standard Smagorinsky model. In this respect we refer to the statement in Ref. [4], p.105: *The connection between turbulent fluctuation and the choice [...] of the Smagorinsky model eddy viscosity seems tenuous.* Nevertheless, for the basic case of homogeneous isotropic turbulence, we apply a Smagorinsky-type model for the velocity subgrid model, see Subsection 4.1. Then we adapt a argument of Lilly for the calibration of parameters of the model to the special case of a one-level projection method on Cartesian grids, see Subsections 4.2–4.3.

4.1. Parametrization of the subgrid model

A next step is the choice of the parameters $\nu_T^K(\mathbf{u}_h)$ and $\gamma_K(\mathbf{u}_h)$, which where introduced in Section 2.2. To be as close as possible to the most common turbulence model, the Smagorinsky model Ref. [30], one would like to have

$$\nu_T(\mathbf{u}_h) = C_{\mathrm{VMS}}\Delta^2 \left|\kappa \mathbb{D}\mathbf{u}_h\right|_F$$

as a pointwise function in Ω with a fluctuation operator κ, see Definition 2.3. Here C_{VMS} is a user defined constant, Δ is the filter width and $|\cdot|_F$ is the Frobenius norm. An approximation of this function is already available from the implementation aspects of Section 2.3 with the matrices in (7). It holds

$$\left\| \left|\kappa \mathbb{D}\mathbf{u}_h\right|_F \right\|^2_{L^2(K)} = \left\|\kappa \mathbb{D}\mathbf{u}_h\right\|^2_{L^2(K)} = \mathbf{u}^T_{loc.vect}(K_K - T_K^T M_K^{-1} T_K)\mathbf{u}_{loc.vect}, \tag{20}$$

where $\mathbf{u}_{loc.vect}$ may be the local velocity vector from the last time step on the cell K with respect to the used basis. That is why there is no additional numerical effort for $\nu_T^K(\mathbf{u}_h)$ when it is constant per cell with

$$\nu_T^K(\mathbf{u}_h) = C_{\mathrm{VMS}} \frac{\Delta^2}{\sqrt{vol(K)}} \left\|\kappa \mathbb{D}\mathbf{u}_h\right\|_{L^2(K)}$$

for all $K \in \mathcal{T}_h$. The division by the volume of the cell $vol(K)$ is to scale the local L^2 norm and to loose a dependency of the cell in that factor. Hence, the variational formulation of (1) with turbulence model reads:

$$\begin{aligned}
(\partial_t \mathbf{u}_h, \mathbf{v}_h) + 2\nu\left(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h\right) + b_S(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (\nabla\cdot\mathbf{v}_h, p_h) + \sum_{K\in\mathcal{T}_h} \gamma_K(\mathbf{u}_h)(\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h)_K \\
+ \sum_{K\in\mathcal{T}_h} C_{\mathrm{VMS}} \frac{\Delta^2}{\sqrt{vol(K)}} \left\|\kappa\mathbb{D}\mathbf{u}_h\right\|_{L^2(K)} (\kappa\mathbb{D}\mathbf{u}_h, \kappa\mathbb{D}\mathbf{v}_h)_K = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h \\
(q_h, \nabla\cdot\mathbf{u}_h) = 0 \qquad \forall q_h \in Q_h.
\end{aligned} \tag{21}$$

The parametrization of the pressure subgrid model is a problem of ongoing discussion. The paper Ref. [25] discusses several variants for the linearized Navier-Stokes problem. Here we restrict ourselves to the choice

$$\gamma_K(\mathbf{u}_h) \equiv \gamma \quad \forall K \in \mathcal{T}_h$$

with a user-defined constant $\gamma \geq 0$. A more sophisticated choice is $\gamma_K(\mathbf{u}_h) = \gamma_0 \frac{\|(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h\|_{L^2(K)}}{\|\nabla\mathbf{u}_h\|_{L^2(K)}}$. This quantity will be small in a shear flow with $(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h \approx 0$.

Let us check whether the assumptions (9) and (11) for the application of Theorem 3.5 to problem (21) are valid.

Corollary 4.1. *Let $\mathbf{u}$ be the solution of (2) and $\mathbf{u}_h$ be the solution of (21). Let $\mathbf{f} \in L^1(0,T;L^2(\Omega))$, $\mathbf{u}_0 \in [L^2(\Omega)]^3$ and (9) be true. Then we get the error estimate from Theorem 3.5 with interpolation operators I_h onto $L^4(0,T;V_{h,div})$ and J_h onto $L^2(0,T;Q_h)$.*

Proof. The only thing to prove is the L^1 regularity of the terms stemming from the turbulence model, such that the assumptions (11) are fulfilled. We obtain

$$
\begin{aligned}
\int_0^t \sum_{K\in\mathcal{T}_h} \nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{u}-I_h\mathbf{u})\,\|\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^2\,dt &= \sum_{K\in\mathcal{T}_h}\int_0^t \frac{C_{\mathrm{VMS}}\Delta^2}{\sqrt{vol(K)}}\,\|\kappa\mathbb{D}\mathbf{u}_h\|_{L^2(K)}\,\|\kappa\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^2\,dt \\
&\leq \sqrt[3]{\int_0^t \sum_{K\in\mathcal{T}_h} \frac{C_{\mathrm{VMS}}\Delta^2}{\sqrt{vol(K)}}\,\|\kappa\mathbb{D}\mathbf{u}_h\|_{L^2(K)}^3\,dt}\;\sqrt[3]{\int_0^t \sum_{K\in\mathcal{T}_h} \frac{C_{\mathrm{VMS}}\Delta^2}{\sqrt{vol(K)}}\,\|\kappa\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^3\,dt}^{\,2} \\
&\leq \sqrt[3]{\|\mathbf{u}_0\|_0^2+\frac{3}{2}\,\|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}^2}\;\sqrt[3]{\int_0^t \sum_{K\in\mathcal{T}_h} \frac{C_{\mathrm{VMS}}\Delta^2}{\sqrt{vol(K)}}\,\|\kappa\mathbb{D}(\mathbf{u}-I_h\mathbf{u})\|_{L^2(K)}^3\,dt}^{\,2} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \sum_{K\in\mathcal{T}_h} \nu_{\mathrm{VMS}}^K(\mathbf{u}_h,\mathbf{u})\,\|\mathbb{D}\mathbf{u}\|_{L^2(K)}^2\,dt \\
&\qquad\leq \sqrt[3]{\|\mathbf{u}_0\|_0^2+\frac{3}{2}\,\|\mathbf{f}\|_{L^1(0,t;L^2(\Omega))}^2}\;\sqrt[3]{\int_0^t \sum_{K\in\mathcal{T}_h} \frac{C_{\mathrm{VMS}}\Delta^2}{\sqrt{vol(K)}}\,\|\kappa\mathbb{D}\mathbf{u}\|_{L^2(K)}^3\,dt}^{\,2} < \infty,
\end{aligned}
$$

where the Hölder inequality, the stability Lemma 3.1 and (9) were used. This proves the claim. □

4.2. Modeling of the Eddy Viscosity for homogeneous isotropic turbulence

An important issue is the choice of the model parameters. For the Smagorinsky model the parameters C_{VMS} and Δ were derived in Ref. [22]. In particular, the dependence of these parameters on the polynomial degree of the velocity approximation is "*still a largely open issue*", see Ref. [32]. At this point we want to adapt the analysis of Lilly Ref. [22] to the VMS approach, what came up in Ref. [13]. The main assumption is that the turbulent kinetic energy production and dissipation are in balance. Additionally the famous law of the Kolmogorov energy spectrum can be used for isotropic turbulence, see Ref. [33], p.11. This law is illustrated by Figure 1, where we obtain how the spectral amplitude of the kinetic energy $E(k)$ depends on the wave number.

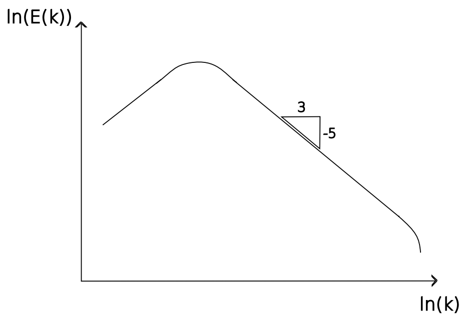

Figure 1: Kolmogorov energy spectrum

Under these assumption one can derive a formula for C_{VMS} similar to the calculations in Ref. [13], formula (63) and Remark 2 of Section 6.4. For our case this formula is

$$
C_{\mathrm{VMS}} = \left(\frac{4}{3\alpha}\right)^{3/2}\frac{1}{\pi^2}\left(1-\left(\frac{k_c}{k_f}\right)^{4/3}\right)^{-3/2}, \tag{22}
$$

where k_c is the resolution limit wave number of the coarse space L_H with respect to the velocity, see Definition 2.2, and k_f is the resolution limit wave number of the whole space V_h. Note that C_{VMS} only depends on the Kolmogorov constant α and the ratio $\frac{k_c}{k_f}$. Let us point out that with this formula we are able to determine a discretization parameter a priori, which is very hard to choose otherwise. The exact procedure how to calculate all parameters of the turbulence model will be given in Subsection 4.3 and 5.2.

4.3. Choice of the Model Parameters

In (22) we derived a design for the model parameters C_{VMS}. In the next section we will compare these theoretical results to numerical tests for the problem of decaying homogeneous isotropic turbulence. At the end of Subsection 4.2, we saw that C_{VMS} depends on the ratio between the resolution limit wave numbers k_c and k_f. Here k_c is the largest wave number which can be represented in the coarse space L_H with respect to the velocity and k_f is the largest wave number which can be represented in the whole space V_h. The indices shall indicate that k_c is in a coarse space whereas k_f is in a fine space.
In our numerical experiments below, we apply Taylor-Hood FE pairs $V_h/Q_h = \mathbb{Q}_k/\mathbb{Q}_{k-1}$, $k \geq 2$, for velocity and pressure. They fulfill the discrete inf-sup condition (4). Moreover, we have to select a coarse space L_H for the VMS method, with respect to Definition 2.2. Here we consider a *one-level* approach with $H = h$, hence with $\mathcal{T}_H = \mathcal{T}_h$, and consider the choice $L_H = \mathbb{Q}_q^{disc}$. The deformation tensor or the gradient of a continuous, piecewise tensor-polynomial function may be discontinuous across the edges and the order is reduced by one. Therefore, in practice one should take $q < k-1$, since comparing the spaces $\mathbb{D}V_h = \mathbb{D}\mathbb{Q}_k$ and $L_H = \mathbb{Q}_{k-1}^{disc}$, one will see no big difference. Then there would be no turbulence modelling. Later on we will take $V_h \in \{\mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_4\}$ and try all of the possible coarse spaces L_H satisfying $q < k-1$.

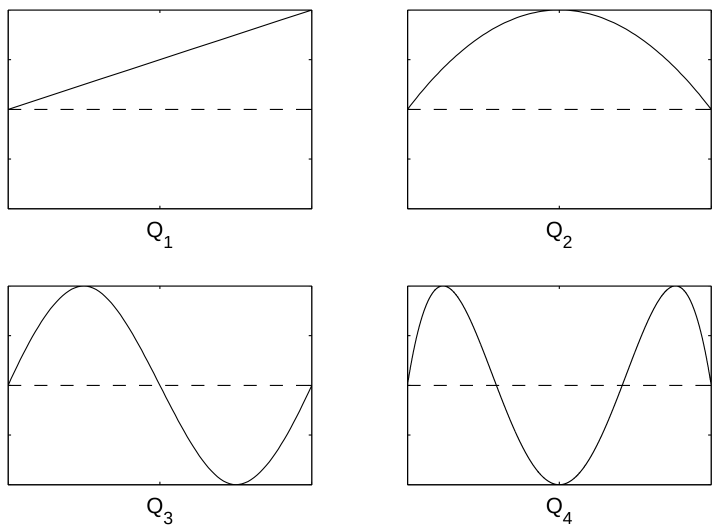

Figure 2: Representations of oscillating functions in different finite element spaces on one cell in one dimension

Now it is left to calculate the ratio $\frac{k_c}{k_f}$. Therefore we have to know what the resolution limit wave number k of a space is, i.e. how many waves can be represented per cell. Figure 2 illustrates that the $\mathbb{Q}_1$ element can represent a quarter of a wave per cell and direction, such that its resolution limit wave number is $k \propto \pi/2$. Moreover we obtain π for $\mathbb{Q}_2$, 2π for $\mathbb{Q}_3$ and 3π for $\mathbb{Q}_4$. The ratio $\frac{k_c}{k_f}$ contains the wave number k_c of the coarse space with respect to the velocity, but our space considers the deformation tensor of the velocity. If we want to calculate for example k_c of $L_H = \mathbb{Q}_0^{disc}$, we will use $\mathbb{Q}_1$ for the velocity and therefore $k_c \propto \pi/2$ due to the explanations above. With these wave numbers and formula (22) for C_{VMS} we get Table 1, where $\alpha = 1.4$ is taken for the Kolmogorov constant.

element V_h, L_H	$\mathbb{Q}_2$, $\mathbb{Q}_0^{disc}$	$\mathbb{Q}_3$, $\mathbb{Q}_1^{disc}$	$\mathbb{Q}_3$, $\mathbb{Q}_0^{disc}$	$\mathbb{Q}_4$, $\mathbb{Q}_2^{disc}$	$\mathbb{Q}_4$, $\mathbb{Q}_1^{disc}$	$\mathbb{Q}_4$, $\mathbb{Q}_0^{disc}$
$\frac{k_c}{k_f}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
C_{VMS}	≈ 0.2010	≈ 0.2010	≈ 0.1218	≈ 0.3489	≈ 0.1397	≈ 0.1088

Table 1: Ratios $\frac{k_c}{k_f}$ and correspondig values of C_{VMS} for different finite element spaces

In the following numerical tests, we also consider the original Smagorinsky model. In the framework of the parameter design we presented here, the value of the Smagorinsky constant C_S is calculated by plugging in the ratio $\frac{k_c}{k_f} = 0$. We obtain $C_S \approx 0.0942$.

Remark 4.2. *If one wants to use the presented values for the variational multiscale approach in original parameter design of the Smagorinsky model, one can apply a simple formula. The original terms added for the turbulence model are* $2(\tilde{C}_S\Delta)^2 \left|\sqrt{2}\mathbb{D}\mathbf{u}_h\right|_F (\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h)$, *respectively* $2(\tilde{C}_{VMS}\Delta)^2 \left|\sqrt{2}\kappa\mathbb{D}\mathbf{u}_h\right|_F (\kappa\mathbb{D}\mathbf{u}_h, \kappa\mathbb{D}\mathbf{v}_h)$ *for the variational multiscale approach. That is why one can say* $\tilde{C}_{VMS} = \sqrt{\frac{C_{VMS}}{\sqrt{2}^3}}$ *and calculates for the original Smagorinsky constant* $\tilde{C}_S \approx 0.1825$, *see also Ref. [13] p. 51.*

5. Numerical experiments

In this section, we will check whether the values for the model parameters from Section 4 fit for numerical calculations. Our test case will be the benchmark of decaying homogeneous isotropic turbulence Ref. [7]. Moreover, in the numerics we will compare the results of the variational multiscale approach to the common Smagorinsky model, which is designed especially for the test case of homogeneous isotropic turbulence Ref. [22].

5.1. Description of the method and the benchmark problem

Our benchmark problem is that of decaying homogeneous isotropic turbulence in $\Omega = [0, 2\pi]^3$ with periodic boundary conditions on all sides of Ω and right hand side $\mathbf{f} = \mathbf{0}$. Please notice that the analysis of Section 3 is still applicable as the validity of the inequalities of Korn and Poincaré-Friedrichs can be shown for this case as well, see Remark 2.1. For comparison, we consider the experimental results of Ref. [7] which provide energy spectra at three different times. We take the first for calculating the turbulent initial data and compare the numerical solution to the remaining two energy spectra. Therefore we apply a Fourier transform $\hat{\mathbf{u}}(\mathbf{k}, t) = \int_\Omega \mathbf{u}(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$ and get the values of the energy spectrum of the numerical solution $E(k, t) = \frac{1}{2}\sum_{k-\frac{1}{2}\leq|\mathbf{k}|\leq k+\frac{1}{2}} \hat{\mathbf{u}}(\mathbf{k}, t) \cdot \hat{\mathbf{u}}(\mathbf{k}, t)$ for a given time t. The experiment in Ref. [7] is prescribed by a Taylor scale Reynolds number $Re_\lambda = 150$ and $\nu = 1.494 \times 10^{-5}$ (Reynolds number for air).
For the simulations we apply the FE library deal.II, see Refs. [2, 3]. The model (21) is discretized by an IMEX time discretization with a second order DIRK scheme, see Ref. [1]. The time-step size is taken as $\Delta t = 0.0174$, since smaller values showed no improvement. We apply the Taylor-Hood element $\mathbb{Q}_k/\mathbb{Q}_{k-1}$ for $k \in \{2, 4\}$ for the discretization of velocity and pressure in space. The pre-calculated values for C_{VMS} of the velocity turbulence model from Table 1 were developed under the assumption of isotropic turbulence, which is fulfilled in this test case.
To illustrate the behaviour of the decaying turbulence we show some results on the development of the kinetic energy, approximated by $\|\mathbf{u}_h\|_0^2$, in Figure 3. The energy should follow a $t^{-1.4}$ law in the long-run, see Ref. [6], Section 4, which is well observed in our numerics. The kinetic energy of the approximated solution is shown for the $\mathbb{Q}_2/\mathbb{Q}_1$ element with 32^3 degrees of freedom (dofs) for the grad-div stabilization with and without the full Smagorinsky model. Moreover, we observe that a turbulence model is really necessary, since the energy does not follow the $t^{-1.4}$ law if all turbulence models are switched off. The vertical lines indicate the benchmark points $t = 0.87$ and $t = 2.0$ from Ref. [7]. The values for the Smagorinsky and the grad-div stabilization are obtained later as optimal values from Figures 4 and 7. We observe a reasonable behaviour of the kinetic energy for grad-div stabilization alone, although these results are not satisfying for the energy spectra as we will see later.

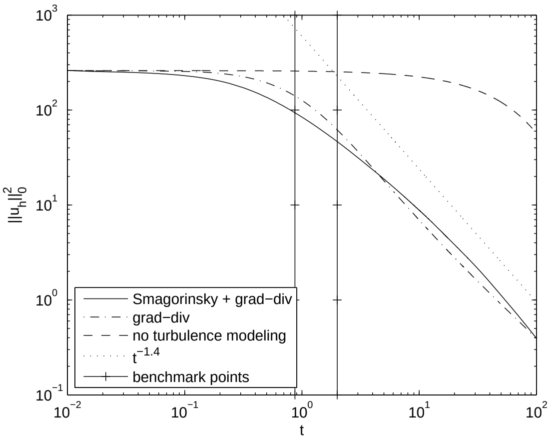

Figure 3: Long-run energy behaviour with the $\mathbb{Q}_2/\mathbb{Q}_1$ element for 32^3 dofs and optimal values for the full Smagorinsky model and the grad-div stabilization and without any turbulence modeling.

5.2. Choice of the Model Parameters

Next we discuss the concrete choice of the model parameters from a numerical point of view and compare it to the analytically derived values and recommendations from Table 1. Therefore we compute the energy spectra $E(k,t)$ at given mode k and time t of the numerical approximations and compare them to the experimental values of Ref. [7]. To this goal we use the error functional

$$J(C_{\mathrm{VMS}}) := \frac{1}{\sqrt{M}} \left(\sum_{t \in \{0.87, 2\}} \sum_{i=1}^{M} \left(E_{C_{\mathrm{VMS}}}(k_i, t) - E_{exp}(k_i, t) \right)^2 \right)^{1/2} . \qquad (23)$$

The model is described by (21). Up to now, we have left open the choice of the filter width Δ of the velocity turbulence model. The filter width should depend on the polynomial degree q of the FE space of the velocity, i.e. $V_h = \mathbb{Q}_q$, since the theoretical considerations in (22) predict a connection with respect to the resolution limit wave number k_f. This is also done in Ref. [13], Section 4.1 Remark 1, where it is proposed that $\Delta = h \propto k_f^{-1}$ for linear elements with a mesh parameter h. Let us review the considerations about the resolution limit wave numbers in Section 4.1. Therein we saw $k_f = \frac{\pi}{2h}$ for the $\mathbb{Q}_1$ element and $k_f = \frac{\pi(q-1)}{h}$ for $\mathbb{Q}_q$ and $q \geq 2$. That is why we use

$$\Delta = \frac{\max(\Delta x, \Delta y, \Delta z)}{2(q-1)}$$

with the meshwidths of each direction and the polynomial degree $q \geq 2$ of the FE space of the velocity, i.e. $V_h = \mathbb{Q}_q$.

Figure 4 illustrates the error $J(C_{\mathrm{VMS}})$ for different values of C_{VMS} with different discretizations and dofs. The vertical lines represent the theoretical values for C_{VMS} in the corresponding colour. There is no grad-div stabilization used, i.e. $\gamma_K \equiv 0$. In the left plot one can see the results with the $\mathbb{Q}_2/\mathbb{Q}_1$ element for the full Smagorinsky model and the VMS approach. The optimal numerical values of C_{VMS} are in good agreement to the theoretical values and are even getting better with refinement. The error level of both methods and the sensitivity to the model parameters are nearly the same, even if the Smagorinsky model is designed especially for this test case. On the right we see the results with the $\mathbb{Q}_4/\mathbb{Q}_3$ element. Here, the theoretical values of C_{VMS} are also very good, like for the $\mathbb{Q}_2/\mathbb{Q}_1$ element. Nevertheless, we observe that the error is getting larger if the large scale space L_H in Definition 2.2 becomes larger. It seems that $L_H = \mathbb{Q}_2^{disc}$ is too large as there are not enough scales left in the small scale space for the turbulence model for this benchmark problem.

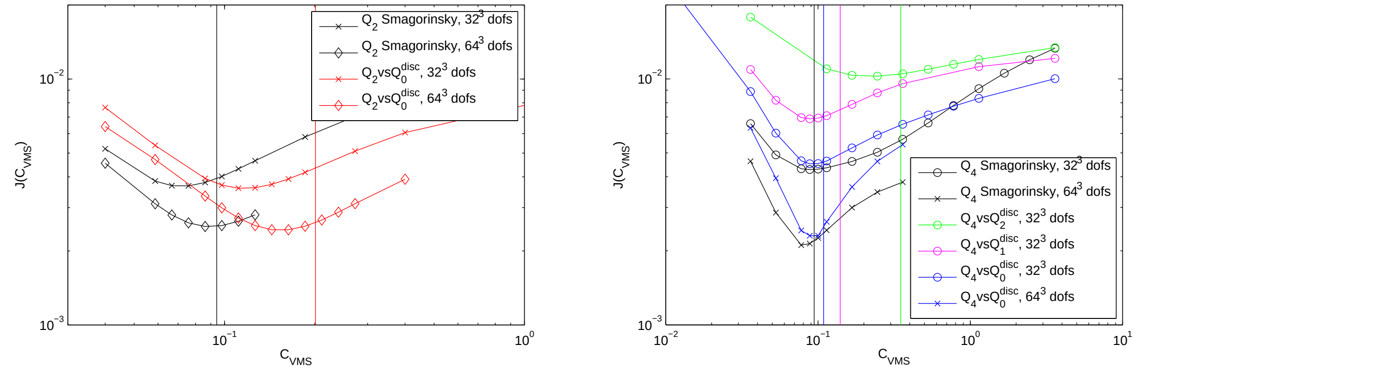

Figure 4: Error functional J in comparison to the theoretical values of the model parameter C_{VMS}, $\mathbb{Q}_2/\mathbb{Q}_1$ element left and $\mathbb{Q}_4/\mathbb{Q}_3$ element right, no grad-div stabilization

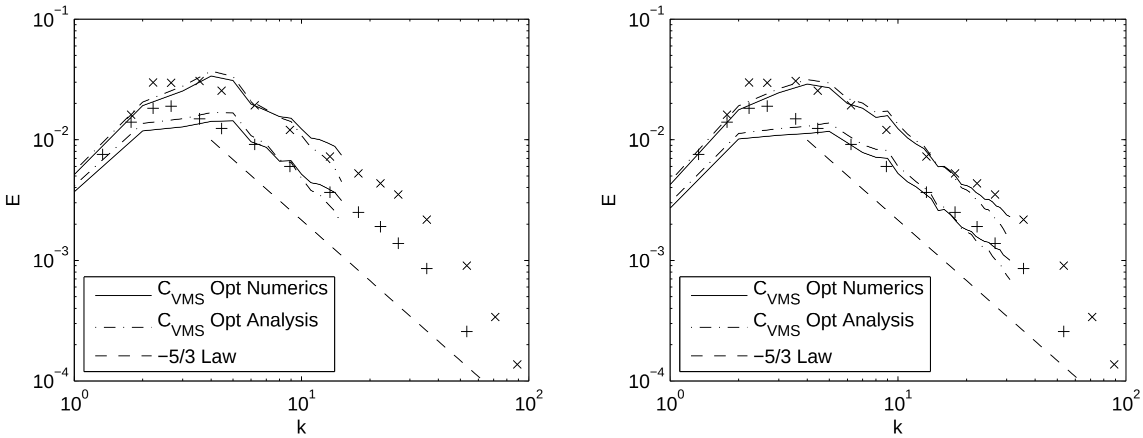

Figure 5: Numerical and analytical optimal values of C_{VMS} for the $\mathbb{Q}_2/\mathbb{Q}_1$ element with large scale space $L_H = \mathbb{Q}_0^{disc}$, 32^3 dofs left and 64^3 dofs right, no grad-div stabilization. 'x' and '+' denote the experimental data from Ref. [7].

In Figures 5 and 6 we compare the energy spectra for the optimal values for C_{VMS} from the numerical and the analytical point of view, where the numerical optimum is obtained by Figure 4. While the numerical optimal spectrum is in good agreement to the experimental results, it seems that the analytical optimum fits well to the Kolmogorov spectrum with slope $-5/3$. This is also observed after refinement.
In Figure 6 we observe for the $\mathbb{Q}_4/\mathbb{Q}_3$ pair a reasonable behaviour for the large scale space $L_H = \mathbb{Q}_0^{disc}$ (left) whereas the less convincing results for $L_H = \mathbb{Q}_2^{disc}$ again show that this space is too rich (right).
Another aspect is the choice of the parameter γ_K for the grad-div stabilization in (21). Up to this point, all results were for $\gamma_K = 0$. For Figure 7 we took the numerical optimal values of C_{VMS} from the calculations presented above and varied the constant parameter $\gamma = \gamma_K$ for all $K \in \mathcal{T}_h$. On the left we present the cases with an improvement of the error. All other variants showed a behaviour similar to the two lines with velocity turbulence model on the right. The error is not getting worse but there is also no improvement. Note that even for $\gamma = 0$, we can interprete the deformation tensor as a term which contains some grad-div stabilization, since $(2\nu\mathbb{D}\mathbf{u}, \mathbb{D}\mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \nu(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})$ for constant ν. That is why the case $\gamma = 10^{-5}$ corresponds to $\gamma = 0$. The third line plotted on the right is for the case, where only grad-div stabilization is used, i.e. $C_{\text{VMS}} = 0$. In Figure 3 we saw that the grad-div stabilization alone results in a good long-run behaviour of the decaying energy, but Figure 7 predicts no good approximation of the energy spectra at certain times. Let us point out that the values of γ for this line were scaled by a factor of 10^{-2}, such that

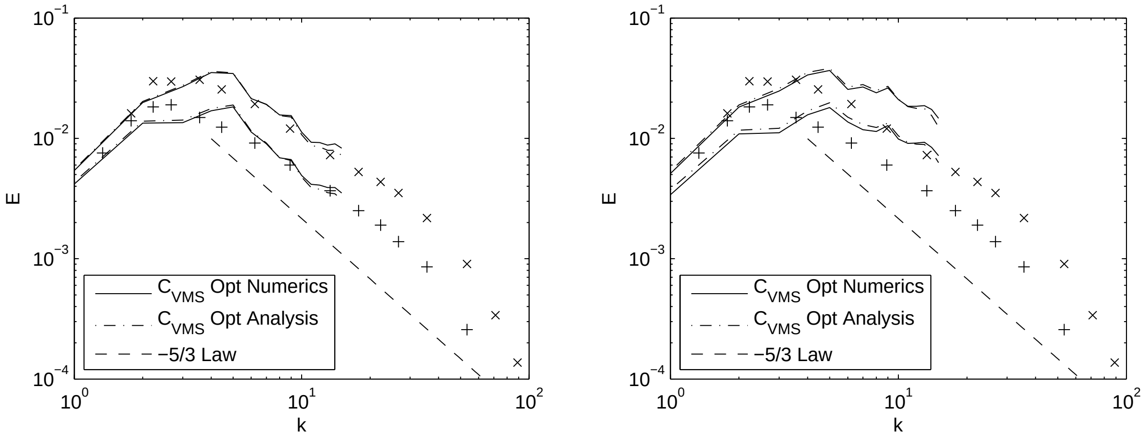

Figure 6: Numerical and analytical optimal values of C_{VMS} for the $\mathbb{Q}_4/\mathbb{Q}_3$ element with 32^3 dofs, $L_H = \mathbb{Q}_0^{disc}$ left and $L_H = \mathbb{Q}_2^{disc}$ right, no grad-div stabilization. 'x' and '+' denote the experimental data from Ref. [7].

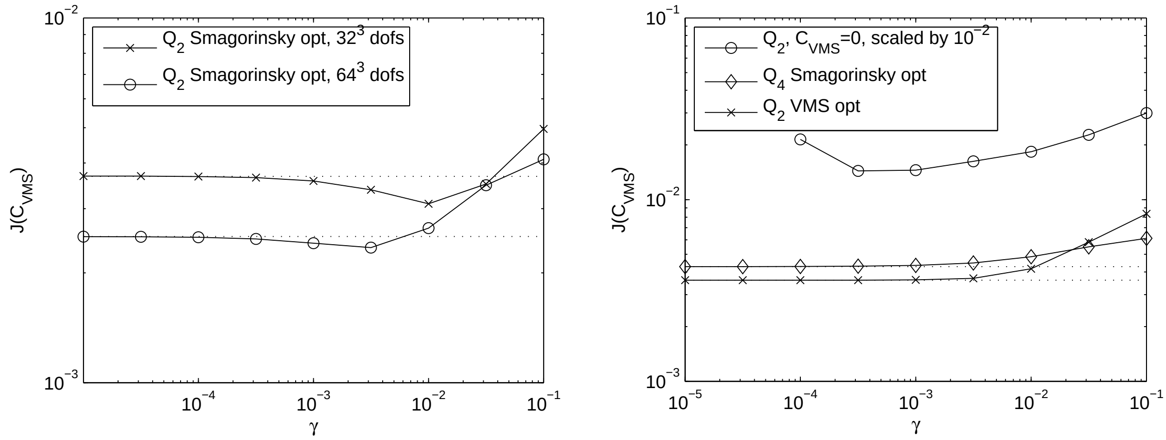

Figure 7: Error J for different values of the constant grad-div stabilization parameter γ together with other turbulence models, dotted lines correspond to $\gamma = 0$

the plot fits into the picture. Hence, the values for γ with $C_{VMS} = 0$ are larger by a factor of 10^2. The dotted lines in Figure 7 correspond to the optimal values with $\gamma = 0$.
In Figure 8 we present the effect of grad-div stabilization on the energy spectra. Therefore, on the left we plotted the numerical optimal spectra with and without grad-div stabilization while using the $\mathbb{Q}_2/\mathbb{Q}_1$ element with the full Smagorinsky model. One can see that the grad-div stabilization adds some energy to the large scales. On the right side we plotted the numerically optimal energy spectra of the corresponding methods without grad-div, which is optimal in these cases. The reason for that is, that there already are some overpredictions of the large scales such that there is no improvement possible with the grad-div stabilization. The dashed line shows the approximation where only grad-div was used. We clearly see, that the small scales are not treated well enough.
Summarizing, we conclude that – due to the isotropic behavior of the turbulence in this case – it seems to be sufficient to model the unresolved pressure scales via the grad-div contributions with scale ν from the viscous term of the Galerkin scheme and with scale $C_{\mathrm{VMS}}\Delta^2/\sqrt{\mathrm{vol}(K)}$ from the velocity subgrid model.

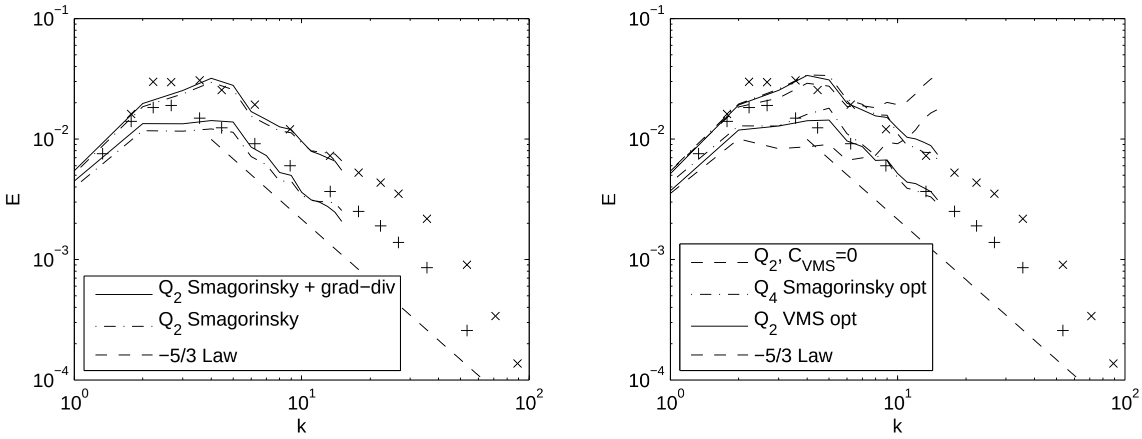

Figure 8: Energy spectra for corresponding numerical optimal model parameters, obtained by Figure 7. 'x' and '+' denote the experimental data from Ref. [7].

6. Summary, outlook

In the present paper, we considered a modified projection-based finite element VMS method which had been presented in a general form first in Ref. [20]. The subgrid model for the unresolved velocity scales is based on the L^2-projection Π_H for the definition of the large scales of the velocity deformation tensor. Opposite to the approach in Ref. [16] the so-called fluctuation operator $I - \Pi_H$ is applied to the velocity deformation tensor whereas the velocity deformation tensor is applied to the fluctuation operator in Ref. [16] first. Another difference to Ref. [16] is the application of the so-called grad-div stabilization as a subgrid model for the unresolved pressure scales. Let us emphasize that the subgrid models for the unresolved velocity and pressure scales contain a rather general nonlinear parametrization.

The theoretical part of the paper provides stability estimates for the discrete solution of the VMS model and a priori error estimates for the resolved velocity and pressure scales after spatial semidiscretization based on inf-sup stable FE pairs for velocity and pressure. The analysis relies on minimal assumptions on the subgrid models. In particular, cellwise constant values of the (nonlinear) subgrid viscosity coefficients are assumed. For the numerical experiments, a specification of the subgrid viscosity coefficients was given for the case of homogeneous isotropic turbulence. A Smagorinsky type subgrid model for the unresolved velocity scales together with a (globally constant) grad-div subgrid model for the unresolved pressure scales were considered. In particular, parameters of the Smagorinsky type model were identified by adapting the approach of Lilly Ref. [22] for the standard Smagorinsky model and taking the polynomial degree into account. Finally we presented results of the application of the approach to the standard benchmark of decaying homogeneous isotropic turbulence. It turned out that the calibration of the Smagorinsky type model following Lilly's argument is in good agreement with the numerical results. On the other hand, the globally constant grad-div subgrid model for the unresolved pressure scales was much less important in this application.

Future work will be devoted to the application of the modified projection-based FE-VMS method to wall-bounded flows, in particular to channel flows and flows over obstacles. In the present article we compared the original Smagorinsky model to the modified projection-based FE-VMS method and observed almost the same errors of the methods, even if the Smagorinsky was designed for the test case of homogeneous isotropic turbulence Ref. [22]. The hope is, that the variational multiscale approach performs even better in wall-bounded flows, since there are several drawbacks known for such flows with the Smagorinsky model Ref. [34]. It remains open whether the grad-div stabilization as a subgrid model for the unresolved pressure scales will be more important for flows with significant flow patterns like streaks, vortices etc.

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