Hardy space infinite elements for scattering and resonance problems

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Waves 2007, University of Reading
introduction

pole condition

Hardy space method

numerical examples
scattering problem

inc. wave  total field  scattered field
setting

\[ - \Delta u(x) - \kappa^2 u(x) = 0, \quad x \in \mathbb{R}^d \setminus K, \]
\[ u(x) = f(x), \quad x \in \partial K, \]
\[ u \text{ satisfies a radiation condition.} \]

Sommerfeld radiation condition:
\[ r^{\frac{d-1}{2}} \left( \frac{\partial u}{\partial r}(x) - i\kappa u(x) \right) \to 0, \quad r = |x| \to \infty. \]

\[ u \text{ has a series representation with Hankel functions:} \]
\[ u(r, \hat{x}) = \sum_{\nu=0}^{\infty} \sum_{\mu} c_{\nu\mu} h^{(1)}_{\nu}(\kappa r) Y_{\nu}^{\mu}(\hat{x}), \quad r > 0, \quad \hat{x} \in \partial B_a. \]
classical infinite element methods

separation ansatz:

\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_\hat{x}} c_\mu u_\mu(r) w_\mu(\hat{x}) \]
classical infinite element methods

separation ansatz:

\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_{\hat{x}}} c_{\mu} u_{\mu}(r) w_{\mu}(\hat{x}) \]

discretization of \( u_{\mu} \):

\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_{\hat{x}}} \sum_{\nu=0}^{N_r} c_{\nu\mu} h^{(1)}_{\nu}(\kappa r) w_{\mu}(\hat{x}) \]
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tensor product structure:

\[ M_{el} = M_{el}^{\hat{x}} \otimes M_{el}^r, \quad S_{el} = S_{el}^{\hat{x}} \otimes S_{el}^r \]
classical infinite element methods

separation ansatz:

\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_{\hat{x}}} c_\mu u_\mu(r) w_\mu(\hat{x}) \]

discretization of \( u_\mu \):

\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_{\hat{x}}} \sum_{\nu=0}^{N_r} c_{\nu \mu} h_\nu^{(1)}(\kappa r) w_\mu(\hat{x}) \]

tensor product structure:

\[ M_{el} = M_{el}^{\hat{x}} \otimes M_{el}^{r} \quad \text{and} \quad S_{el} = S_{el}^{\hat{x}} \otimes S_{el}^{r} \]
Lax & Phillips: Scattering Theory, 1967:

With $K \subset \mathbb{R}^d$ compact, scattering resonances are the square root of eigenvalues $\kappa^2$ of the Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \forall x \in \mathbb{R}^d \setminus K$$

which correspond to eigenfunctions $u$ satisfying a boundary condition on $\partial K$ and a radiation condition at infinity.

Typically the resonances have negative imaginary parts, and the eigenfunctions grow exponentially at infinity.
outline

introduction

pole condition

Hardy space method

numerical examples
publications


pole condition in 1d

The Laplace transform $\mathcal{L}$

$$\hat{u}(s) = \int_0^\infty e^{-sr} u(r) dr , \Re(s) > 0$$

is a standard method for solving ODE's. It is also suitable to characterize outgoing solutions:

$$u(r) = \begin{cases} c \exp(+i\kappa r), \\ c \exp(-i\kappa r) \end{cases}$$

$$\hat{u}(s) = \begin{cases} \frac{c}{s-i\kappa}, \\ \frac{c}{s+i\kappa} \end{cases}$$

\begin{tikzpicture}
  \fill (8,8) circle (2pt) node [below] {pole of outgoing part};
  \fill (8,-8) circle (2pt) node [below] {pole of incoming part};
\end{tikzpicture}
Definition

$u$ satisfies the pole condition if the mapping

$s \mapsto \hat{u}(s, \cdot) = \mathcal{L}\{u(\frac{r+a}{a}, \cdot)\}(s)$ defined on \{\(s \in \mathbb{C} : \Re s > 0\}\} with values in $L^2(\partial B_a)$ has a holomorphic extension to \(\mathbb{C}^- := \{s \in \mathbb{C} : \Im s < 0\}\).
general case

Definition

u satisfies the **pole condition** if the mapping
\[ s \mapsto \hat{u}(s, \cdot) = \mathcal{L}\{u(\frac{r+a}{a}, \cdot)\}(s) \]
defined on \( \{ s \in \mathbb{C} : \Re s > 0 \} \) with values in \( L^2(\partial B_a) \) has a holomorphic extension to \( \mathbb{C}^- := \{ s \in \mathbb{C} : \Im s < 0 \} \).

Theorem

A bounded solution to the Helmholtz equation for \( \kappa > 0 \) satisfies the pole condition if and only if it satisfies the Sommerfeld radiation condition.

Hardy space $H^{-}(\mathbb{R})$

**Definition**
The Hardy space $H^{-}(\mathbb{R})$ consists of all functions $f \in L^{2}(\mathbb{R})$ which are $L^{2}$-boundary values of holomorphic functions $g$ in $\mathbb{C}^{-}$. Equipped with the $L^{2}(\mathbb{R})$-norm it is a Hilbert space.

- **Pole condition:** $\hat{u}(\cdot, \hat{x}) \in H^{-}(\mathbb{R})$ for all $\hat{x} \in \partial B_{a}$,
- **Idea:** Galerkin method in $H^{-}(\mathbb{R})$,
- **Problem:** Appropriate basis of $H^{-}(\mathbb{R})$. 
Hardy space $H^+(S^1)$

**Definition**
The Hardy space $H^+(S^1)$ consists of all functions $F \in L^2(S^1)$ which are $L^2$-boundary values of holomorphic functions $G$ in $D := \{ z \in \mathbb{C} \mid |z| < 1 \}$.

- Equipped with the $L^2(S^1)$-norm, $H^+(S^1)$ is a Hilbert space.
- A orthogonal basis of $H^+(S^1)$ is given via the monomials
  \[ \{ b_\nu(z) = z^\nu \mid \nu = 0, 1, \ldots \}. \]
- $F(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^+(S^1) \Rightarrow (\alpha_n)_{n \in \mathbb{N}} \in \ell_2$. 

Möbius transform

Lemma

The mapping $\mathcal{M} u(z) := (u \circ \varphi)(z) \cdot \frac{\sqrt{-2i\kappa_0}}{z-1}$ is a unitary operator from $\tilde{H}^-(\mathbb{R})$ to $H^+(S^1)$. 
Hardy space infinite element methods

separation ansatz:
\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_\hat{x}} c_\mu u_\mu(r) w_\mu(\hat{x}) \]

discretization of \( u_\mu \):
\[ u(r, \hat{x}) = \sum_{\mu=0}^{N_\hat{x}} \sum_{\nu=0}^{N_r} c_{\nu\mu} \left( \mathcal{L}^{-1} \mathcal{M}^{-1} b_\nu \right)(r) w_\mu(\hat{x}) \]

ansatz functions in radial direction:
\[ \left( \mathcal{L}^{-1} \mathcal{M}^{-1} b_\nu \right)(r) = (-\sqrt{-2i\kappa_0}) e^{i\kappa_0 r} \sum_{j=0}^{\nu} \binom{\nu}{j} \frac{(2i\kappa_0 r)^j}{j!} \]
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transformation of a bilinear form

\[ \int_0^\infty f(r)g(r)dr \]
transformation of a bilinear form

Fourier transform:

\[(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt\]

Laplace transform:

\[(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt\]

\[\int_{0}^{\infty} f(r)g(r) dr = \mathcal{F}\{f^* g^*\}(0)\]
transformation of a bilinear form

Fourier transform:

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

Laplace transform:

$$(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} f(r)g(r)dr = \mathcal{F}\{f^* g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\{f^* e^{-M\cdot}\}(t)\mathcal{F}\{g^* e^{M\cdot}\}(-t)dt$$
transformation of a bilinear form

Fourier transform:

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

Laplace transform:

$$(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} f(r)g(r)dr = \mathcal{F}\{f^* g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}\{f^*\}(t - iM)\mathcal{F}\{g^*\}(-t + iM) dt$$
transformation of a bilinear form

Fourier transform:

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

Laplace transform:

$$(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} f(r)g(r)dr = \mathcal{F}\{f^* g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}\{f\}(it + M)\mathcal{L}\{g\}(-(it + M))dt$$
transformation of a bilinear form

Fourier transform:
\[(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt\]

Laplace transform:
\[(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt\]

\[
\int_{0}^{\infty} f(r)g(r) dr = \mathcal{F}\{f^* g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}\{f\}(it + M)\mathcal{L}\{g\}(-(it + M)) dt
\]

\[
= \frac{-i}{2\pi} \lim_{R \to \infty} \int_{\gamma_1} \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(-s) ds
\]
transformation of a bilinear form

**Fourier transform:**

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

**Laplace transform:**

$$(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} f(r)g(r)dr = \mathcal{F}\{f^*g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}\{f\}(it + M)\mathcal{L}\{g\}(-(it + M))dt$$

$$= -\frac{i}{2\pi} \lim_{R \to \infty} \int_{\gamma_3} \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(-s)ds$$
transformation of a bilinear form

Fourier transform:

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

Laplace transform:

$$(\mathcal{L}f)(s) := \int_{0}^{\infty} e^{-st} f(t) dt$$

$$\int_{0}^{\infty} f(r) g(r) dr = \mathcal{F}\{f^* g^*\}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}\{f\}(it + M)\mathcal{L}\{g\}(-(it + M)) dt$$

$$= \frac{-i}{2\pi} \lim_{R \to \infty} \int_{\gamma_3} \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(-s) ds$$

$$= \frac{1}{2\pi} \int_{S^1} \mathcal{M}\mathcal{L}\{f\}(z) \mathcal{M}\mathcal{L}\{g\}(\overline{z}) |dz|$$
coupling parameter

boundary terms: \[ \int_{\partial B_a} u'_0(\hat{x}) v_0(\hat{x}) d\hat{x} \]
coupling parameter

boundary terms: $\int_{\partial B_a} u'_0(\hat{x})v_0(\hat{x})d\hat{x}$

Laplace transform of $\partial_r u(r, \hat{x})$:

$$\mathcal{L}\{\partial_r u(r, \hat{x})\}(s) = s\mathcal{L}\{u(\cdot, \hat{x})\}(s) - u_0(\hat{x})$$
coupling parameter

boundary terms: \[ \int_{\partial B_a} u_0'(\hat{x}) \nu_0(\hat{x}) d\hat{x} \]

Laplace transform of \( \partial_r u(r, \hat{x}) \):

\[ \mathcal{L}\{ \partial_r u(r, \hat{x}) \}(s) = s\mathcal{L}\{ u(\cdot, \hat{x}) \}(s) - u_0(\hat{x}) \]

limit theorem of the Laplace transform:

\[ u_0(\hat{x}) \approx \lim_{|s| \to \infty} s\mathcal{L}\{ u(\cdot, \hat{x}) \}(s) \approx (\mathcal{M}\mathcal{L}u(\cdot, \hat{x})) (1) \]
coupling parameter

boundary terms: \[ \int_{\partial B_a} u'_0(\hat{x}) v_0(\hat{x}) d\hat{x} \]

Laplace transform of \( \partial_r u(r, \hat{x}) \):
\[
\mathcal{L}\{\partial_r u(r, \hat{x})\}(s) = s\mathcal{L}\{u(\cdot, \hat{x})\}(s) - u_0(\hat{x})
\]

limit theorem of the Laplace transform:
\[
u_0(\hat{x}) \approx \lim_{|s| \to \infty} s\mathcal{L}\{u(\cdot, \hat{x})\}(s) \approx (\mathcal{ML}u(\cdot, \hat{x})) (1)
\]

separation of \( u_0 \) and \( U \):
\[
(\mathcal{ML}u(\cdot, \hat{x}))(z) = c_0 u_0(\hat{x}) + (z - 1)\tilde{U}(z, \hat{x}),
\]
\[
\tilde{U}(\cdot, \hat{x}) \in H^+(S^1) \cap C^\infty(S^1)
\]
variational formulation

ansatz space: $X = H^1(\Omega_{int}) \times (H^+(S^1) \otimes H^1(\partial B_a))$

Find nontrivial eigenpairs $( (u, \tilde{U}) , \kappa^2 ) \in X \times \mathbb{C}$, which fulfil the variational formulation

$$ a( (v, \tilde{V}) , (u, \tilde{U}) ) = \kappa^2 b( (v, \tilde{V}) , (u, \tilde{U}) ) , \quad (v, \tilde{V}) \in X. $$
variational formulation

ansatz space: \[ X = H^1(\Omega_{\text{int}}) \times (H^+(S^1) \otimes H^1(\partial B_a)) \]

Find nontrivial eigenpairs \((u, \tilde{U}), \kappa^2 \) \( \in X \times \mathbb{C} \), which fulfil the variational formulation

\[
a( (v, \tilde{V}), (u, \tilde{U}) ) = \kappa^2 b( (v, \tilde{V}), (u, \tilde{U}) ), \quad (v, \tilde{V}) \in X.
\]

\[
a( (v, \tilde{V}), (u, \tilde{U}) ) := \int_{\Omega_{\text{int}}} \nabla v \nabla u \, dx + \frac{d-1}{2a} \int_{\partial B_a} v_0 u_0 \, d\hat{x}
\]

\[
- \frac{\kappa^2}{2\pi} \int_{\partial B_a} \int_{S^1} (v_0, \tilde{V}) \tilde{T}_+^\top \tilde{T}_+ (u_0, \tilde{U}) \, ^\top \! |dz|d\hat{x}
\]

\[
- \frac{C_d}{2\pi} \int_{\partial B_a} \int_{S^1} (v_0, \tilde{V}) \tilde{T}_-^\top \tilde{D}_a^{-2} \tilde{T}_- (u_0, \tilde{U}) \, ^\top \! |dz|d\hat{x}
\]

\[
+ \frac{a^{1-d}}{2\pi} \int_{\partial B_a} \int_{S^1} (\nabla_{\hat{x}} v_0, \nabla_{\hat{x}} \tilde{V}) \tilde{T}_-^\top \tilde{D}_a^{-2} \tilde{T}_- (\nabla_{\hat{x}} u_0, \nabla_{\hat{x}} \tilde{U}) \, ^\top \! |dz|d\hat{x},
\]

\[
b( (v, \tilde{V}), (u, \tilde{U}) ) := \int_{\Omega_{\text{int}}} v \, u \, dx + \frac{a^{1-d}}{2\pi} \int_{\partial B_a} \int_{S^1} (v_0, \tilde{V}) \tilde{T}_-^\top \tilde{T}_- (u_0, \tilde{U}) \, ^\top \! |dz|d\hat{x}.
\]
local element matrices

tensor product structure:

\[ M_{el} = M_{el}^{x} \otimes M_{el}^{r} \]
\[ S_{el} = S_{el}^{x} \otimes S_{el}^{r} \]

\[ M_{el}^{r} = \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \cdot \begin{bmatrix} x & x \\ x & x \\ x & x \end{bmatrix} \]
\[ S_{el}^{r} = \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \cdot \left( \begin{bmatrix} x & x \\ x & x \\ x & x \end{bmatrix} \right)^{-2} \cdot \begin{bmatrix} x & x \\ x & x \\ x & x \end{bmatrix} \]
convergence of the HSM

operator equation for fixed $\hat{x}$:

$$(T_f - K)(u_0, \tilde{U}) = h(u'_0)$$

$K$ is a compact operator and $T_f$ a Toeplitz operator with symbol $f(z) = -\kappa_0^2|z + 1|^2 - \kappa^2|z - 1|^2$.

projection method:

$$P_{Nr} (T_f - K) P_{Nr} (u_0^{(Nr)}, \tilde{U}^{(Nr)}) = P_{Nr} h(u'_0)$$
convergence of the HSM

operator equation for fixed \( \hat{x} \):

\[
(T_f - K)(u_0, \tilde{U}) = h(u'_0)
\]

\( K \) is a compact operator and \( T_f \) a Toeplitz operator with symbol
\[
f(z) = -\kappa_0^2|z + 1|^2 - \kappa^2|z - 1|^2.
\]

projection method:

\[
P_{N_r}(T_f - K)P_{N_r}(u_0^{(N_r)}, \tilde{U}^{(N_r)}) = P_{N_r}h(u'_0)
\]

Theorem

The projection method converges super-algebraically, i.e. for arbitrary \( p \in \mathbb{N} \) there holds the error estimation

\[
|u_0^{(N_r)} - u_0| \leq \frac{C_p}{(N_r)^p}.
\]
Toeplitz operator

Definition
Given $f : S^1 \to \mathbb{C}$ and the projection $P : L^2(S^1) \to H^+(S^1)$, the Toeplitz operator $T_f : H^+(S^1) \to H^+(S^1)$ with symbol $f$ is defined by $T_f \varphi := P(f \cdot F)$, $F \in H^+(S^1)$.

Theorem
If $f(z) \neq 0 \ \forall z \in S^1$, then $T_f$ is a Fredholm-operator with index $\text{ind}(T_f) = -\text{wn}(f)$, where $\text{wn}(f)$ is the winding number of $f$.

outline

introduction

pole condition

Hardy space method

numerical examples
convergence test

\[ \kappa = 1 \]  \quad \kappa = 5 \quad \kappa = 25

![Graphs showing convergence test for different \( \kappa \) values.](image)
resonances of an open square

QF ≈ 1, 98 \cdot 10^7

QF ≈ 199, 6

QF ≈ 2, 24
resonance frequencies I

\( \kappa = 3.6979 \)

plot range: \([-2 \cdot 10^5, 2 \cdot 10^5]\)

\( \kappa = 3.7665 \)

plot range: \([-10^5, 10^5]\)

\( \kappa = 3.8 \)

plot range: \([-3 \cdot 10^4, 3 \cdot 10^4]\)
Comparison HSM - PML

Resonances of an open square

Resonances of an one-sided open square
resonance frequencies II

\[ \kappa = 3.6979 \]  \hspace{1cm}  \[ \kappa = 3.7665 \]  \hspace{1cm}  \[ \kappa = 3.8 \]

plot range:

\[ [-10^{10}, 10^{10}] \]  \hspace{1cm}  \[ [-2 \cdot 10^5, 2 \cdot 10^5] \]  \hspace{1cm}  \[ [-4 \cdot 10^5, 4 \cdot 10^5] \]
resonance frequencies III

\[ \kappa = 3.7665 \]

plot range: \([-10^5, 10^5]\)

\[ \kappa = 3.8 \]

plot range: \([-3 \cdot 10^5, 3 \cdot 10^5]\)
Laplace transform of the Helmholtz equation

\[- \Delta u(x) - \kappa^2 u(x) = 0\]
Laplace transform of the Helmholtz equation

\[- \Delta u(x) - \kappa^2 u(x) = 0\]

\[\downarrow \text{polar coordinates: } u_a(r, \hat{x}) := (r + a)^{d-1/2} u((r + a)\hat{x})\]

\[- \partial_r^2 u_a(r, \hat{x}) - \left( \kappa^2 + \frac{C_d + \Delta \hat{x}}{(r + a)^2} \right) u_a(r, \hat{x}) = 0\]
Laplace transform of the Helmholtz equation

\[ - \Delta u(x) - \kappa^2 u(x) = 0 \]

\[ \downarrow \text{polar coordinates: } u_a(r, \hat{x}) := (r + a)^{d-1} u ((r + a)\hat{x}) \]

\[ - \partial_r^2 u_a(r, \hat{x}) - \left( \kappa^2 + \frac{C_d + \Delta_{\hat{x}}}{(r + a)^2} \right) u_a(r, \hat{x}) = 0 \]

\[ \downarrow \text{Laplace transform: } \hat{u}_a(\cdot, \hat{x}) := \mathcal{L}_a(\cdot, \hat{x}) \]

\[ - (s^2 + \kappa^2) \hat{u}_a(s, \hat{x}) - (C_d + \Delta_{\hat{x}}) \hat{I}_a^2 \hat{u}_a(s, \hat{x}) = -su_0(\hat{x}) - u'_0(\hat{x}) \]
Laplace transform of the Helmholtz equation

\[- \Delta u(x) - \kappa^2 u(x) = 0\]

\[
\downarrow \quad \text{polar coordinates: } u_a(r, \hat{x}) := (r + a)^{\frac{d-1}{2}} u((r + a)\hat{x})
\]

\[- \frac{\partial^2}{\partial r^2} u_a(r, \hat{x}) - \left( \kappa^2 + \frac{C_d + \Delta \hat{x}}{(r + a)^2} \right) u_a(r, \hat{x}) = 0\]

\[
\downarrow \quad \text{Laplace transform: } \hat{u}_a(\cdot, \hat{x}) := \mathcal{L} u_a(\cdot, \hat{x})
\]

\[- (s^2 + \kappa^2)\hat{u}_a(s, \hat{x}) - (C_d + \Delta \hat{x}) l^2_a \hat{u}_a(s, \hat{x}) = -su_0(\hat{x}) - u'_0(\hat{x})\]

\[
\downarrow \quad \text{Möbius transform: } U_a(\cdot, \hat{x}) := M \hat{u}_a(\cdot, \hat{x})
\]

\[
\left( \kappa_0^2(z + 1)^2 - \kappa^2(z - 1)^2 \right) U_a(z, \hat{x}) + (z - 1)^2(C_d + \Delta \hat{x}) l^2_a U_a(z, \hat{x}) = i\kappa_0 \sqrt{-2i\kappa_0(z + 1)} u_0(\hat{x}) - i\kappa_0(z + 1)(z - 1) u'_0(\hat{x})
\]
Laplace transform of the Helmholtz equation

\[- \Delta u(x) - \kappa^2 u(x) = 0\]

\[\downarrow \text{polar coordinates: } u_a(r, \hat{x}) := (r + a) \frac{d-1}{2} u((r + a)\hat{x})\]

\[- \partial_r^2 u_a(r, \hat{x}) - \left( \kappa^2 + \frac{C_d + \Delta \hat{x}}{(r + a)^2} \right) u_a(r, \hat{x}) = 0\]

\[\downarrow \text{Laplace transform: } \hat{u}_a(\cdot, \hat{x}) := \mathcal{L} u_a(\cdot, \hat{x})\]

\[- (s^2 + \kappa^2) \hat{u}_a(s, \hat{x}) - (C_d + \Delta \hat{x}) I_a^2 \hat{u}_a(s, \hat{x}) = -su_0(\hat{x}) - u_0'(\hat{x})\]

\[\downarrow \text{Möbius transform: } U_a(\cdot, \hat{x}) := \mathcal{M} \hat{u}_a(\cdot, \hat{x})\]

\[(T_f + K) U_a(\cdot, \hat{x}) = h(u_0(\hat{x}), u_0'(\hat{x}))\]

with \(f(z) := \kappa_0^2(z + 1)^2 - \kappa^2(z - 1)^2\)