Theory of Power Kernels

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Preface

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First of all, I thank Professor Robert Schaback for his support, patience and understanding while supervising my studies. His enthusiasm, insight, precision and criticism of gratuitous abstraction have enormously influenced my development as a mathematician in the field of using radial basis function for global approximation. In spite of his many commitments he has always been generous with his time. He taught me how to do research and how to present it in a precise way. Also, he has always been prepared to answer questions and to help over difficulties.
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This dissertation is dedicated to my parents.

Chapter 1

Introduction

In practice one often faces the problem of reconstructing an unknown function f from a finite set of discrete data. These data consist of data sites $X = \{x_1, \dots, x_N\}$ and data values $f_j = f(x_j)$,$1 \le j \le N$, and the reconstruction has to approximate the data values at the data sites. Moreover, in many cases the data sites are scattered, i.e. they bear no regular structure at all. In some applications, the data sites even come from a very high dimensional space. Hence, for an unifying approach, methods have to be developed which are capable of meeting these claims.

One possible way to reconstruct a function from discrete data is interpolation. Suppose $X = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$ is a set of distinct data sites and $f_1, \dots, f_N$ are certain data values which should be interpolated at the sites. In other words, one is interested in finding a continuous function $s : \mathbb{R}^d \to \mathbb{R}$ with $s(x_j) = f_j$, $1 \le j \le N$. At this point it is not necessary that the data values $\{f_j\}$, $1 \le j \le N$, actually stem from a function f, but we will keep this in mind for later reasons.

In the univariate case $d = 1$, a space of interpolating functions can be chosen to consist of all polynomials p of degree at most $N - 1$. However, by a well-known theorem of Mairhuber, if we are working in space dimension $d \ge 2$, it is impossible to fix an N-dimensional function space beforehand which works for all sets of N distinct data sites. The space of interpolating functions must depend on the data sites. On the other hand, probably no one with some experience in approximation theory would even in the univariate case try to interpolate in one hundred thousand points with a polynomial of degree 99999.

Hence one has to go away from polynomials and use another framework in higher dimensions. A well-established technique called *finite elements* triangulates the data sites and defines piecewise polynomial and compactly supported functions locally on parts of the triangulations. This requires triangulations, which are a serious problem in higher dimensions, and the assembly of smooth functions from locally defined pieces requires a great amount of additional work.

However, there is a remarkably beautiful theory, where all space dimensions can be handled in the same way, without any triangulation needed, and where the use of arbitarily smooth functions is no problem. It starts with a smooth univariate function $\phi : [0, \infty) \to \mathbb{R}$ and the Euclidean distance $\|x - y\|_2$ of two vectors $x, y \in \mathbb{R}^d$ to define a multivariate *radial basis function* of the form $\phi(\|x - y\|_2)$ on $\mathbb{R}^d \times \mathbb{R}^d$. Then this function is shifted to the data sites $x_j,\ 1 \leq j \leq N$ such that interpolation is done by the span of the functions $\phi(\|x - x_j\|_2),\ 1 \leq j \leq N$. This easily generates a data-dependent space of functions of any number of variables and of any smoothness, if ϕ is chosen properly, e.g. as the *Gaussian* $\phi(r) = e^{-r^2}$.

Radial basis functions have become an increasingly popular mathematical discipline which started with the practical work of Hardy [12, 13] in 1971, and with the theoretical work of Duchon [5, 6, 7], and which developed in some domain of mathematics and physics by for example Dyn [8, 9], Jackson [14], Powell [21], Buhmann [2, 3, 4] and Schaback and Wendland [25, 27, 31]. Only recently, two books on the subject have appeared (Buhmann [3], Wendland [30]).

The theory of radial basis functions generalizes easily to the theory of *reproducing kernel Hilbert spaces* of functions on domains Ω, where *kernels* K reproduce functions f in the Hilbert spaces H with inner products $(\cdot, \cdot)_H$ via

$$f(x) = (f, K(x, \cdot))_H \text{ for all } f \in H,\ x \in \Omega.$$

The connection to radial basis functions is made via $K(x, y) = \phi(\|x - y\|_2)$ if there is Euclidean invariance in the Hilbert space H.

These reproducing kernels of Hilbert spaces of functions form the context of this thesis, as indicated by part of the title. But since the kernels sometimes are given without any Hilbert space background, we first focus on kernels themselves. They usually are *conditionally* positive definite of a certain *order* m, but the cases $m = 0$ and $m > 0$ require different techniques. To compare these two situations, and to overcome the differences will be the topic of the next two chapters. Chapter 2 describes the standard setting of interpolation problems using conditionally positive definite kernels, and it confines itself to methods not using the Hilbert space background. We study the general case $m \geq 0$ and check the spans and dimensions of spaces of functions occurring there. Our main result will be on the linear independence of shifts even in case $m > 0$. The next chapter then introduces Hilbert spaces, and we explain how any given conditionally positive kernel defines a *native* Hilbert space of functions in which it is reproducing in a certain way. Furthermore, it relates the cases $m = 0$ and $m > 0$ by describing the transition from $m > 0$ to $m = 0$ via a change of the initial kernel to a "normalized" kernel. This transition occurs in rather incomplete versions at various isolated places in the literature, but since it is basic to what follows later, we provide a full account of it here.

Chapters 4 and 5 are the core of the thesis. The main idea is to decompose large interpolation problems into smaller ones by introducing data-dependent kernels called *power kernels* because of their close relation to what is called the *power function* in the classical theory. These new kernels must have *native* Hilbert spaces along the lines of Chapter 3, and we show how those are related to the native Hilbert space of the original kernel. Except for the situation $m > 0$, which causes some (solvable) problems, it turns out that there is an orthogonal decomposition of the original native Hilbert space, involving the native space of the power kernel. Chapter 5 shows how to do this decomposition recursively. It may be used to split large interpolation problems into smaller ones with different kernels, and this was the main background motivation for this work.

Chapter 2

Interpolation Problems

2.1 Kernels and radial basis functions

This preliminary chapter serves as an introduction for readers working in applications. It describes kernels and radial basis functions as useful tools for the construction of multivariate functions that satisfy certain specified conditions, and it provides the guidelines for their use. We do this mainly by standard methods of linear algebra. The following chapter will then provide the Hilbert space background.

Definition 2.1 *If a univariate real-valued function* $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ *is used as a symmetric multivariate function* $\Phi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ *via*

$$\Phi(x, y) = \phi(\|x - y\|_2) \tag{2.1}$$

for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$*, then* ϕ *is called a* **radial basis function** *(RBF) and* Φ *will be called the* **associated kernel**.

From Definition 2.1 on, we use the notation $\|.\|_2$ for the standard Euclidean norm in $\mathbb{R}^d$, and thus the function $\Phi(x, y)$ is independent of translations and rotations acting on both vectors x and y simultaneously.
In the following, we shall often write the argument of ϕ in (2.1) as $r = \|x - y\|_2$, as in Table 2.1.

Remarks

a) Any radial basis function ϕ is univariate as a function of r, but $\Phi(x, y) = \phi(r) = \phi(\|x-y\|_2)$ is multivariate as a function of the vectors x and y. If the space dimension d is large, then this is a tremendous computational advantage.

b) The use of the Euclidean norm leads to good invariance properties with respect to the Euclidean geometry.

c) Since certain radial basis functions like $\phi(r) = 0$ are useless, and since there is a vast variety of conceivable radial basis functions, we need additional arguments for their classification.

□

RBF	ϕ	m
poly-harmonic thin-plate spline	$\phi(r) = r^2 log(r)$	2
poly-harmonic spline	$\phi(r) = r^3$	2
multi-quadric	$\phi(r) = (1 + r^2)^{1/2}$	1
inverse multi-quadric	$\phi(r) = (1 + r^2)^{-1/2}$	0
Gaussian	$\phi(r) = exp(-r^2)$	0
Wendland	$\phi(r) = (1 - r)_+^4(1 + 4r)$	0

Table 2.1: Conditionally positive definite radial basis functions of order m

We now ask whether arbitrary data specified at nodes $x_1, \dots, x_N$ in $\mathbb{R}^d$ can be interpolated by a function of the form $s(x) = \sum_{j=1}^N \alpha_j \Phi(x, x_j)$. The answer is yes if the symmetric matrix $A_{\Phi,X} = \Big(\Phi(x_j, x_k)\Big)_{j,k}$ is nonsingular. Our goal is to characterize the functions Φ for which this matrix is positive definite. Along the way we prove some theorems about the positive definiteness and review a bit of matrix theory.

Definition 2.2 *Let Ω be a subset of $\mathbb{R}^d$. A function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called* **positive semidefinite** *on Ω, if for any finite set $X = \{x_1, \dots, x_N\}$ of points* *in Ω the matrix $A_{\Phi,X} = (\Phi(x_j, x_k))_{j,k}$ is positive semidefinite. i.e.*

$$\alpha^T A_{\Phi,X}\alpha = \sum_{j=1}^{N}\sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j, x_k) \geq 0 \tag{2.2}$$

for all $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ in $\mathbb{R}^d$. And if, in addition,

$$\alpha^* A_{\Phi,X}\alpha = 0 \text{ implies } \alpha \equiv 0$$

then we say that Φ is (strictly) **positive definite**.

We may call such a Φ rather a *kernel* than a *function.*

Note that in some older parts of the literature a function is called positive definite when it is positive semidefinite in our notation. But we want to align the notion of positive definiteness of functions to the notion of positive definiteness of matrices, and we do not want to repeat bad notation just because it is used in the literature.

In what follows, let $\mathbb{P}_m^d$ denote the space of d–variate polynomials over $\mathbb{R}$ with degree up to $m-1$ or to *order* up to m. Note that $\mathbb{P}_0^d = \{0\}$. The dimension of $\mathbb{P}_m^d$ will be denoted by $Q \geq 0$. From chapter 3 on, we shall replace $\mathbb{P}_m^d$ by a general Q-dimensional space of functions on $\mathbb{R}^d$. This could also be done in this chapter, but we wanted to stay close to the original definition of conditional positive definiteness of order m, as follows:

Definition 2.3 *Let Ω be a set of $\mathbb{R}^d$. A function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called* **conditionally positive semi-definite** *on Ω of order m if for any finite set of points $X = \{x_1, \ldots, x_N\}$ in Ω and for all $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ in $\mathbb{R}^N$ satisfying the additional condition*

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0 \text{ for all } p \in \mathbb{P}_m^d \tag{2.3}$$

the inequality (2.2) holds. A function Φ will be called **conditionally positive definite** *of order m, if the quadratic form in (2.2), defined on the subspace of vectors α satisfying the above moment condition, is positive unless α is zero.*

Proposition 2.4 *Every function that is conditionally positive (semi-)definite function of order m is also conditionally positive (semi-)definite function of order $l \geq m$.*
A function which is conditionally positive (semi-)definite of order $m = 0$ is positive (semi-)definite.

We need some statements about symmetric positive definite matrices later:

Lemma 2.5 *If a symmetric matrix is positive definite, then its eigenvalues are positive.*

Proof: Let A be a symmetric positive definite matrix, and let λ be one of its eigenvalues. Then there exists a corresponding eigenvector $V \neq 0$ in $\mathbb{R}^d$. Since $V \neq 0$ then

$$0 < V^T A V = V^T \lambda V = \lambda \|V\|_2^2.$$

Hence $\lambda > 0$. □

2.2 Interpolation of multivariate functions

Generally a discrete set $X = \{x_1, \ldots, x_N\}$ of points in $d-$dimensional space $\mathbb{R}^d$ and real valued data $f(x_1), \ldots, f(x_N)$ are given, and the task is to construct a continuous or sufficiently differentiable function $s : \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfies the interpolation equations

$$s(x_j) = f(x_j), \quad j = 1, \ldots, N \tag{2.4}$$

If s depends linearly on N parameters, these equations define a $N \times N$ system of linear equations. We now let the interpolant s be a linear combination of translates of a conditionally positive definite kernel $(x, y) \mapsto \Phi(x, y)$ for $x, y \in \mathbb{R}^d$. Written explicity, s has the form:

$$s(x) = \sum_{j=1}^{N} \alpha_j \Phi(x, x_j), \quad x \in \mathbb{R}^d. \tag{2.5}$$

If we define $A_{X,\Phi}$ to be the $N \times N$ matrix that has the elements $\Phi(x_i, x_j)$, and f_X to be the vector $(f(x_1), \ldots, f(x_N))^T$ in $\mathbb{R}^N$ whose elements are the right hand sides of the interpolation equations (2.4), and α to be the vector $(\alpha_1, \ldots, \alpha_N)^T$ in $\mathbb{R}^N$, then the interpolation equations (2.4) provide the linear system

$$A_{X,\Phi}\alpha = f_X. \tag{2.6}$$

For several important choices of Φ, the matrix $A_{X,\Phi}$ is invertible under rather mild conditions on the positions of the N interpolation points $x_1, \ldots, x_N$. In fact, this will hold by definition, if Φ is positive definite.

However, there are well-defined and useful kernels for which the matrix $A_{X,\Phi}$ is not always invertible. One example is the thin plate spline basis function $\phi(r) = r^2 \log r$ which was introduced by Duchon [5, 6]. If one data point lies at the centre of the unit sphere and the others are distinct points on the unit sphere, then one row and one column of $A_{X,\Phi}$ consist entirely of zeros and thus $A_{X,\Phi}$ is singular. Fortunately, it is possible to remove this difficulty by augmenting (2.5) by adding a polynomial of degree at most one or of order at most two. This leads to the notion of conditionally positive definite functions of positive order m, as defined before.

In general, let $p_1, \ldots, p_Q$ be a basis of the space $\mathbb{P}_m^d$ of polynomials of order up to m on $\mathbb{R}^d$, i.e:

$$\mathbb{P}_m^d = \text{ span } \{p_1, \ldots, p_Q\} \;\; Q = \text{ dim } \;\; \mathbb{P}_m^d = \begin{pmatrix} m-1+d \\ d \end{pmatrix}.$$

Now we represent the interpolation function s in the form

$$s_{f,X}(x) = \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) + \sum_{k=1}^{Q} \beta_k p_k(x). \tag{2.7}$$

The interpolation conditions are then given by

$$f_i = \sum_{j=1}^{N} \alpha_j \Phi(x_i, x_j) + \sum_{k=1}^{Q} \beta_k p_k(x_i), \quad \text{for all } \; 1 \le i \le N.$$

We add the following condition

$$\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0, 1 \le k \le Q \tag{2.8}$$

for the coefficients in order to bind the remaining degrees of freedom, and to arrive at the condition (2.3) used in the definition of conditional positive definiteness.

The two equations (2.7) and (2.8) form the system

$$\begin{pmatrix} A_{X,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix} = \begin{pmatrix} f_X \\ 0 \end{pmatrix} \tag{2.9}$$

with

$$\begin{aligned} A_{X,\Phi} &= (\Phi(x_i, x_j))_{1\le i,j\le N} \\ P_X &= (p_k(x_j))_{1\le j\le N,\, 1\le k\le Q} \\ f_X &= (f(x_1), \dots, f(x_N))^T \\ \alpha_X &= (\alpha_1, \dots, \alpha_N)^T \\ \beta_X &= (\beta_1, \dots, \beta_Q)^T. \end{aligned}$$

In case of $m = 0$ i.e. when the radial basis function Φ is positive definite, the interpolant $s_{f,X}$ reduces to (2.5) without applying the additional condition (2.8).

The general theory of the solvability of the interpolation problem is very well developed and we only collect the basics here for later use. The interested reader can look at the paper of Light [15] and Powell [21] for an enlarged overview.

Proposition 2.6 *The system (2.9) has a unique solution if the matrix P_X is injective, and if Φ is a conditionally positive definite function of order m.*

Proof: We look at the homogeneous system of the form (2.9):

$$\begin{pmatrix} A_{X,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then we have $0 = A_{X,\Phi}\alpha_X + P_X\beta_X$, which implies $\alpha_X^T A_{X,\Phi}\alpha_X + \alpha_X^T P_X\beta_X = 0$, and $P_X^T\alpha_X = 0 = \alpha_X^T P_X$, hence $\alpha_X^T A_{X,\Phi}\alpha_X = 0$. Since $A_{X,\Phi}$ is a conditionally positive definite function we get $\alpha_X = 0$, and $P_X\beta_X = 0$. And since P_X is injective then we finally have $\beta_X = 0$. Since the homogeneous system has only the trivial solution, the general system (2.9) is uniquely solvable. □

In the rest of this thesis, we shall always assume the hypotheses of Proposition 2.6 to be satisfied, unless stated otherwise.

Proposition 2.7 *If the polynomials in $\mathbb{P}_m^d$ satisfy the property*

$$\textit{If} \quad p \in \mathbb{P}_m^d \quad \textit{satisfies} \quad p(x_i) = 0 \textit{ for all } x_i \in X \textit{ then } p \equiv 0$$

then P_X is injective.

Proof: Again, we assume $\mathbb{P}_m^d = \text{span}\,\{p_1, \dots, p_Q\}$ with dim $\mathbb{P}_m^d = Q$, and then all polynomials $p \in \mathbb{P}_m^d$ have the form $p = \sum_{k=1}^Q \beta_k p_k$ for some $\beta = \{\beta_1, \dots, \beta_Q\}$. And P_X is injective when $P_X \beta_X = 0$ implies $\beta_X = 0$. This means that $p(x_j) = \sum_{k=1}^Q \beta_k p_k(x_j) = 0$ for all $1 \leq j \leq N$ implies $\beta_k = 0$ for all $1 \leq k \leq Q$ or $p = 0$. But this follows from our assumption. □

2.3 Lagrange form of interpolation

To rewrite an interpolant (2.9) in Lagrange form, we now introduce the system

$$\tilde{A}_{X,\Phi} \begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} = \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} = \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix} \qquad (2.10)$$

with

$$\begin{aligned} \tilde{A}_{\Phi,X} &= \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \\ V_X(x) &= (v_j(x))_{1 \leq j \leq Q}^T \\ p(x) &= (p_j(x))_{1 \leq j \leq Q}^T \\ U_X(x) &= (u_j(x))_{1 \leq j \leq N}^T \\ \Phi_X(x) &= (\Phi(x, x_j))_{1 \leq j \leq N}^T. \end{aligned}$$

The system (2.10) is uniquely solvable due to the argument given in Proposition 2.6 under the assumptions there. This allows us to rewrite the interpolant in the following way:

Proposition 2.8 *Under the assumptions of Proposition 2.6, we can rewrite the interpolant from (2.7) in the form*

$$s_{f,X}(x) = \sum_{j=1}^N u_j(x) f(x_j).$$

Proof: We start with the definition (2.7) of the interpolant and do some linear algebra:

$$
\begin{aligned}
s_{f,X}(x) &= \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) + \sum_{k=1}^{Q} \beta_k p_k(x) \\
&= \alpha_X^T \Phi_X(x) + \beta_X^T p(x) \\
&= \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix}^T \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix}^T \tilde{A}_{\Phi,X} \begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T & , & \alpha_X^T P_X \end{pmatrix} \begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} \\
&= (\alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T) U_X(x) + \alpha_X^T P_X V_X(x) \\
&= (\alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T) U_X(x) \\
&= f_X^T U_X(x) \\
&= \sum_{j=1}^{N} u_j(x) f(x_j).
\end{aligned}
$$

□

Proposition 2.9 *Under the assumptions of Proposition 2.6, there is a unique Lagrange-type representation associated to the system (2.10) in X characterized by*

$$
\begin{aligned}
u_i(x_j) &= \delta_{i,j},\ 1 \le i, j \le N && (2.11) \\
v_k(x_j) &= 0,\ 1 \le j \le N,\ 1 \le k \le Q && (2.12) \\
p_k(x) &= \sum_{i=1}^{N} u_j(x) p_k(x_j),\ 1 \le k \le Q. && (2.13)
\end{aligned}
$$

Proof: We start with the system (2.10):

$$
\begin{aligned}
A_{X,\Phi} U_X(x) + P_X V_X(x) &= \Phi_X(x) && (2.14) \\
P_X^T U_X(x) &= p(x). && (2.15)
\end{aligned}
$$

Hence, we have for $x = x_i,\ 1 \le i \le N$

$$\begin{aligned}\sum_{i=1}^{N} u_j(x_i)\Phi(x_i,x_j) + \sum_{k=1}^{Q} p_k(x_j)v_k(x_i) &= \Phi(x_i,x_j)\\ \sum_{i=1}^{N} u_j(x)p_k(x_j) &= p_k(x).\end{aligned}$$

Then we conclude that since (2.10) is uniquely solvable, the equations $u_j(x_i) = \delta_{i,j}$ and $v_k(x_j) = 0$ must hold. □

And we get the uniqueness from

Theorem 2.10 *The vector* $U(x) = \Big(u_1(x),\dots,u_N(x)\Big)^T$ *formed by the values of the Lagrange basis functions* $u_1,\dots,u_N$ *of Proposition 2.8 at* $x \in \mathbb{R}^d$ *coincides with the solution* $U^*(x)$ *of the conditional minimization problem*

$$\min\Big\{U_X^T A_{X,\Phi} U_X - 2U_X^T \Phi_X(x) + \Phi(x,x)\ \mid\ P_X^T U_X(x) = p(x)\Big\}. \tag{2.16}$$

Proof: If (2.16) is solved by Lagrange multiplier techniques, and if the solution is written in a matrix form, then there exists a vector $V_X^*(x) = \Big(v_1^*(x),\dots,v_N^*(x)\Big)^T$ of Lagrange multipliers such that the system

$$\begin{pmatrix} A_{X,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix}\begin{pmatrix} U_X^* \\ V_X^* \end{pmatrix} = \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix} \tag{2.17}$$

holds for the solution $U_X^*(x) = \Big(u_1^*(x),\dots,u_N^*(x)\Big)^T$ of (2.16). This follows from standard results in quadratic optimization. But the system (2.17) is the same as (2.10) and uniquely solvable, thus the solutions must agree. □

Corollary 2.11 *The equation* $P_X^T U_X(x) = p(x)$ *implies that every polynomial of* $\mathbb{P}_m^d$ *can be reproduced .*

Proof: This equation is equivalent to $p_k(x) = \sum_{j=1}^{N} u_j(x)p_k(x_j)$ for all $1 \le k \le Q$, which is the same as (2.13). Since a basis is reproduced, all elements of the space are reproduced. □

The results that we have collected up to now are

$$\begin{array}{rcll} u_i(x_j) & = & \delta_{ij}, & 1 \leq i, j \leq N \\ v_k(x_j) & = & 0, & 1 \leq j \leq N,\ 1 \leq k \leq Q \\ p_k(x) & = & \sum_{j=1}^{N} u_j(x) p_k(x_j), & 1 \leq k \leq Q \\ s_{f,X}(x) & = & \sum_{j=1}^{N} u_j(x) f(x_j), & 1 \leq j \leq N. \end{array} \tag{2.18}$$

2.4 Stability of the solution

We now look at perturbations of the system (2.9) when the right-hand side varies. In standard perturbation notation, this means

$$\begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \alpha_X + \Delta\alpha_X \\ \beta_X + \Delta\beta_X \end{pmatrix} = \begin{pmatrix} f_X + \Delta f_X \\ 0 \end{pmatrix}. \tag{2.19}$$

Subtracting (2.9) from (2.19) gives

$$\begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \Delta\alpha_X \\ \Delta\beta_X \end{pmatrix} = \begin{pmatrix} \Delta f_X \\ 0 \end{pmatrix}. \tag{2.20}$$

This implies

$$\begin{array}{rcl} P_X^T \Delta\alpha_X & = & 0, \\ \Delta\alpha_X^T A_{X,\,\Phi} \Delta\alpha_X & = & \Delta\alpha_X^T \Delta f_X. \end{array}$$

We denote

$$F_{\alpha_X}(A_{X,\,\Phi}) = \frac{\alpha_X^T A_{X,\,\Phi} \alpha_X}{\alpha_X^T \alpha_X},$$

$$L_X = \{\alpha_X \in \Omega \setminus \{0\} \ \mid \ P_X^T \alpha_X = 0\}. \tag{2.21}$$

With this notation we can define

$$\lambda_{min}(A_{\Phi,X}) = \inf_{\alpha \in L_X} F_{\alpha_X}(A_{X,\,\Phi}),$$

Since the quadratic form $\alpha_X^T A_{X,\,\Phi} \alpha_X$ is positive on L_X because $A_{\Phi,X}$ is conditionally positive definite, then from Lemma 2.5 we have

$$\lambda_{min}(A_{\Phi,X}) > 0.$$

Thus we can give

Proposition 2.12 *The perturbation of the solution of (2.19) has the property*

$$\|\Delta\alpha_X\|_2 \le \frac{1}{\lambda_{min}(A_{X,\,\Phi})}\|\Delta f_X\|_2.$$

Proof: From the systems (2.9) and (2.20), we get

$$(\Delta\alpha_X)^T A_{X,\,\Phi}(\Delta\alpha_X) \;\;=\;\; (\Delta\alpha_X)^T(\Delta f_X).$$

Now the kernel Φ is conditionally positive definite, and $\Delta\alpha_X$ is in L_X. Then from our definition of $\lambda_{min}(A_{\Phi,X})$ we see that

$$0 < \lambda_{min}(A_{\Phi,X})\|\Delta\alpha_X\|_2^2 \le \Delta\alpha_X^T A_{\Phi,X}\Delta\alpha_X = (\Delta\alpha_X)^T(\Delta f_X)$$

holds. This implies the assertion. □

2.5 Inverses of interpolation matrices

The systems (2.9) and (2.10) have the matrix

$$\begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix}$$

of which we only know that the upper left part defines a positive definite form on a subspace. The full matrix itself has no apparent definiteness properties. Here, we shall study properties of the inverse matrix

$$\begin{pmatrix} B_{X,\,\Phi} & Q_{X,\Phi} \\ Q_{X,\Phi}^T & C_{X,\Phi} \end{pmatrix}$$

where $B_{X,\,\Phi}$ is $N \times N$, $Q_{X,\Phi}$ is $Q \times N$, and $C_{X,\Phi}$ is $Q \times Q$. At this point it is an interesting question to ask which matrices actually depend on Φ. For the solution of the standard system, we get

$$\begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix} = \begin{pmatrix} B_{X,\,\Phi} & Q_{X,\Phi} \\ Q_{X,\Phi}^T & C_{X,\Phi} \end{pmatrix} \begin{pmatrix} f_X \\ 0 \end{pmatrix}$$

This system allows us to have the equations

$$\begin{aligned} \alpha_X &= B_{X,\,\Phi} f_X \\ \beta_X &= Q_{X,\Phi}^T f_X \end{aligned} \tag{2.22}$$

that will be very useful. The inversion property gives

$$\begin{aligned}\begin{pmatrix} I_N & 0 \\ 0 & I_Q \end{pmatrix} &= \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} B_{X,\,\Phi} & Q_{X,\Phi} \\ Q_{X,\Phi}^T & C_{X,\Phi} \end{pmatrix} \\ &= \begin{pmatrix} A_{X,\,\Phi}B_{X,\,\Phi} + P_X Q_{X,\Phi}^T & A_{X,\,\Phi}Q_{X,\Phi} + P_X C_{X,\Phi} \\ P_X^T B_{X,\,\Phi} & P_X Q_{X,\Phi} \end{pmatrix}.\end{aligned}$$

Thus we arrive at

$$A_{X,\,\Phi}B_{X,\,\Phi} + P_X Q_{X,\Phi}^T = I_N \tag{2.23}$$
$$A_{X,\,\Phi}Q_{X,\Phi} + P_X C_{X,\Phi} = 0 \tag{2.24}$$
$$P_X^T B_{X,\,\Phi} = 0 \tag{2.25}$$
$$P_X^T Q_{X,\Phi} = I_Q. \tag{2.26}$$

We now multiply the equation (2.23) by f_X from the right side and α_X^T from the left side. Then we get

$$\alpha_X^T A_{X,\,\Phi} B_{X,\,\Phi} f_X = \alpha_X^T f_X - \alpha_X^T P_X Q_{X,\Phi}^T f_X.$$

Now we replace the value of $B_{X,\,\Phi} f_X$ from (2.22) and use $P_X^T \alpha_X = 0$. Then we get a nice equation

$$\alpha_X^T A_{X,\,\Phi} \alpha_X = f_X^T B_{X,\,\Phi} f_X. \tag{2.27}$$

Lemma 2.13 *The matrix $B_{X,\,\Phi}$ is positive semidefinite.*

Proof: Take an arbitrary vector f_X and form the system (2.9) with a conditionally positive definite kernel Φ. Then we have (2.27), and the additional condition (2.8) allows us to conclude $f_X^T B_{X,\,\Phi} f_X \geq 0$ for all $f_X \in \mathbb{R}^d$. □

The kernel of the matrix $B_{X,\,\Phi}$ can be determined:

Theorem 2.14 *$f_X^T B_{X,\,\Phi} f_X$ is zero if and only if the vector f_X is a vector of values of a polynomial on X of order at most m. This is equivalent to the fact that the interpolant $s_{f,X}$ is a polynomial of order at most m.*

Proof: Let $f_X^T B_{X,\,\Phi} f_X = 0$. Again, we use this f_X and solve the system (2.9) with it to get a coefficient vector α_X satisfying (2.8). From (2.27), we get $\alpha_X^T A_{X,\,\Phi}\alpha_X = 0$, and since Φ is conditionally positive definite, we get $\alpha_X = 0$. Using (2.9), we get $P_X\beta_X = f_X$. Hence, f_X is a vector of values of a polynomial of order at most m, and unique interpolation implies that $s_{f,X}$ is exactly this polynomial.

To prove the converse, let f_X be a vector of values of a polynomial p of order at most m on X. Then interpolation via the system (2.8) leads to $p = s_{f,X}$ and $\alpha_X = 0$. Since (2.27) holds, we get the result $f_X^T B_{X,\,\Phi} f_X = 0$. But this could have followed directly from (2.25) and (2.22). □

Hence, $B_{X,\,\Phi}$ has Q vanishing eigenvalues, and $N - Q$ eingenvalues are positive.

The system (2.9) causes $\alpha_X = B_{X,\Phi} f_X$ to satisfy $P_X^T \alpha_X = 0$ for every $f_X \in \mathbb{R}^d$. That means, it must satisfy $P_X^T B_{X,\Phi} f_X = 0$ for all $f_X \in \mathbb{R}^d$, which is equivalent to the condition $P_X^T B_{X,\Phi} = 0$. The symmetry of $B_{X,\Phi}$ yields the new condition

$$B_{X,\Phi} P_X = 0$$

associated to the matrix $B_{X,\Phi}$. This is another way of expressing (2.25) or one direction of Theorem 2.14.

Remarks

a) With a notation to become better known in the next chapter, we can describe the "energy" of an interpolant $s_{f,X}$ solving the system (2.9) as $\|s_{f,X}\|_\Phi^2 := \alpha_X^T A_{X,\Phi}\alpha_X$. This is related to the matrix $B_{X,\Phi}$ in the following way. We have

$$\|s_{f,X}\|_\Phi^2 = \alpha_X^T A_{X,\Phi}\alpha_X = \alpha_X^T f_X$$

due to $P_X\alpha_X = 0$. If we multiply the equation (2.22) by f_X^T, we get $f_X^T\alpha_X = f_X^T B_{X,\Phi} f_X$. Thus

$$\|s_{f,X}\|_\Phi^2 = f_X^T B_{X,\Phi} f_X$$

generalizes the obvious identity

$$\|s_{f,X}\|_\Phi^2 := \alpha_X^T A_{X,\Phi}\alpha_X = f_X^T (A_{X,\Phi})^{-1} f_X$$

which holds in case $m = 0$.

b) If we multiply (2.23) from the left with $B^T_{X,\Phi}$, we get

$$B^T_{X,\Phi} A_{X,\Phi} B_{X,\Phi} + B^T_{X,\Phi} P_X Q^T_{X,\Phi} = B^T_{X,\Phi}.$$

If we now use (2.25), then

$$B^T_{X,\Phi} A_{X,\Phi} B_{X,\Phi} = B^T_{X,\Phi}.$$

c) If we multiply (2.24) from the left with $B^T_{X,\Phi}$, we get

$$B^T_{X,\Phi} A_{X,\Phi} Q_{X,\Phi} + B^T_{X,\Phi} P_X C_{X,\Phi} = 0.$$

Using (2.25) again, we get

$$B^T_{X,\Phi} A_{X,\Phi} Q_{X,\Phi} = 0. \tag{2.28}$$

d) If we multiply (2.24) from the left with $Q^T_{X,\Phi}$, we get

$$Q^T_{X,\Phi} A_{X,\Phi} Q_{X,\Phi} + Q^T_{X,\Phi} P_X C_{X,\Phi} = 0$$

Using (2.26), we get

$$C_{X,\Phi} = Q^T_{X,\Phi} A_{X,\Phi} Q_{X,\Phi}. \tag{2.29}$$

At this point, there is no indication that any of the submatrices of the inverse does not depend on Φ.

Finally, we look at the problem of interpolation in Lagrange form by components of the vector $U_X(x)$ given with the system (2.10). We take the same notation of the inverse matrix. Then we get

$$\begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} = \begin{pmatrix} B_{X,\Phi} & Q_{X,\Phi} \\ Q^T_{X,\Phi} & C_{X,\Phi} \end{pmatrix} \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}.$$

Thus,

$$U_X(x) = B_{X,\Phi}\Phi_X(x) + Q_{X,\Phi}p(x) \tag{2.30}$$
$$V_X(x) = Q^T_{X,\Phi}\Phi_X(x) + C_{X,\Phi}p(x). \tag{2.31}$$

Now we use (2.23) and multiply it from the left with $U_X(x)^T$ and from the right with $\Phi_X(y)$. Then we get

$$U_X(x)^T A_{X,\Phi} B_{X,\Phi} \Phi_X(y) + U_X(x)^T P_X Q_{X,\Phi}^T \Phi_X(y) = U_X(x)^T \Phi_X(y).$$

If we use (2.30) and (2.15), we get

$$U_X(x)^T A_{X,\Phi} U_X(y) - U_X(x)^T A_{X,\Phi} Q_{X,\Phi} p(y) + p^T(x) Q_{X,\Phi}^T \Phi_X(y) = U_X(x)^T \Phi_X(y).$$

If we replace the value of $U_X^T(x)$ from (2.30) we get

$$\begin{aligned} & U_X(x)^T A_{X,\Phi} U_X(y) - U_X(x)^T A_{X,\Phi} Q_{X,\Phi} p(y) + p^T(x) Q_{X,\Phi}^T \Phi_X(y) \\ = \; & \Phi_X^T(x) B_{X,\Phi} \Phi_X(y) + p^T(x) Q_{X,\Phi}^T p(y). \end{aligned}$$

Hence,

$$\begin{aligned} U_X(x)^T A_{X,\Phi} U_X(y) \; = \; & \Phi_X^T(x) B_{X,\Phi} \Phi_X(y) \\ & + U_X(x)^T A_{X,\Phi} Q_{X,\Phi} p(y). \end{aligned} \tag{2.32}$$

This is still somewhat unsymmetric, but we can insert (2.30) into the final term again, use (2.28) and (2.29) to arrive at a symmetric kernel

$$\begin{aligned} U_X(x)^T A_{X,\Phi} U_X(y) \; = \; & \Phi_X^T(x) B_{X,\Phi} \Phi_X(y) \\ & + p(x)^T C_{X,\Phi} p(y) \end{aligned}$$

which might be worth investigating.

2.6 Space decompositions

Here we want to derive some properties of the span of spaces spanned by the Lagrange basis function $U_X(x)$ and the complementary vector $V_X(x)$ of functions.

As before, let Φ be a conditionally positive definite function on Ω of order m, and let $\mathbb{P}_m^d$ the space of d-variate polynomials of order up to m with dimension Q. Furthermore, we keep our standard hypothesis relating m and X as stated after Proposition 2.6.

We define some linear spaces by :

$$\begin{aligned} U^* &= \text{span } \{u_1 \ldots, u_N\} \text{ with } \dim U^* = N \\ V^* &= \text{span } \{v_1 \ldots, v_Q\} \text{ with } \dim V^* \le Q \\ P^* &= \text{span } \{p_1 \ldots, p_Q\} \text{ with } \dim P^* = Q \\ W^* &= \text{span } \{\Phi(x_1, \cdot), \ldots, \Phi(x_N, \cdot)\} \text{ with } \dim W^* \le N. \end{aligned}$$

The first dimension equation $\dim U^* = N$ follows from the linear independence of Lagrange bases, while $\dim P^* = Q$ is true by definition. Our goal here is to prove

$$U^* + V^* = W^* + P^*$$

and it is hypothesized that the dimension of that space is $N + Q$. This question is not addressed in the literature. In case $m = 0$ we have $U^* = W^*$ with dimension N and there is nothing to do. Thus we assume $m > 0$ in this section.

For handling the additional condition for positive order m, we define L_X as in (2.21) by

$$L_X = \{\alpha \in \mathbb{R}^N \mid \sum_{i=1}^{N} \alpha_i p(x_i) = 0 \quad \text{for all} \quad p \in \mathbb{P}_m^d\}. \tag{2.33}$$

With these notions and hypotheses, one can define the interpolation space as

$$S^* = \mathbb{P}_m^d + \left\{ \sum_{j=1}^{N} \alpha_j \Phi(x_j, \cdot) \mid \alpha \in L_X \right\} \tag{2.34}$$

and some mappings by

$$\begin{aligned} T &: f \mapsto (f(x_1) \ldots, f(x_N)) \\ I_1 &: Tf \mapsto \sum_{j=1}^{N} u_j(\cdot) f(x_j) \\ I_2 &: Tf \mapsto \sum_{j=1}^{N} \alpha_j \Phi(x_j, \cdot) + \sum_{k=1}^{Q} \beta_k p_k(\cdot) \end{aligned} \tag{2.35}$$

where $\alpha \in L_X$ and $\beta \in \mathbb{R}^d$ are solution vectors of the system (2.8) when we replace f_X by $T(f)$ there. Then the mappings have the domains and ranges

$$\begin{aligned} I_1 &: \mathbb{R}^N \rightarrow U^* \\ I_2 &: \mathbb{R}^N \rightarrow S^*. \end{aligned}$$

Lemma 2.15 *We have $I_1 = I_2$.*

Proof: It is easy to start with the definition of the map I_2 :

$$\begin{aligned}
I_2(Tf) &= \sum_{j=1}^{N} \alpha_j \Phi_X(\cdot, x_j) + \sum_{k=1}^{Q} \beta_k p_k(\cdot) \\
&= \alpha_X^T \Phi_X(\cdot) + \beta_X^T p(\cdot) \\
&= \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix}^T \begin{pmatrix} \Phi_X(\cdot) \\ p(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_X \\ \beta_X \end{pmatrix}^T \tilde{A}_{\Phi,X} \begin{pmatrix} U_X(\cdot) \\ V_X(\cdot) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T \ , & \alpha_X^T P_X \end{pmatrix} \begin{pmatrix} U_X(\cdot) \\ V_X(\cdot) \end{pmatrix} \\
&= (\alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T) U_X(\cdot) + \alpha_X^T P_X V_X(\cdot) \\
&= (\alpha_X^T A_{\Phi,X} + \beta_X^T P_X^T) U_X(\cdot) \\
&= f_X^T U_X(\cdot) \\
&= \sum_{j=1}^{N} u_j(\cdot) f(x_j) \\
&= I_1(Tf).
\end{aligned}$$

□

Lemma 2.16 *The maps I_1 and I_2 are isomorphisms $\mathbb{R}^N \to U^*$ and $\mathbb{R}^N \to S^* = U^*$.*

Proof: We first prove that the map I_1 is injective. If $I_1(Tf) = 0$, we use the Lagrange property $u_j(x_k) = \delta_{jk}$ to get $I_1(Tf)(x_j) = f(x_j) = 0$ for all j, thus $Tf = 0$. Consequently, also I_2 is injective. Furthermore, the mapping I_1 clearly is surjective, again due to the Lagrange property. Thus both maps are isomorphisms between R^N and $U^* \subseteq S^*$. For the surjectivity of I_2, take some s in S^* and interpolate its values $f_X := T(s)$ using the system (2.8). Since interpolation is unique, we get $s(x) = s_{s,X}(x)$ for all $x \in \Omega$. Then $s(x) = I_2(f_X)(x) = I_1(T(s))(x) = s(x)$ for all $x \in \Omega$, and s is in the range of I_1 and I_2. Finally, this proves $U^* = S^*$. □

Corollary 2.17 *We have*

$$dim\ S^* = N = \ dim\ U^*.$$

Lemma 2.18 *We have*

$$dim\ L_X = N - Q.$$

Proof: From the map $P_X^T : \mathbb{R}^N \to \mathbb{R}^Q$ we conclude

$$N = \ \dim \mathbb{R}^N = \ \dim kern(P_X^T) + \ \dim P_X^T(\mathbb{R}^Q).$$

Since our general assumptions following Proposition 2.6 imply rank $P_X = Q \leq N$, we have $\dim\ (P_X^T(\mathbb{R}^Q)) = Q$. Hence

$$\dim kern(P_X^T) = \ \dim L_X = N - Q.$$

□

Theorem 2.19 *We have*

$$U^* + V^* = W^* + P^*$$

Proof: We look at the system (2.10). Since the full matrix $\tilde{A}_{\Phi,X}$ is non-singular (invertible), then the space that is spanned by the components of $U_X(x)$ and $V_X(x)$ equals the space that is spanned by $\Phi_X(x)$ and $p(x)$. This means

$$U^* + V^* = W^* + P^*.$$

□

The next step would be to prove that

$$\dim\ (U^* + V^*) = \ \dim\ (W^* + P^*) = N + Q.$$

This is not easy, because it requires a proof that functions in W^* are linearly independent and the sum in $W^* + P^*$ is direct. It is an open problem to prove this in general.

To end this chapter, we give a partial proof in case of translation-invariant Fourier-transformable kernels.

In particular, our first goal is to prove that $\dim W^* = N$. This requires to prove that the functions $\Phi(x_1, .), \ldots, \Phi(x_N, .)$ are linearly independent over Ω. For that, we specialize to the case where $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ is a univariate real-valued function used as a symmetric (conditionally) positive multivariate function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ via $\Phi(x, y) = \phi(\|x - y\|_2)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ in order to employ for our proof the analysis of Fourier. First, we will define the Fourier transform of a function as

Definition 2.20 *For $f \in L_1$ and $\omega \in \mathbb{R}^d$ we define its Fourier transform by*

$$\widehat{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\omega) e^{-ix^T\omega} d\omega$$

And we define the inverse Fourier transform by

$$\check{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\omega) e^{ix^T\omega} d\omega.$$

To reach our proof, we have to modify the notion of the Schwartz space $\mathcal{S}$ is the sense of this definition:

Definition 2.21 *For $m \in \mathbb{N}_0$ the set of all functions $\gamma \in \mathcal{S}$ which satify $\gamma(\omega) = o(\|\omega\|_2^m)$ for $\|\omega\|_2 \to 0$ will be denoted by $\mathcal{S}_m$. i.e:*

$$\mathcal{S}_m = \{\gamma \in \mathcal{S} \mid \gamma(\omega) = o(\|\omega\|_2^m), \;\; \textit{for all} \;\; \|\omega\|_2 \to 0\}.$$

Now we give this theorem

Theorem 2.22 *Let $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ be a univariate real-valued function used as a symmetric (conditionally) positive definite multivariate function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$ via $\Phi(x, y) = \phi(\|x - y\|_2)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $\widehat{\phi}$ exists. Then the functions $\Phi(x_1, .), \ldots, \Phi(x_N, .)$ are linearly independent on $\mathbb{R}^d$.*

Proof: Let $\{\alpha_1, \ldots, \alpha_N\} \in \mathbb{R}^N$ be such that

$$\sum_{j=1}^{N} \alpha_j \Phi(x_j, .) = 0$$

and we will prove that $\alpha_j = 0$ for all $1 \leq j \leq N$. We then have

$$0 = \sum_{j=1}^{N} \alpha_j \Phi(x_j, \omega) = \sum_{j=1}^{N} \alpha_j \phi(\|x_j - \omega\|_2) \text{ for all } \omega \in \mathbb{R}^d.$$

For every test function $\gamma \in \mathcal{S} \cap C_0^\infty(\mathbb{R}^d \setminus \{0\})$, we have

$$0 = \int_{\mathbb{R}^d} \sum_{j=1}^{N} \alpha_j \phi(\|\omega - x_j\|_2) \widehat{\gamma}(\omega) d\omega.$$

Then

$$\widehat{0} = 0 = \int_{\mathbb{R}^d} \widehat{\phi(\|.\|_2)}(\omega)\gamma(\omega) \sum_{j=1}^{N} \alpha_j e^{-i\omega^T x_j} d\omega$$

since γ is in $C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Then we get for all $\omega \neq 0$

$$\widehat{\phi(\|.\|_2)}(\omega) \sum_{j=1}^{N} \alpha_j e^{-i\omega^T x_j} = 0.$$

Since $\widehat{\phi(\|.\|_2)} > 0$ (because Φ is (conditionally) positive definite), then

$$\text{for all } \omega \in \mathbb{R}^d : \sum_{j=1}^{N} \alpha_j e^{-i\omega^T x_j} = 0.$$

But since the $e^{-i\omega^T x_j}$ are linearly independent over any open set, we get $\alpha_j = 0$.

□

Now we want to prove that the sum of W^* and P^* is direct. For that we use this lemma:

Lemma 2.23 *Let p be a polynomial of degre less than m. Then for every test function $\gamma \in \mathcal{S}_m$ we have*

$$\int_{\mathbb{R}^d} p(x)\widehat{\gamma}(x)dx = 0.$$

Proof: Let us assume that p has the representation $p(x) = \sum_{|\eta|<m} a_\eta x^\eta$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} p(x)\widehat{\gamma}(x)dx &= \sum_{|\eta|<m} a_\eta i^{-|\eta|} \int_{\mathbb{R}^d} (ix)^\eta \widehat{\gamma}(x)dx \\
&= \sum_{|\eta|<m} a_\eta i^{-|\eta|} \int_{\mathbb{R}^d} \widehat{D^\eta \gamma}(x)dx \\
&= (2\pi)^{d/2} \sum_{|\eta|<m} a_\eta i^{-|\eta|} D^\eta \gamma(0) \\
&= 0
\end{aligned}$$

because γ is in $\mathcal{S}_m$. □

Proposition 2.24 *Let $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ be a univariate real-valued function used as a symmetric (conditionally) positive definite multivariate function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ via $\Phi(x,y) = \phi(\|x-y\|_2)$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $\widehat{\phi}$ exists. Then under the preceding notation, the sum of W^* and P^* is direct.*

Proof: We just must prove that the $\sum_{j=1}^N \alpha_j \Phi(\cdot, x_j)$ is never a polynomial. For that, we suppose that we have

$$\sum_{j=1}^N \alpha_j \Phi(\cdot, x_j) = p(\cdot).$$

Then for all $\omega \in \mathbb{R}^d$, we have

$$\sum_{j=1}^N \alpha_j \phi(\|\omega - x_j\|_2) = p(\omega).$$

If we choose a test function γ in the space $\mathcal{S}_m \cap C_0^\infty(\mathbb{R}^d \setminus \{0\})$, we get for all $\omega \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} p(x)\widehat{\gamma}(x)dx = \int_{\mathbb{R}^d} \sum_{j=1}^N \alpha_j \phi(\|x - x_j\|_2)\widehat{\gamma}(x)dx.$$

Using Lemma 2.23, this vanishes, and we get

$$\widehat{0} = 0 = \int_{\mathbb{R}^d} \widehat{\phi}(\|.\|_2)(\omega) \sum_{j=1}^N \alpha_j e^{-i\omega^T x_j} \gamma(\omega) d\omega$$

since γ is in $C_0^\infty(\mathbb{R}^d \setminus \{0\})$. Then we get for all $\omega \neq 0$

$$\widehat{\phi}(\|.\|_2)(\omega) \sum_{j=1}^N \alpha_j e^{-i\omega^T x_j} = 0,$$

and the rest of the proof proceeds like in the previous theorem. □

Corollary 2.25 *Let* $\phi : \mathbb{R}_+ \mapsto \mathbb{R}$ *be a univariate real-valued function used as a symmetric (conditionally) positive definite multivariate function* $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *via* $\Phi(x, y) = \phi(\|x - y\|_2)$ *for all* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ *such that* $\widehat{\phi}$ *exists. Then under the preceding notation, we have*

$$dim\ V^* = Q.$$

Proof: From 2.22 and 2.24, one now knows that the sum of W^* and P^* is direct. Hence, applying 2.19, one gets $\dim U^* + \dim V^* = \dim W^* + \dim P^* = N + Q$. Since $\dim U^* = N$ then we get $\dim V^* = Q$. □

Chapter 3

Native Hilbert spaces for kernels

In the previous chapters, the discussion focused on the general interpolation problem in a context of Linear Algebra. Here, we look at the background in Functional Analysis. On one hand, all Hilbert spaces of functions with continuous point evaluation lead to a positive semidefinite kernel, but, on the other hand, all conditionally positive definite kernels are (in a somewhat generalized sense) reproducing kernels in a specific Hilbert space called the *native space* for the given kernel.

Consequently, the first section of the chapter proceeds from a Hilbert space of functions to its reproducing kernel, while the second section starts from a kernel and constructs its native Hilbert space. The results are mainly taken from the literature and compiled here for use in the following chapters.

Then we describe the transition from a conditionally positive definite kernel to an unconditionally positive definite kernel, and characterize their related native spaces. This is done in two steps, and the new kernels are called the *normalized* and the *extended* kernel. The first case is closely related to the *power kernels* of the next chapter.

3.1 Reproducing kernel Hilbert spaces

Let Ω be some domain in $\mathbb{R}^d$.

Definition 3.1 *Let H be a space of functions $f : \Omega \to \mathbb{R}$. We define* **the point evaluation functionals** *δ_x for f at $x \in \Omega$ by $\delta_x(f) = f(x)$. Then all δ_x are in the algebraic dual space of H for all $x \in \Omega$.*

Theorem 3.2 *Let H be a (real) Hilbert function space over Ω with an inner product $(.,.)_H$, and assume that all δ_x for $x \in \Omega$ are continuous, i.e. they are in the topological dual space H^*. Then, for all $x \in \Omega$, there exists a unique element $k(x,\cdot) \in H$ such that the* reproduction equation

$$(f, k(x,\cdot))_H = f(x) = \delta_x(f) \text{ for all } x \in \Omega,\ f \in H \tag{3.1}$$

holds.

Proof: The Riesz representation theorem for Hilbert spaces gives the existence and the uniqueness of the function $k(x,\cdot)$. □

Definition 3.3 *Let H be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. A function $k : \Omega \times \Omega \to \mathbb{R}$ is called* **reproducing kernel** *for H if*

a) $k(x,\cdot) \in H$ *for all* $x \in \Omega$

b) $f(x) = \Big(f, k(x,.)\Big)_H$ *for all* $f \in H$ *and* $x \in \Omega$.

A Hilbert space with a reproducing kernel is called a reproducing kernel Hilbert space.

Theorem 3.4 *Let H be a (real) Hilbert function space over Ω with a reproducing kernel k. Then the kernel has the properties*

$$\begin{array}{lll} a. & k(x,y) & = (k(x,\cdot), k(y,\cdot))_H \text{ for all } x,y \in \Omega \\ b. & k(x,y) & = k(y,x) \text{ for all } x,y \in \Omega \\ c. & k(x,x) & = \|k(x,\cdot)\|_H^2 \geq 0 \text{ for all } x \in \Omega \\ d. & |k(x,y)| & \leq \sqrt{k(x,x)}\sqrt{k(y,y)} \text{ for all } x,y \in \Omega. \end{array}$$

Proof: The reproduction property (3.1) implies

$$(k(y,\cdot), k(x,\cdot))_H = k(y,x) \text{ for all } x,y \in \Omega,$$

and then all assertions follow from the properties of the real-valued inner product on H. In particular, the final assertion is the Cauchy-Schwarz inequality. □

Proposition 3.5 *If a reproducing kernel k of a Hilbert space H exists, then it is uniquely determined.*

Proof: Suppose that there exist two reproducing kernels Φ_1 and Φ_2. Then we have from the definition

$$\Big(f, \Phi_1(x,.) - \Phi_2(x,.)\Big)_H = 0 \text{ for all } f \in H,\ x \in \Omega.$$

If we set $f = \Phi_1(x,.) - \Phi_2(x,.)$ then we get $\|\Phi_1(x,.) - \Phi_2(x,.)\|_H^2 = 0$ for all $x \in \Omega$, hence $\Phi_1 = \Phi_2$. □

Theorem 3.6 *Let be H a Hilbert space of functions $f : \Omega \to \mathbb{R}$. Then the following statements are equivalent:*
a)- *The point evaluation functionals are continuous, i.e. in H^*.*
b)- *H has a reproducing kernel k.*

Proof: **a)-** to **b)-** : This is Theorem 3.2.
b)- to **a)-** : From $\delta_x(f) = \Big(f, k(x,\cdot)\Big)_H$ and the continuity of the inner product in H it immediately follows that δ_x is continuous. □

Theorem 3.7 *Let H be a (real) Hilbert function space over Ω with continuous point evaluations. Then the unique reproducing kernel k of H is a positive semi-definite function in the sense of Definition 2.2. If, in addition, all selections of finitely many point-evaluation functionals in H^* are linearly independent over H, then k is positive definite.*

Proof: We select $N \in \mathbb{N}$ and N different points $x_1, \dots, x_N$ in Ω. Let $\alpha_1, \dots, \alpha_N$ be arbitrary real numbers. Then we have

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j k(x_i, x_j) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i\alpha_j (k(x_i,\cdot), k(x_j,\cdot))_H \\
&= \left(\sum_{i=1}^{N} \alpha_i k(x_i,\cdot), \sum_{j=1}^{N} \alpha_j k(x_j,\cdot)\right)_H \\
&= \|\sum_{i=1}^{N} \alpha_i k(x_i,\cdot)\|_H^2 \\
&\geq 0 \ .
\end{aligned}$$

Now suppose that k is not strictly positive definite. Then there exist N distinct points $x_1, \ldots, x_N$ in Ω and a nonzero vector α in $\mathbb{R}^N$ such that

$$\sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j k(x_i, x_j) = 0 = \| \sum_{i=1}^N \alpha_i k(x_i, \cdot)\|_H^2.$$

Thus we have $\sum_{i=1}^N \alpha_i k(x_i, \cdot) = 0$ as a function in H. From the reproduction property, we get

$$\begin{aligned} 0 &= \left(f, \sum_{i=1}^N \alpha_i k(x_i, \cdot)\right)_H \\ &= \sum_{i=1}^N \alpha_i (f, k(x_i, \cdot))_H \\ &= \sum_{i=1}^N \alpha_i f(x_i) \\ &= \sum_{i=1}^N \alpha_i \delta_{x_i}(f) \end{aligned}$$

for all $f \in H$. This shows that the set of point-evaluation functionals $\delta_{x_1}, \ldots, \delta_{x_N}$ is linearly dependent. □

The following observation will be the starting point of the next section, because it allows to define an inner product, given a positive definite kernel.

Corollary 3.8 *Let H be a real Hilbert space of functions on some domain Ω, and let point evaluations be continuous and linearly independent, such that a positive definite reproducing kernel k exists uniquely. Then the coefficients α_j, β_k of functions of the form*

$$f(x) := \sum_{j=1}^M \alpha_j k(x, x_j),\ g(x) := \sum_{k=1}^N \beta_k k(x, y_k)$$

are uniquely determined, and an inner product can be defined by

$$(f, g) := \sum_{j=1}^M \sum_{k=1}^N \alpha_j \beta_k k(x_j, y_k).$$

This inner product coincides with $(f, g)_H$*, and thus it has a continuous extension to all of* H*.*

Proof: This follows easily by using property a) of Theorem 3.4 and evaluating the inner product like in the proof of Theorem 3.7. □

An early source of further information on reproducing kernel Hilbert spaces is Meschkowski [20].

3.2 Native spaces

The previous section has shown that each Hilbert space H of functions on some domain Ω, which is useful for interpolation, i.e. if it has continuous and linearly independent point evaluations, leads to a positive definite function k being the unique reproducing kernel of H.

Our goal now is to turn this upside down: For each positive definite function Φ on some domain Ω it turns out that there is a Hilbert space $\mathcal{N}_\Phi$, which will be called the **native space** for Φ, such that Φ is the reproducing kernel of H_Φ. Of course, we need as much information as possible about this space, because it is necessary to know the underlying Hilbert space in order to assess the optimality properties of the reconstruction process.

This one-to-one correspondence of positive definite functions and reproducing kernels of Hilbert spaces is nice and useful, but it omits some very important kernels, namely the ones which are *conditionally* positive definite of some *positive* order m, like the multiquadric and the thin-plate spline. Thus one needs an extension of the theory of reproducing kernel Hilbert spaces that allows a one-to-one correspondence to conditionally positive definite kernels via a generalized notion of the reproduction property. For more information about the association of a Hilbert space to each conditionally positive definite function we must go back to the analysis of Madych and Nelson [17, 18, 19]. The practical advantage of all of this is that all useful conditionally positive definite functions Φ, which were constructed without any relation to a Hilbert space, can be investigated thoroughly within their native space, once the latter is defined and characterized. The native space, in the conditionally positive definite case, turns out to be a Hilbert space plus a finite-dimensional space.

Thus this section begins with the construction of the native space associated to a (not necessary radial) basis function or kernel. Starting from the kernels themselves, the general construction of native spaces is carried out for both the unconditionally and the conditionally positive definite case. The first step is to define suitable pre-Hilbert spaces, the second step is their completion, and then we have the Hilbert spaces we want, and which form part of the native space.

For unconditionally positive definite kernels, we could follow the argument from Corollary 3.8 to define an inner product which can be extended to a full Hilbert space by completion. But we want to handle the conditional positive definite case, which is not covered by the previous chapter.

We start with a slight generalization of Definition 2.3, replacing $\mathbb{P}_m^d$ with a general subspace P:

Definition 3.9 *Let Ω be a subset of $\mathbb{R}^d$, and let P be a finite-dimensional subspace of continuous real-valued functions on Ω. A function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called* **conditionally positive semi-definite** *on Ω with respect to P, if for any finite set of points $X = \{x_1, \ldots, x_N\}$ in Ω and for all $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ in $\mathbb{R}^N$ satisfying the additional condition*

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0 \text{ for all } p \in P \tag{3.2}$$

the inequality (2.2) holds. A function Φ will be called **conditionally positive definite** *with respect to P, if the quadratic form in (2.2), defined on the subspace of vectors α satisfying the above moment condition, is positive unless α is zero.*

From here on, and until the end of the dissertation, let $\Phi : \Omega \times \Omega \to \mathbb{R}$ be a conditionally positive definite kernel with respect to P. We remark that the considerations in Chapter 1 remain true if we use the above generalization. In analogy to the previous chapters, we shall always assume interpolation sets $X = \{x_1, \ldots, x_N\}$ to contain a P-unisolvent subset, i.e. there is no nonzero function $p \in P$ which vanishes on all of X.

The key point now is to take (3.2) to define a space of functionals first.

Definition 3.10 *We define the space of all linear functionals $L_P(\Omega)$ with finite support in Ω that vanish on P,i.e:*

$$L_P(\Omega) := \{\lambda_{\alpha,X} \mid \lambda_{\alpha,X}(f) := \sum_{j=1}^{N} \alpha_j f(x_j),\ \lambda_{\alpha,X}(P) = \{0\}\}, \tag{3.3}$$

where arbitrary finite sets $X = \{x_1, \ldots, x_N\} \subseteq \Omega$ and coefficient vectors $\alpha = \{\alpha_1, \ldots, \alpha_N\} \in \mathbb{R}^N$ are allowed within the definition, provided that they satisfy (3.2).

A functional $\lambda_{\alpha,X} \in L_P(\Omega)$ is *supported* on the finite subset $X = \{x_1, \ldots, x_N\}$ of Ω, has the *coefficients* $\alpha \in \mathbb{R}^N$ and satisfies (3.2). Note that N, α and X can vary freely under this constraint.

Proposition 3.11 *Let $\Phi : \Omega \times \Omega \to \mathbb{R}$ be a conditionally positive definite kernel with respect to P. If we define the bilinear form*

$$(\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi := \lambda^x_{\alpha,X} \lambda^y_{\beta,Y} \Phi(x, y) = \sum_{j=1}^{N} \sum_{k=1}^{M} \alpha_j \beta_k \Phi(x_j, y_k) \tag{3.4}$$

on $L_P(\Omega)$ for arbitrary $\lambda_{\alpha,X}, \lambda_{\beta,Y}$ in $L_P(\Omega)$, then the space $L_P(\Omega)$ becomes a pre-Hilbert space equipped with the inner product $(.,.)_\Phi$.

Proof: By elementary calculations. □

We could go to a Hilbert space completion of $L_P(\Omega)$ immediately, but we postpone this step a little while.

The space $L_P(\Omega)$ is a fine pre-Hilbert space, but it contains functionals, not functions. We want a pre-Hilbert space of functions instead. An easy way to go from functionals to functions is to define

$$f_{\lambda_{\alpha,X}}(\cdot) := \lambda^z_{\alpha,X} \Phi(\cdot, z) = \sum_{j=1}^{N} \alpha_j \Phi(\cdot, x_j) \tag{3.5}$$

for arbitrary functionals $\lambda_{\alpha,X} \in L_P(\Omega)$. This can be written via a linear map

$$\begin{array}{rcl} S_\Phi & : & L_P(\Omega) \to S_\Phi(L_P(\Omega)) \\ \lambda_{\alpha,X} & \mapsto & f_{\lambda_{\alpha,X}}(\cdot) = \lambda^z_{\alpha,X} \Phi(\cdot, z) \end{array} \tag{3.6}$$

which yields functions on Ω which have point evaluations, but we do not know whether the point evaluations will turn out to be continuous in our future native space. In particular, point evaluations are generally not contained in $L_P(\Omega)$ unless P is the zero space, and thus we must be careful to apply point evaluation functionals to $f_{\lambda_{\alpha,X}}$. Instead, we should apply functionals of the form $\lambda_{\beta,Y} \in L_P(\Omega)$ to $f_{\lambda_{\alpha,X}}$ in order to get the fundamental identity

$$\lambda_{\beta,Y}(f_{\lambda_{\alpha,X}}) = \lambda_{\beta,Y}(S_\Phi(\lambda_{\alpha,X})) = (\lambda_{\beta,Y}, \lambda_{\alpha,X})_\Phi \text{ for all } \lambda_{\alpha,X},\ \lambda_{\beta,Y} \in L_P(\Omega). \tag{3.7}$$

Since point evaluations need not be continuous in case of conditionally positive kernels of positive order (or, more generally, for the case $P \neq \{0\}$), we need a workaround for point evaluations.

Assumption 3.12 *For our finite-dimensional space P of functions on Ω there is a fixed finite subset $Z = \{z_1, \dots, z_Q\}$ of Ω and a (Lagrange) basis $p_1, \dots, p_Q$ of P which satisfies*

$$p(\cdot) = \pi_P(p)(\cdot) = \sum_{j=1}^{Q} p_j(\cdot)p(z_j).$$

for all $p \in P$ and $p_i(z_j) = \delta_{i,j}$ for all $1 \le i, j \le Q =$ dim P.

Such a set $Z = \{z_1, \dots, z_Q\} \subseteq \Omega$ is often called *unisolvent for P*. We assume the above hypothesis from now on.

Definition 3.13 *On any space H of functions on Ω we define a projector $\pi_P : H \to P$ by*

$$\Big(\pi_P(f)\Big)(x) := \sum_{k=1}^{Q} p_k(x) f(z_k) \quad \text{for all } x \in \Omega,\ f \in H.$$

Then $f - \pi_P(f)$ always vanishes on Z.

Definition 3.14 *For each $x \in \Omega$ let $\delta_{(x)} \in L_P(\Omega)$ be the functional defined by*

$$\begin{aligned} \delta_{(x)}(f) &:= \Big(\delta_x - \textstyle\sum_{k=1}^{Q} p_k(x)\delta_{z_k}\Big)(f) \\ &= f(x) - \pi_P(f)(x) \text{ for all } f \in H,\ x \in \Omega. \end{aligned}$$

This is a very useful variation of the standard point evaluation δ_x at x which is not in $L_P(\Omega)$ while $\delta_{(x)}$ is. In addition, we have

$$\delta_{(z_k)}(f) = 0 \text{ for all } z_k \in Z,\ f \in H \tag{3.8}$$

because $p_k(z_j) = \delta_{j,k},\ 1 \leq j,k \leq Q$.

Lemma 3.15 **a)-** *For all functions f on Ω we have*

$$f \in P \Longleftrightarrow \delta_{(x)}(f) = 0 \text{ for all } x \in \Omega.$$

b)- $P \cap \{f_{\lambda_{\alpha,X}} \ :\ \lambda_{\alpha,X} \in L_P(\Omega)\} = \{0\}$.

Proof: **a)** This follows immediately from Assumption 3.12 and Definition 3.14.
b) Let $f_{\lambda_{\alpha,X}}$ be a function in $P \cap \{f_{\lambda_{\alpha,X}} \ :\ \lambda_{\alpha,X} \in L_P(\Omega)\}$. Then

$$\begin{aligned} f_{\lambda_{\alpha,X}} \in P \quad &\rightarrow \quad \lambda_{\alpha,X}(f_{\lambda_{\alpha,X}}) = \sum_{j=1}^{N} \alpha_j f_{\lambda_{\alpha,X}}(x_j) = 0 \\ &\rightarrow \quad \sum_{j=1}^{N}\sum_{j=1}^{N} \alpha_j \alpha_k \Phi(x_k, x_j) = 0 \end{aligned}$$

or since $\lambda_{\alpha,X} \in L_P(\Omega)$, we conclude $\lambda_{\alpha,X} = 0$. Hence, $f_{\lambda_{\alpha,X}} = 0$. □

Now we can define the map

$$\begin{aligned} R_{\Phi,\Omega} \ :\ L_P(\Omega) \quad &\rightarrow \quad R_{\Phi,\Omega}(L_P(\Omega)) \\ R_{\Phi,\Omega}(\lambda_{\alpha,X})(x) \quad &= \quad \Big(\lambda_{\alpha,X}, \delta_{(x)}\Big)_\Phi \text{ for all } x \in \Omega\ ,\lambda_{\alpha,X} \in L_P(\Omega) \end{aligned} \tag{3.9}$$

where now $R_{\Phi,\Omega}(L_P(\Omega))$ is a space of functions on Ω vanishing in the points $z_1, \ldots, z_Q$.

>From the equation (3.7) we get

Lemma 3.16 *The mapping $R_{\Phi,\Omega}$ can be expressed as*

$$R_{\Phi,\Omega}(\lambda_{\alpha,X})(x) = \delta_{(x)}\Big(f_{\lambda_{\alpha,X}}\Big).$$

Note that we distinguish between S_Φ of (3.6) and $R_{\Phi,\Omega}$. Both maps result in functions on Ω. But usually

$$R_{\Phi,\Omega}(\lambda_{\alpha,X})(x) = \delta_{(x)}(f_{\lambda_{\alpha,X}}) \neq \delta_x(f_{\lambda_{\alpha,X}}) = f_{\lambda_{\alpha,X}}(x) = S_\Phi(\lambda_{\alpha,X})(x) \quad (3.10)$$

unless $P = \{0\}$, $Q = 0$, though both sides are well-defined. The left-hand side of this non-equality will extend to the closure of $R_{\Phi,\Omega}(L_p(\Omega))$, while the right-hand side does not extend easily. The left-hand side always vanishes on the points $z_1, \ldots, z_Q$, while the right-hand side does not necessarily have this property. However, both sides differ only by a function in P.

Theorem 3.17 *For all $\lambda_{\alpha,X}, \lambda_{\beta,Y} \in L_P(\Omega)$ we have the fundamental property*

$$\lambda_{\beta,Y} R_{\Phi,\Omega}(\lambda_{\alpha,X}) = (\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi = \lambda_{\beta,Y}\Big(f_{\lambda_{\alpha,X}}\Big).$$

Proof: For all $\lambda_{\alpha,X}, \lambda_{\beta,Y} \in L_P(\Omega)$ we use Lemma 3.16 to get

$$\begin{aligned}
\lambda_{\beta,Y}\Big(R_{\Phi,\Omega}(\lambda_{\alpha,X})(.)\Big) &= \lambda_{\beta,Y}\Big(\delta_{(.)}(f_{\lambda_{\alpha,X}})\Big)\\
&= \lambda_{\beta,Y}\Big(f_{\lambda_{\alpha,X}}(.)\Big) - \lambda_{\beta,Y}\Big(\pi_P(f_{\lambda_{\alpha,X}})(.)\Big)\\
&= \lambda_{\beta,Y}\Big(\sum_{j=1}^N \alpha_j \Phi(., x_j)\Big) - \lambda_{\beta,Y}\Big(\pi_P(f_{\lambda_{\alpha,X}})(.)\Big)\\
&= \sum_{i=1}^M \sum_{j=1}^N \alpha_j \beta_i \Phi(y_i, x_j)\\
&= (\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi
\end{aligned}$$

because $\pi_P(f_{\lambda_{\alpha,X}}) \in P$ and $\lambda_{\beta,Y} \in L_P(\Omega)$. □

Note the similarity between (3.7) and Theorem 3.17.

Theorem 3.18 *The map $R_{\Phi,\Omega}$ is injective on $L_P(\Omega)$.*

Proof: Take $\lambda_{\alpha,X} = \lambda_{\beta,Y} \in L_P(\Omega)$ in Theorem 3.17 to get

$$\lambda_{\alpha,X}\Big(R_{\Phi,\Omega}(\lambda_{\alpha,X})(.)\Big) = \|\lambda_{\alpha,X}\|_\Phi^2 \geq 0.$$

This proves the assertion. □

Definition 3.19 *We can define an inner product on the space $R_{\Phi,\Omega}(L_P(\Omega))$ by*

$$\Big(R_{\Phi,\Omega}(\lambda_{\alpha,X}), R_{\Phi,\Omega}(\lambda_{\beta,Y})\Big)_\Phi := (\lambda_{\alpha,X}, \lambda_{\beta,Y})_\Phi \text{ for all } \lambda_{\alpha,X},\ \lambda_{\beta,Y} \in L_P(\Omega)$$

which turns the space $R_{\Phi,\Omega}(L_P(\Omega))$ into a pre-Hilbert space which is isometric to $L_P(\Omega)$ via $R_{\Phi,\Omega}$. In both spaces we use the same notion for the inner product.

In Theorems 3.17, 3.18 and in Definition 3.19 it is allowed to use $S_{\Phi,\Omega}$ instead of $R_{\Phi,\Omega}$. However, this replacement fails from now on, because now we want to go from pre-Hilbert spaces to Hilbert spaces by abstract completion.

Definition 3.20 *The completions of the two pre-Hilbert spaces $L_P(\Omega)$ and $R_{\Phi,\Omega}(L_P(\Omega))$ with inner products $(.,.)_\Phi$ are denoted by $\mathcal{L}_{\Phi,P}(\Omega)$ and $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$, respectively. The new inner products are also denoted by $(.,.)_\Phi$ for simplicity, and the continuous extension of the isometry $R_{\Phi,\Omega}$ to the closures will be denoted by $\mathcal{R}_{\Phi,\Omega}$.*

The continuous linear functionals $\delta_{(x)} \in L_P(\Omega)$ for all $x \in \Omega$ extend continuosly to $\mathcal{L}_{\Phi,P}(\Omega)$ with the same notation. Furthermore, Theorem 3.17 extends to the closures to yield

$$\alpha(\mathcal{R}_{\Phi,\Omega}(\beta)) = (\alpha, \beta)_\Phi \text{ for all } \alpha,\ \beta \in \mathcal{L}_{\Phi,P}(\Omega). \tag{3.11}$$

In particular, we can use this identity to interpret an abstract element $\mathcal{R}_{\Phi,\Omega}(\beta) \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ as a function via

$$\mathcal{R}_{\Phi,\Omega}(\beta)(x) := \delta_{(x)}(\mathcal{R}_{\Phi,\Omega}(\beta)) = (\delta_{(x)}, \beta)_\Phi \text{ for all } x \in \Omega,\ \beta \in \mathcal{L}_{\Phi,P}(\Omega). \tag{3.12}$$

In this sense, the abstract space $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ is a space of functions on Ω which vanish on Z. If such a function $g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ is written as $g = \mathcal{R}_{\Phi,\Omega}(\beta)$ with $\beta \in \mathcal{L}_{\Phi,P}(\Omega)$, then

$$g(x) = \delta_{(x)}(\beta) = (\delta_{(x)}, \beta)_\Phi = (\mathcal{R}_{\Phi,\Omega}(\delta_{(x)}), g)_\Phi \text{ for all } x \in \Omega. \tag{3.13}$$

Hence, after this technical interlude, we are able to define the native space of the conditionally positive definite kernel Φ as

Definition 3.21 *The native space to a symmetric kernel Φ which is conditionally positive definite on Ω with respect to P is defined by*

$$\mathcal{N}_\Phi(\Omega) := \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)) + P. \tag{3.14}$$

Lemma 3.22 *The sum $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)) + P$ of spaces of functions on Ω is direct.*

Proof: Any function $p \in P$ which is also in $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ must vanish on Z. Since Z is unisolvent for P, the function p must be zero. □

Lemma 3.23 *For all functions $f \in \mathcal{N}_\Phi(\Omega)$ we have*

$$f - \pi_P(f) \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)).$$

Proof: If we write $f = g + p$ with $p \in P$ and $g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$, we find

$$\pi_P(f) = \pi_P(g + p) = \pi_P(g) + \pi_P(p) = p = f - g,$$

and thus $f - \pi_P(f) = g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$. □

Definition 3.24 *The native space $\mathcal{N}_\Phi(\Omega)$ has a semi-inner product defined as*

$$(f, g)_{\mathcal{N}_\Phi(\Omega)} := (\mathcal{R}^{-1}_{\Phi,\Omega}(f - \pi_P(f)), \mathcal{R}^{-1}_{\Phi,\Omega}(g - \pi_P(g)))_\Phi \text{ for all } f, g \in \mathcal{N}_\Phi(\Omega),$$

$$(p, .)_{\mathcal{N}_\Phi(\Omega)} := 0 \quad \text{for all } p \in P.$$

Definition 3.24 and Lemma 3.23 allow us to ensure that

$$\|f\|_{\mathcal{N}_\Phi(\Omega)} = \|f - \pi_P(f)\|_\Phi \text{ for all } f \in \mathcal{N}_\Phi(\Omega).$$

Note that $\|f\|_\Phi$ is undefined for $f \in \mathcal{N}_\Phi(\Omega)$ if Φ is Conditionally positive definite kernel.

To derive further properties of the native space, we need to include a generalized notation of the reproduction equation (3.1). We do this first by sticking to the original kernel Φ, while the next section provides a new kernel which allows a somewhat nicer reproduction formula.

We use the isometry property of $\mathcal{R}_{\Phi,\Omega}$ to get

$$(\alpha, \delta_{(x)})_\Phi = \Big(\mathcal{R}_{\Phi,\Omega}(\alpha), \mathcal{R}_{\Phi,\Omega}(\delta_{(x)})\Big)_\Phi \text{ for all } \alpha \in \mathcal{L}_{\Phi,P}(\Omega),\ x \in \Omega. \quad (3.15)$$

With Lemma 3.23 we can take an arbitrary $f \in \mathcal{N}_\Phi(\Omega)$ and insert $\alpha := \mathcal{R}^{-1}_{\Phi,\Omega}(f - \pi_P(f))$ to find

$$\begin{aligned}(\mathcal{R}^{-1}_{\Phi,\Omega}(f - \pi_P(f)), \delta_{(x)})_\Phi &= \Big(\mathcal{R}_{\Phi,\Omega}(\mathcal{R}^{-1}_{\Phi,\Omega}(f - \pi_P(f))), \mathcal{R}_{\Phi,\Omega}(\delta_{(x)})\Big)_\Phi \\ &= \Big(f - \pi_P(f), \mathcal{R}_{\Phi,\Omega}(\delta_{(x)})\Big)_\Phi \\ &= (f - \pi_P(f))(x)\end{aligned} \quad (3.16)$$

for all $\alpha \in \mathcal{L}_{\Phi,P}(\Omega)$, $x \in \Omega$, using (3.13). This proves

Theorem 3.25 *Every function f in the native space $\mathcal{N}_\Phi(\Omega)$ of conditionally positive definite function Φ on a some domain Ω with respect to a finite-dimensional function space P has the representation*

$$\begin{aligned} f(x) &= (\pi_P(f))(x) + (f - \pi_P(f), \mathcal{R}_{\Phi,\Omega}(\delta_{(x)}))_\Phi \\ &= (\pi_P(f))(x) + (f, \mathcal{R}_{\Phi,\Omega}(\delta_{(x)}))_{\mathcal{N}_\Phi(\Omega)}. \end{aligned}$$

for all $x \in \Omega$.

This is very much like a Taylor formula. Furthermore, the above reproduction formula suggests to use the function $\mathcal{R}_{\Phi,\Omega}(\delta_{(x)})(y) = \delta_{(y)}\mathcal{R}_{\Phi,\Omega}(\delta_{(x)})$ as a kernel. We shall do this in the next section. Before that, we still have to look at the dual space of the native space:

Theorem 3.26 *The functionals which are continuous on $\mathcal{N}_\Phi(\Omega)$ are exactly those which vanish on P and coincide with functionals from $\mathcal{L}_{\Phi,P}(\Omega)$ on $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$. Thus the dual space $\mathcal{N}_\Phi(\Omega)^*$ of $\mathcal{N}_\Phi(\Omega)$ can be identified with $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$, defining the values of functionals from the latter space to be zero on P.*

Proof: Let λ be a continuous linear functional on $\mathcal{N}_\Phi(\Omega)$. Then there is a nonnegative constant C such that

$$|\lambda(\mathcal{R}_{\Phi,\Omega}(\beta) + p)| \leq C\|\mathcal{R}_{\Phi,\Omega}(\beta) + p\|_{\mathcal{N}_\Phi(\Omega)} = C\|\mathcal{R}_{\Phi,\Omega}(\beta)\|_\Phi$$

for all $p \in P$ and all $\beta \in \mathcal{L}_{\Phi,P}(\Omega)$. Thus λ vanishes on P and coincides with a continuous linear functional on $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$. □

We now generalize the reproduction formula to general functionals:

Theorem 3.27 *For all elements $\alpha \in \mathcal{N}_\Phi(\Omega)^* = \mathcal{L}_{\Phi,P}(\Omega)$ and all functions $f \in \mathcal{N}_\Phi(\Omega)$ we have*

$$\begin{array}{rcl} \alpha(f) & = & \alpha(f - \pi_P(f)) \\ & = & (\alpha, \mathcal{R}^{-1}_{\Phi,\Omega}(f - \pi_P(f)))_\Phi \\ & = & (\mathcal{R}_{\Phi,\Omega}(\alpha), f - \pi_P(f))_\Phi \\ & = & (\mathcal{R}_{\Phi,\Omega}(\alpha), f)_{\mathcal{N}_\Phi(\Omega)}. \end{array}$$

Proof: Use Theorem 3.26, Lemma 3.23, equation (3.11) and the isometry property of $\mathcal{R}_{\Phi,\Omega}$. □

We end with the general formulation of the standard orthogonality condition stating that an interpolant is always orthogonal to the error function:

Theorem 3.28 *Define the space*

$$S_{X,\Phi} := P + \{R_{\Phi,\Omega}(\lambda_{\alpha,X}) \ : \ \lambda_{\alpha,X} \in L_P(\Omega)\}$$

for a fixed finite set X satisfying the hypotheses of Proposition 2.6. By the second chapter, this is the space spanned by all interpolants to functions $f \in \mathcal{N}_\Phi(\Omega)$ by functions from P and translates of Φ. Then the sum

$$\mathcal{N}_\Phi(\Omega) = S_{X,\Phi} + \{f \in \mathcal{N}_\Phi(\Omega) \ : \ f(X) = \{0\}\}$$

is orthogonal with respect to $(.,.)_{\mathcal{N}_\Phi(\Omega)}$. In particular, if $s_{f,X,\Phi}$ interpolates f on X using Φ, then

$$\begin{array}{ccc} f - s_{f,X,\Phi} & \perp_{\mathcal{N}_\Phi(\Omega)} & S_{X,\Phi} \\ \|f\|^2_{\mathcal{N}_\Phi(\Omega)} & = & \|f - s_{f,X,\Phi}\|^2_{\mathcal{N}_\Phi(\Omega)} + \|s_{f,X,\Phi}\|^2_{\mathcal{N}_\Phi(\Omega)}. \end{array} \tag{3.17}$$

Proof: Take an $f \in \mathcal{N}_\Phi(\Omega)$ with $f(X) = \{0\}$ and an $s = p + R_{\Phi,\Omega}(\lambda_{\alpha,X}) \in S_{X,\Phi}$ with $\lambda_{\alpha,X} \in L_P(\Omega)$. Then

$$\begin{array}{rcl} (f,s)_{\mathcal{N}_\Phi(\Omega)} & = & (f - \pi_P(f), s - \pi_P(s))_\Phi \\ & = & (f - \pi_P(f), R_{\Phi,\Omega}(\lambda_{\alpha,X}))_\Phi \\ & = & \lambda_{\alpha,X}(f - \pi_P(f)) \\ & = & \lambda_{\alpha,X}(f) - \lambda_{\alpha,X}(\pi_P(f)) \\ & = & 0. \end{array}$$

□

Note that it is in general not valid to replace $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ by $(.,.)_\Phi$ here, not even for the term $\|f - s_{f,X,\Phi}\|$. The reason is that $f - s_{f,X,\Phi} = (Id - \pi_X)(f)$, with $s_{f,X,\Phi} = \pi_X$, is not necessarily of the form $g - \pi_P(g)$ with $g \in \mathcal{N}_\Phi(\Omega)$, and this is equivalent to the fact that $\pi_P(f - \pi_X(f))$ does not necessarily vanish, because the two projectors π_X and π_P do not necessarily commute. However, if $Z \subset X$, one has $\|f - s_{f,X,\Phi}\|_\Phi = \|f - s_{f,X,\Phi}\|_{\mathcal{N}_\Phi(\Omega)}$ because the above argument does not fail.

Another similar pitfall occurs when interpolating a function $f \in \mathcal{N}_\Phi(\Omega)$ with $f(Z) = \{0\}$, e.g. a function from $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$. Then the interpolant $s_{f,X}$ will not necessarily lie in $R_{\Phi,\Omega}(L_P(\Omega))$ or vanish on Z unless $Z \subseteq X$.

3.3 Normalized kernels

We keep all notations of the preceding section.

Definition 3.29 *We define a kernel function* $h : \Omega \times \Omega \to \mathbb{R}$ *by*

$$h(x,y) = \delta_{(y)} R_{\Phi,\Omega}(\delta_{(x)}) = (R_{\Phi,\Omega}(\delta_{(x)}))(y) \text{ for all } x, y \in \Omega.$$

This kernel will be called the **normalization** *of* Φ.

Note that we can take $R_{\Phi,\Omega}$ here and do not need the continuous extension $\mathcal{R}_{\Phi,\Omega}$, because the functionals $\delta_{(x)}$ and $\delta_{(y)}$ are in $L_P(\Omega)$.

Proposition 3.30 *From (3.13) and (3.17) we get another representation of* $h(.,.)$ *by*

$$h(x,y) = \Big(\delta_{(x)}, \delta_{(y)}\Big)_\Phi = (R_{\Phi,\Omega}(\delta_{(x)}), R_{\Phi,\Omega}(\delta_{(y)}))_{\mathcal{N}_\Phi} \text{ for all } x, y \in \Omega.$$

furthermore

$$\pi_P\Big(h(.,x)\Big) = 0 \quad \text{for all } x \in \Omega. \;\square$$

Proposition 3.31 *The function* $h(.,.)$ *satisfies the property*

$$h(x,y) = (h(x,\cdot), h(\cdot,y))_\Phi \text{ for all } x, y \in \Omega.$$

Proof: This combines Proposition 3.30 and Definition 3.29. □

Proposition 3.32 *Let Φ be a conditionally positive definite kernel on Ω. Under the preceding assumptions, the function $h(x,y) = (\delta_{(x)}, \delta_{(y)})_\Phi$ can be expressed as*

$$\begin{aligned} h(x,y) &= \Phi(x,y) - \sum_{j=1}^{Q} p_j(x)\Phi(z_j,y) - \sum_{k=1}^{Q} p_k(y)\Phi(x,z_k) \\ &\quad + \sum_{j=1}^{Q}\sum_{k=1}^{Q} p_j(x)p_k(y)\Phi(z_j,z_k) \\ &= \Big(Id - \pi_P\Big)^x \Big(Id - \pi_P\Big)^y \Phi(x,y) \end{aligned}$$

for all $x, y \in \Omega$.

Proof: We start with the form of h in Proposition 3.30 and get

$$\begin{aligned} h(x,y) &= \Big(\delta_{(x)}, \delta_{(y)}\Big)_\Phi \\ &= \Phi(x,y) - \sum_{j=1}^{Q} p_j(x)\Phi(z_j,y) - \sum_{k=1}^{Q} p_k(y)\Phi(x,z_k) \\ &\quad + \sum_{j=1}^{Q}\sum_{k=1}^{Q} p_j(x)p_k(y)\Phi(z_j,z_k) \end{aligned}$$

from (3.4) and Definition 3.14. □

Our reason for giving some of the details here is that we need to observe from (3.8) that for all $1 \le k \le Q$ we have $h(.,z_k) = 0$ because $p_1, \ldots, p_Q$ is a Lagrange basis for P with respect to the points $z_1, \ldots, z_Q$.

We now can rewrite Theorem 3.25 in a simpler form:

Theorem 3.33 *For all functions $f \in \mathcal{N}_\Phi(\Omega)$, and under the assumption 3.12, we have*

$$f(x) = (\pi_P(f))(x) + \Big(h(x,.), f\Big)_{\mathcal{N}_\Phi(\Omega)} \text{ for all } x \in \Omega. \ \square$$

Theorem 3.34 *The bilinear form $(.,.)_{\mathcal{N}_\Phi}$ defines an inner product on the Hilbert space*

$$\mathcal{M}_\Phi = \mathcal{N}_\Phi \cap \{f \in \mathcal{N}_\Phi(\Omega) : f(z_k) = 0, \ \ 1 \le k \le Q\}$$

which has $h(.,.)$ as reproducing kernel.

Proof: The space $\mathcal{M}_\Phi$ coincides with $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ due to Lemma 3.23. Thus it is a Hilbert space under $(.,.)_\Phi$ which is isometric via $\mathcal{R}_{\Phi,\Omega}$ to $\mathcal{L}_{\Phi,P}(\Omega)$. But for all $g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ we get from Theorem 3.25 and Lemma 3.23 that

$$g(x) = (g, h(x,\cdot))_\Phi = (g, h(x,\cdot))_{\mathcal{N}_\Phi(\Omega)}$$

holds for all $x \in \Omega$. The inner products $(.,.)_\Phi$ and $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ coincide on this Hilbert subspace of the native space for Φ. □

Since h is a reproducing kernel, it is a positive semidefinite function due to Theorem 3.7, if the point evaluation functionals are continuous. But we can prove more:

Proposition 3.35 *The function h is unconditionally positive definite on $\Omega \setminus Z$.*

Proof: Let $\lambda_{\alpha,X} = \{\alpha_1, \dots, \alpha_M\} \in L_P(\Omega)$ be a functional with the support $X = \{x_1, \dots, x_M\} \subseteq \Omega \setminus Z$. Then the functional defined as $\lambda_{\alpha,X}(Id - \pi_P) \in L_P(\Omega)$ is supported on Ω and vanishes on P. Hence if we look at the quadratic form of the function h we get for all x and y in Ω

$$\lambda^x_{\alpha,X}\lambda^y_{\alpha,X} h(x,y) = \Big(\lambda_{\alpha,X}(Id - \pi_P)\Big)^x \Big(\lambda_{\alpha,X}(Id - \pi_P)\Big)^y \Phi(x,y) \geq 0.$$

Thus h is positive semi-definite. We now assume $\lambda^x_{\alpha,X}\lambda^y_{\alpha,X} h(x,y) = 0$. Then $\lambda_{\alpha,X}(Id - \pi_P) = 0$ because Φ is conditionally positive definite. Thus, for all functions f on Ω we have

$$\begin{aligned} 0 = \lambda_{\alpha,X}(Id - \pi_P)(f) &= \sum_{j=1}^M \alpha_j \Big(f(x_j) - \sum_{k=1}^Q p_k(x_j) f(z_k) \Big) \\ &= \sum_{j=1}^M \alpha_j f(x_j) - \sum_{k=1}^Q f(z_k) \Big(\sum_{j=1}^M \alpha_j p_k(x_j) \Big). \end{aligned}$$

Then, we get automatically

$$\sum_{j=1}^M \alpha_j f(x_j) = \sum_{k=1}^Q f(z_k) \Big(\sum_{j=1}^M \alpha_j p_k(x_j) \Big).$$

And since the functional $\lambda_{\alpha,X}$ is supported on $X \in \Omega \backslash Z$, we have $\sum_{j=1}^{M} \alpha_j p_k(x_j) = 0$ for all $k,\ 0 \leq k \leq Q$. This implies $\lambda_{\alpha,X} \in L_P(\Omega)$. Hence,

$$\lambda_{\alpha,X}(Id - \pi_P) = \lambda_{\alpha,X} Id = 0 \text{ implies } \lambda_{\alpha,X} = 0.$$

□

Corollary 3.36 *Using Theorem 3.34 and Proposition 3.35, the space $\mathcal{M}_\Phi$ is the native space of the unconditionally positive kernel $h(.,.)$ on $\Omega \setminus Z$.* □

We add a result concerning continuity of functions from the native space:

Proposition 3.37 *Let $\Phi : \Omega \times \Omega \to \mathbb{R}$ be a continuous conditionally positive definite function on Ω with respect to a space P of continuous functions on Ω, and assume 3.12. Then all functions f in $R_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ are continuous on Ω.*

Proof: Since Φ and all functions in P are continuous, then applying (3.25) one can deduce

$$\begin{aligned}
|f(x) - f(y)| &= \Big| \pi_P f(x) + \Big(f - \pi_P(f), R_{\Phi,\Omega}\delta_{(x)}\Big)_\Phi | \\
&\quad - \pi_P(f)(y) - \Big(f - \pi_P(f), R_{\Phi,\Omega}\delta_{(y)}\Big)_\Phi \Big| \\
&\leq |\pi_P f(x) - \pi_P f(y)| + |\Big(f - \pi_P(f), R_{\Phi,\Omega}\delta_{(x)} - R_{\Phi,\Omega}\delta_{(y)}\Big)_\Phi | \\
&\leq \sum_{k=1}^{Q} |p_k(x) - p_k(y)||f(z_k)| \\
&\quad + \|f - \pi_P(f)\|_\Phi \|R_{\Phi,\Omega}\delta_{(x)} - R_{\Phi,\Omega}\delta_{(y)})\|_\Phi.
\end{aligned}$$

If the final expression is continuous in x and y, then f is continuous on Ω. But we have

$$\begin{aligned}
\|R_{\Phi,\Omega}\delta_{(x)} - R_{\Phi,\Omega}\delta_{(y)})\|_\Phi^2 &= \|R_{\Phi,\Omega}\delta_{(x)}\|_\Phi^2 + \|R_{\Phi,\Omega}\delta_{(y)})\|_\Phi^2 - 2h(x,y) \\
&= h(x,x) + h(y,y) - 2h(x,y)
\end{aligned}$$

and this is continuous due to Proposition 3.32. □

3.4 Extended kernels

In the previous section, we started from a conditionally positive definite kernel Φ on Ω with respect to P and we constructed a native space $\mathcal{N}_\Phi(\Omega)$ for Φ on which the normalized kernel function $h(.,.)$ is a generalized reproducing kernel in the sense of Theorem 3.25. The native space for Φ, however, was not a Hilbert space, because it carried only a semi-inner product. The new kernel had a native Hilbert space, but on $\Omega \setminus Z$, where Z was a unisolvent set for P. This calls for a new kernel Φ_P, now unconditionally positive definite on all of Ω, such that the native space $\mathcal{N}_\Phi(\Omega)$ of Φ coincides as a vector space with the native space of Φ_P, carrying now an inner product that is closely related to the previous semi-inner product $(.,.)_\Phi$.

Definition 3.38 *Under the assumptions made so far, we pick a fixed unisolvent set $Z = \{z_1, \dots, z_Q\}$ for P and a Lagrange basis $p_1, \dots, p_Q$ of P. Then we define*

$$\begin{aligned} \Big(f, g\Big)_P &:= \sum_{k=1}^{Q} f(z_k) g(z_k) \\ \Phi_P(x, y) &:= h(x, y) + \sum_{l=1}^{Q} p_l(x) p_l(y) \end{aligned}$$

for all functions $f, g \in \mathcal{N}_\Phi(\Omega)$ and all $x, y \in \Omega$.

At this point, one is tempted to use Φ instead of h in the above definition of the kernel, in order to avoid the point set Z to enter into the kernel. However, it turns out that one has difficulties proving positive definiteness in that case.

Theorem 3.39 *The native space $\mathcal{N}_\Phi(\Omega)$ to a conditionally positive definite kernel Φ on Ω with respect to P carries the inner product*

$$\begin{aligned} (f, g)_{\Phi_P} &:= (f, g)_{\mathcal{N}_\Phi(\Omega)} + \Big(f, g\Big)_P \\ &= (f - \pi_P(f), g - \pi_P(g))_\Phi + \Big(\pi_P(f), \pi_P(g)\Big)_P \end{aligned}$$

for all $g, f \in \mathcal{N}_\Phi(\Omega)$. Under this inner product, the decomposition of the native space $\mathcal{N}_\Phi(\Omega)$ of Φ in (3.14) is an orthogonal decomposition. The bilinear forms $(.,.)_\Phi$, $(.,.)_{\Phi_P}$, and $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ coincide on $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$.

Proof:
The bilinear form $(.,.)_{\Phi_P}$ is positive definite because for an $f \in \mathcal{N}_\Phi(\Omega)$ with $(f,f)_{\Phi_P} = 0$ we have

$$0 = (f,f)_{\Phi_P} = (f,f)_{\mathcal{N}_\Phi(\Omega)} + \sum_{k=1}^{Q} |f(z_k)|^2.$$

But then $(f,f)_{\mathcal{N}_\Phi(\Omega)} = 0$ and $f \in P$, and $f(z_k) = 0$ hence $f = 0$. Thus $(.,.)_{\Phi_P}$ is positive definite. The rest is simple. □

Theorem 3.40 *The native space $\mathcal{N}_\Phi(\Omega)$ to a conditionally positive definite kernel Φ with respect to a finite-dimensional subspace P is a Hilbert space with the extended kernel Φ_P from Definition 3.38 as reproducing kernel, if the inner product $(.,.)_{\Phi_P}$ from Theorem 3.39 is used.*

Proof: For proving the reproducing kernel property, we have

$$\begin{aligned}
(f, \Phi_P(x,.))_{\Phi_P} &= (f, \Phi_P(x,.))_{\mathcal{N}_\Phi(\Omega)} + (f, \Phi_P(x,.))_P \\
&= (f, h(x,.))_{\mathcal{N}_\Phi(\Omega)} + 0 + \sum_{k=1}^{Q} f(z_k) h(x, z_k) \\
&\quad + \sum_{k=1}^{Q} f(z_k) \sum_{l=1}^{Q} p_l(x) p_l(z_k) \\
&= f(x) - \pi_P(f)(x) + 0 + \sum_{k=1}^{Q} f(z_k) p_k(x) \\
&= f(x).
\end{aligned}$$

From the reproduction property and Theorem 3.6 we then know that Φ_P is positive semidefinite. To prove positive definiteness, we take a set of points $Y = \{y_1, \ldots, y_M\} \in \Omega \setminus Z$ and vectors $\beta = (\beta_1, \ldots, \beta_M) \in \mathbb{R}^M$ and $\gamma = (\gamma_1, \ldots, \gamma_Q) \in \mathbb{R}^Q$. Then we look at the quadratic form

$$\begin{aligned}
&\sum_{j,k=1}^{M} \beta_j \beta_k \Phi_P(y_j, y_k) + 2 \sum_{j=1}^{M} \sum_{k=1}^{Q} \beta_j \gamma_k \Phi_P(y_j, z_k) + \sum_{j,k=1}^{Q} \gamma_j \gamma_k \Phi_P(z_j, z_k) \\
= &\sum_{j,k=1}^{M} \beta_j \beta_k h(y_j, y_k) + 0 + 0 +
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{l=1}^{Q} \sum_{j,k=1}^{M} \beta_j \beta_k p_l(y_j) p_l(y_k) + 2 \sum_{l=1}^{Q} \sum_{j=1}^{M} \sum_{k=1}^{Q} \beta_j \gamma_k p_l(y_j) p_l(z_k) \\
& + \sum_{l=1}^{Q} \sum_{j,k=1}^{Q} \gamma_j \gamma_k p_l(z_j) p_l(z_k) \\
= & \sum_{j,k=1}^{M} \beta_j \beta_k h(y_j, y_k) + \\
& + \sum_{l=1}^{Q} \left(\sum_{j=1}^{M} \beta_j p_l(y_j) + \sum_{k=1}^{Q} \gamma_k p_l(z_k) \right)^2 \\
= & \sum_{j,k=1}^{M} \beta_j \beta_k h(y_j, y_k) + \sum_{l=1}^{Q} \left(\gamma_l + \sum_{j=1}^{M} \beta_j p_l(y_j) \right)^2
\end{aligned}
$$

and this is nonnegative, since h is positive definite on $\Omega \setminus Z$ due to Theorem 3.35. If the quadratic form is zero, then we use Theorem 3.35 again to conclude that β vanishes. Finally, also γ must vanish. □

Proposition 3.41 *The interpolation associated to Φ_P does in general* **not** *coincide with the interpolation associated to Φ.*

Proof: An interpolating function s_P to some function $f \in \mathcal{N}_\Phi(\Omega)$ in data points $x_1, \ldots, x_N$ and associated to Φ_P has the form

$$
s_P(x) = \sum_{j=1}^{N} \alpha_j \Phi_P(x, x_j)
$$

with no additional conditions on the coefficients. Then we rewrite the interpolant as

$$
\begin{aligned}
s_P(x) &= \sum_{j=1}^{N} \alpha_j \Phi_P(x, x_j) \\
&= \sum_{j=1}^{N} \alpha_j h(x, x_j) + \sum_{l=1}^{Q} p_l(x) \underbrace{\sum_{j=1}^{N} \alpha_j p_l(x_j)}_{=:\gamma_l}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) \\
&\quad - \sum_{l=1}^{Q} p_l(x) \sum_{j=1}^{N} \alpha_j \Phi(z_l, x_j) - \sum_{l=1}^{Q} \Phi(z_l, x) \sum_{j=1}^{N} \alpha_j p_l(x_j) \\
&\quad + \sum_{m,l=1}^{Q} p_l(x)\Phi(z_l, z_m) \sum_{j=1}^{N} \alpha_j p_m(x_j) + \sum_{l=1}^{Q} \gamma_l p_l(x) \\
&= \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) - \sum_{l=1}^{Q} \Phi(z_l, x)\gamma_l \\
&\quad + \sum_{l=1}^{Q} p_l(x) \left(\gamma_l - \sum_{j=1}^{N} \alpha_j \Phi(z_l, x_j) + \sum_{m=1}^{Q} \Phi(z_l, z_m)\gamma_m \right) \\
&= s_P(x) - \pi_P(s_P(x)) + \pi_P(s_P(x)).
\end{aligned}
$$

The part $s_P(x) - \pi_P(s_P(x))$ must necessarily be in $\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$, and its coefficients will necessarily satisfy the constraints

$$
\sum_{j=1}^{N} \alpha_j p(x_j) - \sum_{l=1}^{Q} \gamma_l p(z_l) = 0,
$$

which can also be read off the definition of the γ_l after putting in the basis of the functions p_m. It turns out that the interpolant uses functions $\Phi(z_l, \cdot)$ unless all $\gamma_l = 0$. But when inserting z_l into the definition of s_P, we see that $s_P(z_l) = \gamma_l$ holds. Thus it cannot be expected that all of the γ_l vanish, if the set Z is not contained in $X = \{x_1, \ldots, x_N\}$. □

Corollary 3.42 *In contrast to the previous theorem, the interpolation associated to Φ_P does coincide with the interpolation associated to Φ, if the data points include the point set Z.*

Proof: Under the above additional assumption, the proof argument of the preceding theorem shows that the interpolant with respect to Φ_P is of the same form as the one with respect to Φ. Since the latter is unique, the two interpolants coincide. □

Corollary 3.43 *If the data point set $X = \{x_1, \ldots, x_N\}$ for interpolation includes the point set Z, the interpolant $s_{f,X}$ to some function $f \in \mathcal{N}_\Phi(\Omega)$ on the data X with respect to Φ or Φ_P can be calculated as follows:*

1. *Calculate $\pi(f)$. This interpolates f on Z.*
2. *Interpolate $f - \pi(f)$ in $X \setminus Z$ using the kernels Φ_P or h with no functions from P added, and with no conditions on the coefficients. Call the resulting function s.*
3. *Then $s + \pi(f) = s_{f,X}$ is the solution.*

Proof: Clearly, the function s is an interpolant to $f - \pi(f)$ on $X \setminus Z$, and it vanishes on Z. Thus $s + \pi(f)$ interpolates f on all of X. It lies in the same interpolation subspace as $s_{f,X}$ since it uses nothing else than functions from P and from the span of the functions $\Phi(\cdot, x_j)$ with $x_j \in X$. The additional conditions on the coefficients must automatically be satisfied because everything lies in the native space. Thus $s + \pi(f)$ must coincide with $s_{f,X}$. □

Chapter 4

Power kernels

This chapter introduces another class of kernels. These are data–dependent, and in the next chapter they will allow to split larger interpolation problems into smaller ones. They originate from [23] and are closely related to what is called the *power function* in the literature. The latter associates to each point x and each fixed set of points $X = \{x_1, \dots, x_N\}$ of a domain Ω and each quasi-interpolation process $f \longmapsto s_{f,X}$ the norm $P_X(x)$ of the error functional $f \longmapsto f(x) - s_{f,X}(x)$ with respect to the native space $\mathcal{N}_\Phi(\Omega)$ equipped with the bilinear form $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ where Φ is a conditionally positive definite function. Note that the power function can be associated to almost every linear process of approximation or interpolation.

Throughout this chapter, we assume that the function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is conditionally positive definite on $\mathbb{R}^d$ with respect to some finite-dimensional space P of functions on $\mathbb{R}^d$.

Looking at the definition of the power function, for example in ([23, 24, 30]), we define a bilinear form which will be denoted by $K_X : \Omega \times \Omega \to \mathbb{R}$ generated from a given conditionally positive definite kernel Φ and which characterizes the power function. Several properties of this bilinear form will be presented, and in particular, it turns out to be a positive definite *"power"* kernel on $\Omega \setminus X$. The final chapter of this thesis will then apply the power kernel to solve large interpolation problems recursively.

4.1 Construction of power kernels

In this section, we replay the same analysis that we used in the preceding definition of the native space to create and define a new kernel which will be denoted as K_X for a set of points $X = \{x_1, \ldots, x_N\}$.

> **For the rest of this thesis, we assume $Z \subseteq X$ to avoid complications like in Theorem 3.41.**

The starting point is to define on $\Omega \subseteq \mathbb{R}^d$ the point evaluation functional δ_x for all x in Ω on a space H of functions on Ω as in Definition 3.1, by

$$\delta_x(f) = f(x), \qquad \text{for all } x \in \Omega.$$

Let $\Phi : \Omega \times \Omega \to \mathbb{R}$ be a conditionally positive definite kernel on Ω with respect to a finite-dimensional space P, as in Definition 3.9. For this kernel, we define the native space $\mathcal{N}_\Phi(\Omega)$ as in Definition 3.21, using the bilinear forms arising from (3.4) and Definition 3.19.

Definition 4.1 *Let $X = \{x_1, \ldots, x_N\}$ be a set of points in Ω and let U_X be the vector of Lagrange basis functions as defined in Proposition 2.9. Then, we define a projector π_X on any space H of functions on Ω by*

$$\pi_X(f)(x) := \sum_{j=1}^{N} u_j^X(x) f(x_j) =: s_{f,X}(x), \quad \textit{for all } x \in \Omega,\ f \in H.$$

For all sets of points $X = \{x_1, \ldots, x_N\} \subseteq \Omega$ and corresponding Lagrange-type basis $u_1^X, \ldots, u_N^X$ from Proposition 2.9, we define

Definition 4.2 *For each $x \in \Omega$ we define $\delta_{x,X} \in L_P(\Omega)$ as*

$$\delta_{x,X}(f) = f(x) - \pi_X(f)(x) = f(x) - s_{f,X}(x) \textit{ for all } f \in H.$$

As we already showed for the functionals $\delta_{(x)}$ from Definition 3.14, this is a very useful variation

$$\delta_{x,X}(f) = f(x) - \pi_X(f)(x) = \Big(\delta_x - \sum_{j=1}^{N} u_j^X(x)\delta_{x_j}\Big)(f).$$

of the standard point evaluation at x defined for all functions f on Ω. From (2.13) and (3.3) we conclude that $\delta_{x,X}$ is in $L_P(\Omega)$. Furthermore, we have

$$\delta_{x_j,X} = 0 \text{ for all } x_j \in X. \tag{4.1}$$

Now we can invoke the map of (3.9)to define a function K_X as

Definition 4.3 *We define the* **power kernel** $K_X(.,.)$ *of* Φ *with respect to* $X = \{x_1, \ldots, x_N\}$ *for all* $x, y \in \Omega$ *by*

$$K_X(x,y) = \delta_{y,X} R_{\Phi,\Omega}(\delta_{x,X}).$$

Note that $R_{\Phi,\Omega}(\delta_{x,X})$ is always a well-defined function on Ω, but the functions

$$\begin{array}{rcl}
(S_{\Phi,\Omega}(\delta_{x,X}))(y) & := & \delta^s_{x,X}\Phi(s,y) \\
& = & (Id - \pi_X)^s_x \Phi(x,y) \\
& = & \Phi(x,y) - \sum_{j=1}^N u_j(x)\Phi(x_j,y) \\
(R_{\Phi,\Omega}(\delta_{x,X}))(y) & := & \delta_{(y)}(R_{\Phi,\Omega}(\delta_{x,X})) \\
& = & (\delta_{(y)}, \delta_{x,X})_\Phi \\
& = & (Id - \pi_P)^s_y (Id - \pi_X)^t_x \Phi(s,t) \\
\delta_{y,X}(R_{\Phi,\Omega}(\delta_{x,X})) & := & (Id - \pi_X)^s_y (Id - \pi_X)^t_x \Phi(s,t)
\end{array}$$

will usually be different as functions of y. We introduce a notation for the first case above:

Definition 4.4 *We define the functions* $f_{x,X}$ *on* Ω *by*

$$\begin{array}{rcl}
f_{x,X}(y) & := & \delta^s_{x,X}\Phi(s,y) = (S_{\Phi,\Omega}(\delta_{x,X}))(y) \\
& = & \Phi(x,y) - \sum_{i=1}^N u_i^X(x)\Phi(x_i,y) \\
& = & \Phi(x,y) - U_X^T(x)\Phi_X(y)
\end{array}$$

for all $y \in \Omega$*, with* $\Phi_X = \{\Phi(x_1,.), \ldots, \Phi(x_N,.)\}$ *from (2.10) and* $U_X = \{u_1, \ldots, u_N\}$ *from Proposition 2.9.*

Later, we shall make frequent use of the fact that $R_{\Phi,\Omega}(\delta_{x,X})$ and $S_{\Phi,\Omega}(\delta_{x,X}) = f_{x,X}$ just differ by a function from P.

Proposition 4.5 *If the conditionally positive kernel* Φ *is continuous on* Ω*, and if all functions from* P *are continuous on* Ω*, then the functions* $f_{x,X}$ *are continuous on* Ω*. They always are in the native space* $\mathcal{N}_\Phi(\Omega)$ *because of* $\delta_{x,X} \in L_P(\Omega)$*.*

Proof: Since the u_j^X are continuous and Φ is continuous, then the functions $f_{x,X}$ are also continuous on Ω. $\square$

4.2 Properties of power kernels

Proposition 4.6 *There are other representations of $K_X(.,.)$ as*

$$\begin{aligned} K_X(x,y) &= \Big(\delta_{x,X}, \delta_{y,X}\Big)_\Phi \\ &= (R_{\Phi,\Omega}(\delta_{x,X}), R_{\Phi,\Omega}(\delta_{y,X}))_\Phi \\ &= (S_{\Phi,\Omega}(\delta_{x,X}), S_{\Phi,\Omega}(\delta_{y,X}))_\Phi \\ &= (f_{x,X}, f_{y,X})_\Phi \text{ for all } x, y \in \Omega. \end{aligned} \tag{4.2}$$

Proof: This follows from Theorem 3.17, Definition 4.3, from the isometry property of $R_{\Phi,\Omega}$, and from the fact that the difference of $R_{\Phi,\Omega}(\delta_{x,X})$ and $S_{\Phi,\Omega}(\delta_{x,X}) = f_{x,X}$ is a function from P. □

Furthermore, due to (4.1), we have

Corollary 4.7 *Since $u_i^X(x_j) = \delta_{i,j}$ for all $1 \le j \le N$, we have for all $x \in \Omega$*

$$K_X(x, x_j) = K_X(x_j, x) = 0.$$

Proposition 4.8 *Under the preceding assumptions, the function $K_X(x,y) = (\delta_{x,X}, \delta_{y,X})_\Phi$ can be expressed as*

$$\begin{aligned} K_X(x,y) &= \delta_{x,X}^r \delta_{y,X}^s \Phi(r,s) \\ &= \Phi(x,y) - \sum_{i=1}^N u_i^X(x)\Phi(x_i,y) - \sum_{j=1}^N u_j^X(y)\Phi(x,x_j) \\ &\quad + \sum_{i=1}^N \sum_{j=1}^N u_i^X(x) u_j^X(y) \Phi(x_i,x_j). \end{aligned}$$

Proof: This is just (4.2) inserted into (3.4). □

Definition 4.9 *For any pointwise continuous linear interpolation or approximation process in normed linear spaces, the* power function *is the pointwise norm of the error functional.*

Theorem 4.10 *For the interpolation process using a conditional positive definite function Φ on a set X of interpolation data, the power function P_X is given by*

$$P_X^2(x) = K_X(x,x) \text{ for all } x \in \Omega.$$

Proof: Note that the pointwise error functional is just $\delta_{x,X}$ due to Definition 4.2. We then evaluate the definition of the power function as

$$\begin{array}{rcl} P_X^2(x) & := & \|\delta_{x,X}\|_\Phi^2 \\ & = & K_X(x,x) \end{array}$$

using Proposition 4.6. □

The relation of the power function and the power kernel to the interpolation process can be based on the following representation of the interpolation error:

Proposition 4.11 *Under the above assumptions, the interpolation error to any $f \in \mathcal{N}_\Phi(\Omega)$ at any point $x \in \Omega$ can be written as*

$$\begin{array}{rcl} f(x) - s_{f,X}(x) & = & (R_{\Phi,\Omega}(\delta_{x,X}), f - \pi_P(f))_\Phi \\ & = & (S_{\Phi,\Omega}(\delta_{x,X}), f - \pi_P(f))_\Phi \\ & = & (S_{\Phi,\Omega}(\delta_{x,X}), f)_{\mathcal{N}_\Phi(\Omega)}. \\ & = & (f_{x,X}, f)_{\mathcal{N}_\Phi(\Omega)}. \end{array} \qquad (4.3)$$

Proof: We use Theorem 3.27 to get

$$\begin{array}{rcl} \delta_{x,X}(f) & = & f(x) - s_{f,X}(x) \\ & = & \delta_{x,X}(f - \pi_P(f)) \\ & = & (R_{\Phi,\Omega}(\delta_{x,X}), f - \pi_P(f))_\Phi \\ & = & (R_{\Phi,\Omega}(\delta_{x,X}), f)_{\mathcal{N}_\Phi(\Omega)} \end{array}$$

for all $f \in \mathcal{N}_\Phi(\Omega)$ and all $x \in \Omega$. The assertion then follows from the fact that $R_{\Phi,\Omega}(\delta_{x,X})$ and $S_{\Phi,\Omega}(\delta_{x,X}) = f_{x,X}$ just differ by a function from P. □

Note that in (4.3) we have to distinguish between $(.,.)_\Phi$ and $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ carefully, while in (4.2) we could get away with $(.,.)_\Phi$ alone.

Proposition 4.12 *For all $x, y \in \mathbb{R}^d$ the function $K_X(\cdot,\cdot)$ can be written in the form*

$$K_X(x,y) = f_{x,X}(y) - p^T(y)V_X(x).$$

$$i.e. \quad K_X(x,y) = \Phi_X(x,y) - U_X^T(x)\Phi_X(y) - p^T(y)V_X(x).$$

Here, we used the notation of (2.10).

Proof: From the system (2.10) we have

$$\begin{aligned} A_{X,\Phi}U_X(x) + P_X V_X(x) &= \Phi_X(x), \quad \text{for all } x \in \mathbb{R}^d \\ P_X^T U_X(x) &= p(x), \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

Then

$$\begin{aligned} U_X^T(y)A_{X,\Phi}U_X(x) + U_X^T(y)P_X V_X(x) &= U_X^T(y)\Phi_X(x) \\ U_X^T(y)P_X &= p^T(y) \end{aligned}$$

and

$$p^T(y)V_X(x) = U_X^T(y)\Phi_X(x) - U_X^T(y)A_{X,\Phi}U_X(x). \tag{4.4}$$

From Proposition 4.8 we conclude

$$\begin{aligned} K_X(x,y) &= \Phi(x,y) - \sum_{j=1}^N u_j^X(x)\Phi(y,x_j) - \sum_{k=1}^N u_k^X(y)\Phi(x,x_k) \\ &\quad + \sum_{j=1}^N \sum_{k=1}^N u_j^X(x)u_k^X(y)\Phi(x_j,x_k) \\ &= \Phi(x,y) - U_X^T(x)\Phi_X(y) - U_X^T(y)\Phi_X(x) \\ &\quad + U_X^T(y)A_{X,\Phi}U_X(x) \\ &= \Phi(x,y) - U_X^T(x)\Phi_X(y) - p^T(y)V_X(x) \\ &= f_{x,X}(y) - p^T(y)V_X(x), \qquad \text{for all } x, y \in \mathbb{R}^d. \end{aligned}$$

□

Corollary 4.13 *Under the above assumptions, the interpolation error to any $f \in \mathcal{N}_\Phi(\Omega)$ at any point $x \in \Omega$ can be written as*

$$f(x) - s_{f,X}(x) = (K_X(x,\cdot), f)_{\mathcal{N}_\Phi(\Omega)}. \tag{4.5}$$

Proof: Just combine the two preceding propositions. □

Proposition 4.14 *For all $x, y \in \mathbb{R}^d$ the function $K_X(\cdot,\cdot)$ can be written in the form :*

$$K_X(x,y) = f_{x,X}(y) - s_{f_{x,X},X}(y)$$

Proof: Let x and y be in $\mathbb{R}^d$. Then

$$\begin{aligned} K_X(x,y) &= \delta^t_{y,X}\delta^s_{x,X}\Phi(s,t) \\ &= \delta^t_{y,X} f_{x,X}(t) \\ &= f_{x,X}(y) - s_{f_{x,X},X}(y). \end{aligned}$$

□

From these two representations we can deduce the strange identity

$$s_{f_{x,X},X}(y) = p^T(y)V_X(x)$$

showing a polynomial in y times a function of x vanishing on all points of X.

Proposition 4.15 *The function K_X satisfies the property*

$$\Big(K_X(x,.), K_X(y,.)\Big)_{\mathcal{N}_\Phi(\Omega)} = K_X(x,y)$$

for all x, y in Ω.

Proof: From (4.3) for the function $K_X(.,y)$ we get

$$(K_X(.,y), f_{x,X})_{\mathcal{N}_\Phi(\Omega)} = K_X(x,y) - s_{K_X(.,y),X}(x).$$

Since $K_X(x_j,y) = 0$ for all $1 \le j \le N$, we then get

$$(K_X(.,y), f_{x,X})_{\mathcal{N}_\Phi(\Omega)} = K_X(x,y). \tag{4.6}$$

We now substitute the value of $f_{x,X}$ from Proposition 4.14 into (4.6). We get

$$\begin{aligned} K_X(x,y) &= \Big(K_X(.,y), f_{x,X}(.)\Big)_{\mathcal{N}_\Phi(\Omega)} \\ &= \Big(K_X(.,y), K_X(x,.) + s_{f_{x,X},X}(.)\Big)_{\mathcal{N}_\Phi(\Omega)} \\ &= \Big(K_X(.,y), K_X(x,.)\Big)_{\mathcal{N}_\Phi(\Omega)} + \Big(K_X(.,y), s_{f_{x,X},X}(.)\Big)_{\mathcal{N}_\Phi(\Omega)} \\ &= \Big(K_X(.,y), K_X(x,.)\Big)_{\mathcal{N}_\Phi(\Omega)} \end{aligned}$$

using that $s_{f_x,X,X}(.)$ is a function in P. □

Applying the Cauchy-Schwarz inequality to Proposition 4.15, we get

Proposition 4.16 *The power kernel K_X and the power function P_X satisfy the property*

$$|K_X(x,y)| \leq P_X(x)P_X(y) \text{ for all } x,y \in \Omega.$$

Proposition 4.17 *The kernel $K_X(\cdot,\cdot)$ is an unconditionally positive definite function on $\Omega \setminus X$.*

Proof: We take any discrete point set $Y = \{y_1,\ldots,y_M\} \in \Omega \setminus X$ and any vector $\beta = (\beta_1,\ldots,\beta_M)$ and we consider the quadratic form defined as

$$\begin{aligned}
\sum_{j=1}^{M}\sum_{k=1}^{M}\beta_j\beta_k K_X(y_j,y_k) &= \sum_{j=1}^{M}\sum_{k=1}^{M}\beta_j\beta_k(f_{y_j,X},f_{y_k,X})_\Phi \\
&= (\sum_{j=1}^{M}\beta_j f_{y_j,X},\sum_{k=1}^{M}\beta_k f_{y_k,X})_\Phi \\
&= \|\sum_{j=1}^{M}\beta_j f_{y_j,X}\|_\Phi^2 \\
&\geq 0.
\end{aligned}$$

This proves that $K_X(\cdot,\cdot)$ is a positive semi-definite function on Ω.
We now suppose $\sum_{j=1}^{M}\sum_{k=1}^{M}\beta_j\beta_k K_X(y_j,y_k) = 0$. Then

$$\begin{aligned}
0 &= \sum_{j=1}^{M}\sum_{k=1}^{M}\beta_j\beta_k K_X(y_j,y_k) \\
&= \|\sum_{j=1}^{M}\beta_j f_{y_j,X}\|_\Phi^2, \\
0 &= \sum_{j=1}^{M}\beta_j f_{y_j,X}(.) \\
&= \sum_{j=1}^{M}\beta_j\Phi(y_j,.) - \sum_{j=1}^{M}\beta_j\sum_{k=1}^{N}u_k^X(y_j)\Phi(x_k,.)
\end{aligned}$$

$$= \sum_{j=1}^{M} \beta_j \Phi(y_j, .) + \sum_{k=1}^{N} \gamma_k \Phi(x_k, .).$$

with $\gamma_k = -\sum_{j=1}^{M} \beta_j u_k^X(y_j)$ for all $1 \le k \le N$. We now look at the quadratic form with respect to Φ where the coefficients at y_j are β_j, and at x_k are γ_k for all $1 \le j \le M$ and $1 \le k \le N$. That means, we use the quadratic form of Φ on $Z = \{x_1, \ldots, x_N, y_1, \ldots, y_M\}$ with the coefficients $\alpha = \{\alpha_1, \ldots, \alpha_{N+M}\} = \{\gamma_1, \ldots, \gamma_N, \beta_1, \ldots, \beta_M\}$ to get

$$\sum_{j=1}^{M+N} \sum_{k=1}^{M+N} \alpha_j \alpha_k \Phi(z_j, z_k) = 0.$$

To check that these coefficients α_j are admissible in the sense of the equation (3.2), we evaluate them on a function $p \in P$ in order to get

$$\begin{aligned} \sum_{i=1}^{M+N} \alpha_i p(z_i) &= \sum_{j=1}^{M} \beta_j p(y_j) + \sum_{k=1}^{N} \gamma_k p(x_k) \\ &= \sum_{j=1}^{M} \beta_j p(y_j) - \sum_{k=1}^{N} \sum_{j=1}^{M} \beta_j u_k^X(y_j) p(x_k) \\ &= \sum_{j=1}^{M} \beta_j \Big(p(y_j) - \sum_{j=1}^{M} u_k^X(y_j) p(x_k) \Big) \\ &= 0. \end{aligned}$$

Then $\alpha \in L_P(\Omega)$, and we conclude that $\alpha = 0$ and $\beta = \gamma = 0$. □

4.3 Native spaces of power kernels

In the previous chapter we have shown that each Hilbert space setting of a recovery problem leads to a specific positive semidefinite function acting as a reproducing kernel. We also showed a partially converse result: each positive definite kernel is the reproducing kernel of a Hilbert space of functions. Since we have proven in Proposition 4.17 that the power kernel K_X is positive definite on $\Omega \setminus X$, we now can apply this result to get a native Hilbert space for K_X. However, we want to relate this space as closely to the native space of Φ as we can.

We know from Proposition 4.17 that K_X is a symmetric unconditionally positive definite kernel on the set $\Omega \setminus X$. Because we are in the unconditional situation, we can cut the argument from the previous chapter somewhat short and look directly at the space

$$L_{K_X}(\Omega \setminus X) = \text{ span } \{K_X(.,y) : y \in \Omega \setminus X\}$$

and equip it with a bilinear form

$$\Big(\sum_{j=1}^{M} \alpha_j K_X(.,y_j), \sum_{k=1}^{S} \beta_k K_X(.,z_k)\Big)_{K_X} := \sum_{j=1}^{M}\sum_{k=1}^{S} \alpha_j \beta_k K_X(y_j, z_k).$$

with $Y = \{y_1, \ldots, y_M\} \subset \Omega \setminus X,\ Z = \{z_1, \ldots, z_S\} \subset \Omega \setminus X$ and $\alpha = (\alpha_1, \ldots, \alpha_M),\ \beta = (\beta_1, \ldots, \beta_S)$.

The previous chapter now implies

Proposition 4.18 *Let $K_X : \Omega \setminus X \times \Omega \setminus X \to \mathbb{R}$ be the symmetric unconditionally positive definite power kernel. Then the bilinear form $(.,.)_{K_X}$ defines an inner product on $L_X(\Omega \setminus X)$. Furthermore, the space $L_X(\Omega \setminus X)$is a pre-Hilbert space with the reproducing kernel K_X.* □

The completion $\mathcal{L}_{K_X}(\Omega \setminus X)$ of this pre-Hilbert function space with respect to the norm $\|.\|_{K_X}$ is the first candidate for a Hilbert space with reproducing kernel K_X.

Definition 4.19 *The native space for the symmetric unconditionally positive definite function K_X on $\Omega \setminus X$ is given by*

$$\mathcal{N}_{K_X}(\Omega \setminus X) = \mathcal{L}_{K_X}(\Omega \setminus X) = clos_{\|.\|_{K_X}}\Big(L_{K_X}(\Omega \setminus X)\Big).$$

equipped with the inner product $(.,.)_{\mathcal{N}_{K_X}}$ by extension to the completion, and the reproducing property

$$f(y) = (f, K_X(y,.))_{K_X}.$$

for all $f \in \mathcal{N}_{K_X}(\Omega \setminus X),\ y \in \Omega \setminus X$.

Here, the reproduction equation directly shows how to interpret an abstract element from the closure as a function.

Note that the reproduction equation implies that all functions from the native space of K_X will formally vanish on X. Thus they can be extended from $\Omega\backslash X$ to functions on Ω, the reproduction equation being valid there. We shall use this extension in the rest of this thesis, without further mentioning.

Now we want to relate the two native spaces and the norms $\|.\|_\Phi$, $\|.\|_{\mathcal{N}_\Phi(\Omega)}$ and $\|.\|_{K_X}$. For that we start with a function g in $L_X(\Omega \setminus X)$ written as

$$g(x) = \sum_{l=1}^{M} \alpha_l K_X(x, y_l) \tag{4.7}$$

for all $x \in \Omega \setminus X$ with an arbitrary $\alpha \in \mathbb{R}^M$ and a set $Y = \{y_1, \ldots, y_M\} \subset \Omega \setminus X$. Then,

$$\begin{aligned}
g(x) &= \sum_{l=1}^{M} \alpha_l K_X(x, y_l) \\
&= \sum_{l=1}^{M} \alpha_l \Big[\Phi(x, y_l) - \sum_{j=1}^{N} u_j^X(x)\Phi(x_j, y_l) - \sum_{k=1}^{N} u_k^X(y_l)\Phi(x, x_k) \\
&\quad + \sum_{j=1}^{N}\sum_{k=1}^{N} u_j^X(x) u_k^X(y_l)\Phi(x_j, x_k) \Big] \\
&= \sum_{l=1}^{M} \alpha_l \Phi(x, y_l) - \sum_{j=1}^{N} u_j^X(x) \Big(\sum_{l=1}^{M} \alpha_l \Phi(x_j, y_l) \Big) \\
&\quad - \sum_{k=1}^{N} \Phi(x, x_k) \Big(\sum_{l=1}^{M} \alpha_l u_k^X(y_l) \Big) \\
&\quad + \sum_{j=1}^{N} u_j^X(x) \sum_{k=1}^{N} \Phi(x_j, x_k) \sum_{l=1}^{M} \alpha_l u_k^X(y_l).
\end{aligned}$$

We denote

$$\beta_k := \sum_{l=1}^{M} \alpha_l u_k^X(y_l) \tag{4.8}$$

for all $1 \le k \le N$ and we denote

$$g_X(x) := \sum_{k=1}^{N} \beta_k \Phi(x, x_k), \quad g_Y(x) := \sum_{l=1}^{M} \alpha_l \Phi(x, y_l). \tag{4.9}$$

Then :

$$\begin{aligned} g(x) &= g_Y(x) - g_X(x) - \textstyle\sum_{j=1}^N u_j^X(x) g_Y(x_j) + \sum_{j=1}^N u_j^X(x) g_X(x_j) \\ &= (g_Y - g_X)(x) - \textstyle\sum_{j=1}^N u_j^X(x)(g_Y - g_X)(x_j) \\ &= (g_Y - g_X)(x) - s_{g_Y - g_X, X}(x). \end{aligned} \tag{4.10}$$

This equation also proves that $g_Y - g_X$ lies in $\mathcal{N}_\Phi(\Omega)$, because the other two functions are in that space. We even have $g_Y - g_X \in R_{\Phi,\Omega}(L_P(\Omega))$. To prove this, we have to check the moment equations

$$\begin{aligned} & \sum_{r=1}^{M} \alpha_r p(y_r) - \sum_{k=1}^{N} \beta_k p(x_k) \\ = & \sum_{r=1}^{M} \alpha_r \sum_{i=1}^{N} u_i^X(y_r) p(x_i) - \sum_{k=1}^{N} \beta_k p(x_k) \\ = & \sum_{i=1}^{N} p(x_i) \sum_{r=1}^{M} \alpha_r u_i^X(y_r) - \sum_{k=1}^{N} \beta_k p(x_k) \\ = & \sum_{i=1}^{N} \beta_i p(x_i) - \sum_{k=1}^{N} \beta_k p(x_k) \\ = & \ 0. \end{aligned}$$

After this technical interlude we can calculate the value of the norm of $\|.\|_{K_X}$.

$$\begin{aligned} \|g\|_{K_X}^2 &= \sum_{r=1}^{M} \sum_{l=1}^{M} \alpha_r \alpha_l K_X(y_r, y_l) \\ &= \sum_{r=1}^{M} \alpha_r \Big(\sum_{l=1}^{M} \alpha_l K_X(y_r, y_l) \Big) \\ &= \sum_{r=1}^{M} \alpha_r \Big((g_Y - g_X)(y_r) - \sum_{j=1}^{N} u_j^X(y_r)(g_Y - g_X)(x_j) \Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=1}^{M} \alpha_r \Big(g_Y - g_X\Big)(y_r) - \sum_{j=1}^{N} \beta_j \Big(g_Y - g_X\Big)(x_j) \\
&= \sum_{r=1}^{M} \alpha_r g_Y(y_r) + \sum_{j=1}^{N} \beta_j g_X(x_j) \\
&\quad - \sum_{r=1}^{M} \alpha_r g_X(y_r) - \sum_{j=1}^{N} \beta_j g_Y(x_j).
\end{aligned}
$$

If we substitute the value of g_Y and g_X from (4.9), then we get

$$
\begin{aligned}
\|g\|_{K_X}^2 &= \sum_{r=1}^{M} \alpha_r g_Y(y_r) - \sum_{r=1}^{M} \alpha_r \sum_{k=1}^{N} \beta_k \Phi(x_k, y_r) \\
&\quad - \sum_{r=1}^{M} \alpha_r \sum_{j=1}^{N} \beta_j \Phi(x_j, y_r) + \sum_{j=1}^{N} \beta_j g_X(x_j) \\
&= \|g_Y - g_X\|_\Phi^2.
\end{aligned}
$$

The standard orthogonality property of interpolants from (3.17) then yields

$$
\begin{aligned}
\|g_Y - g_X\|_\Phi^2 &= \|g_Y - g_X\|_{\mathcal{N}_\Phi(\Omega)}^2 \\
&= \|g_Y - g_X - s_{g_Y - g_X, X}\|_{\mathcal{N}_\Phi(\Omega)}^2 + \|s_{g_Y - g_X, X}\|_{\mathcal{N}_\Phi(\Omega)}^2 \\
&= \|g\|_{\mathcal{N}_\Phi(\Omega)}^2 + \|s_{g_Y - g_X, X}\|_{\mathcal{N}_\Phi(\Omega)}^2.
\end{aligned}
$$

We summarize:

Proposition 4.20 *The norm $\|.\|_{K_X}$ of the native space $\mathcal{N}_{K_X}$ of K_X can be expressed via*

$$\|g\|_{K_X}^2 = \|g_Y - g_X\|_\Phi^2 = \|g\|_{\mathcal{N}_\Phi(\Omega)}^2 + \|s_{g_Y - g_X, X}\|_{\mathcal{N}_\Phi(\Omega)}^2 \quad \text{for all } g \in L_{K_X}(\Omega \setminus X) \tag{4.11}$$

with the functions g_Y and g_X as defined in (4.9), satisfying the identity

$$g = g_Y - g_X - s_{g_Y - g_X, X}.$$

Furthermore,

$$L_{K_X}(\Omega \setminus X) \subseteq (Id - \pi_X)(R_{\Phi,\Omega}(L_P(\Omega))). \ \square \tag{4.12}$$

The term $s_{g_Y - g_X, X}$ is unexpected in (4.11), and we should show that it does not vanish in general. We have

$$\begin{aligned} g_Y(x_j) - g_X(x_j) &= \sum_{l=1}^{M} \alpha_l \Phi(x_j, y_l) - \sum_{k=1}^{N} \beta_k \Phi(x_j, x_k) \\ &= \sum_{l=1}^{M} \alpha_l \Phi(x_j, y_l) - \sum_{k=1}^{N} \sum_{l=1}^{M} \alpha_l u_k^X(y_l) \Phi(x_j, x_k) \\ &= \sum_{l=1}^{M} \alpha_l \left(\Phi(x_j, y_l) - \sum_{k=1}^{N} u_k^X(y_l) \Phi(x_j, x_k) \right) \qquad (4.13) \\ &= \sum_{l=1}^{M} \alpha_l \sum_{i=1}^{Q} p_i(x_j) v_i(y_l) \\ &= \sum_{i=1}^{Q} p_i(x_j) \sum_{l=1}^{M} \alpha_l v_i(y_l) \end{aligned}$$

with notation as in (2.10). This shows that $s_{g_Y - g_X, X}$ coincides with a function from P, but it is not necessarily zero.

Our goal now is to prove the converse inclusion to (4.12). We take a function $f \in R_{\Phi,\Omega}(L_P(\Omega))$ and represent the interpolation error as

$$f - s_{f,X} = \sum_{r=1}^{M} \alpha_r \Phi(., y_r) - \sum_{l=1}^{N} \gamma_r \Phi(x_l, .)$$

with disjoint sets $X = \{x_1, \ldots, x_N\}$, $Y = \{y_1, \ldots, y_M\}$ and vectors $\alpha \in \mathbb{R}^M$ and $\gamma \in \mathbb{R}^N$ such that their composition is in $L_P(\Omega)$, i.e.

$$\sum_{r=1}^{M} \alpha_r p(., y_r) - \sum_{l=1}^{N} \gamma_r p(x_l, .) = 0 \text{ for all } p \in P.$$

From the definition of $f - s_{f,X}$, we have the vectors α and γ. Then we define g_Y and g_X from (4.9) with β from (4.8). Then a function g is well defined from (4.7), and it will finally also satisfy (4.10). Then we have

$$g_Y = g + g_X + s_{g_Y - g_X, X}$$

and

$$f - s_{f,X} = g + g_X + s_{g_Y - g_X, X} - \sum_{l=1}^{M} \gamma_r \Phi(x_l, .).$$

Now we denote

$$\tilde{s} := g_X + s_{g_Y - g_X, X} - \sum_{r=1}^{M} \gamma_r \Phi(x_r, .).$$

The function $\tilde{s}$ lies in the space S_X spanned by P and the functions $\Phi(\cdot, x_j)$ for $x_j \in X$. We want to prove that its coefficients are in $L_P(\Omega)$. For this, we can focus on the function $g_X - \sum_{r=1}^{M} \gamma_r \Phi(x_r, .)$ and its coefficients because everything is correct for $s_{g_Y - g_X, X}$. We get

$$\begin{aligned}
& \sum_{k=1}^{N} \beta_k p(x_k) - \sum_{r=1}^{M} \gamma_r p(x_r) \\
= & \sum_{k=1}^{N} \sum_{l=1}^{M} \alpha_l u_k^X(y_l) p(x_k) - \sum_{r=1}^{M} \gamma_r p(x_r) \\
= & \sum_{l=1}^{M} \alpha_l \sum_{k=1}^{N} u_k^X(y_l) p(x_k) - \sum_{r=1}^{M} \gamma_r p(x_r) \\
= & \sum_{l=1}^{M} \alpha_l p(y_l) - \sum_{r=1}^{M} \gamma_r p(x_r) \\
= & \ 0.
\end{aligned}$$

Furthermore, for all $1 \leq j \leq N$ we have

$$0 = (f - s_{f,X})(x_j) = g(x_j) + \tilde{s}(x_j) = \tilde{s}(x_j).$$

because $g(x_j) = \sum_{r=1}^{M} \alpha_r K_X(x_j, y_r) = 0$. Thus $\tilde{s}$ interpolates zero data and must vanish. Then $f - s_{f,X} = g$ and $f - s_{f,X} \in L_X(\Omega \setminus X)$ hold. Thus we have proven

$$L_{K_X}(\Omega \setminus X) = (Id - \pi_X)\left(R_{\Phi,\Omega}(L_P(\Omega))\right). \tag{4.14}$$

Theorem 4.21 *The native space for the symmetric unconditionally positive definite power kernel K_X on $\Omega \setminus X$ is exactly the Hilbert space*

$$\mathcal{N}_{K_X}(\Omega \setminus X) = (Id - \pi_X)(\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))) = (Id - \pi_X)(Id - \pi_P)(\mathcal{N}_\Phi(\Omega)).$$

Proof: In principle, we want to take the closure of the identity (4.14), but we have to be careful with the different inner products.

If all functions of $L_{K_X}(\Omega \setminus X)$ are extended to Ω, then it is clear that the space $L_{K_X}(\Omega \setminus X)$ is in $(Id - \pi_X)(\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)))$. Due to (4.11) we have

$$\|g\|^2_{K_X} \geq \|g\|^2_{\mathcal{N}_\Phi(\Omega)} \quad \text{for all } g \in L_X(\Omega \setminus X).$$

Then we have, starting with the definition of the closure,

$$\begin{aligned}
\mathcal{N}_{K_X}(\Omega \setminus X) &= clos_{\|.\|_{K_X}} \Big(L_{K_X}(\Omega \setminus X) \Big) \\
&\subseteq clos_{\|.\|_{\mathcal{N}_\Phi(\Omega)}} \Big(L_{K_X}(\Omega \setminus X) \Big) \\
&= clos_{\|.\|_{\mathcal{N}_\Phi(\Omega)}} ((Id - \pi_X)(R_{\Phi,\Omega}(L_P(\Omega)))) \\
&= (Id - \pi_X)(\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)))
\end{aligned}$$

without trouble, since $Id - \pi_X$ is an orthogonal projector. Thus

$$\mathcal{N}_{K_X}(\Omega \setminus X) \subseteq (Id - \pi_X)(\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))).$$

Now we do the converse. Assume that there is an abstract function $f \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)) \subset \mathcal{N}_\Phi(\Omega)$ such that $f - s_{f,X}$ is orthogonal to $L_{K_X}(\Omega \setminus X)$ with respect to the inner product $(.,.)_{\mathcal{N}_\Phi(\Omega)}$. We take a function g from $L_{K_X}(\Omega \setminus X)$ and write it as in (4.7). This implies

$$\begin{aligned}
0 &= (f - s_{f,X}, g)_{\mathcal{N}_\Phi(\Omega)} \\
&= (f - s_{f,X}, \sum_{l=1}^{M} \alpha_l K_X(\cdot, y_l))_{\mathcal{N}_\Phi(\Omega)} \\
&= \sum_{l=1}^{M} \alpha_l (f - s_{f,X}, K_X(\cdot, y_l))_{\mathcal{N}_\Phi(\Omega)} \\
&= \sum_{l=1}^{M} \alpha_l (f - s_{f,X}, f_{y_l,X})_{\mathcal{N}_\Phi(\Omega)} \\
&= \sum_{l=1}^{M} \alpha_l ((f - s_{f,X})(y_l) - s_{f - s_{f,X},X}(y_l)) \\
&= \sum_{l=1}^{M} \alpha_l (f - s_{f,X})(y_l).
\end{aligned}$$

Since the points y_l are arbitrary from $\Omega \setminus X$ and the coefficients are arbitrary, we conclude that $f - s_{f,X} = 0$ on all of Ω. Thus the closure of $L_{K_X}(\Omega \setminus X)$ in $(Id - \pi_X)(\mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega)))$ with respect to $(.,.)_{\mathcal{N}_\Phi(\Omega)}$ is the full space. □

Corollary 4.22 *The native space $\mathcal{N}_{K_X}(\Omega \setminus X)$ for K_X can be written as*

$$\mathcal{N}_{K_X}(\Omega \setminus X) = \{f \in \mathcal{N}_\Phi(\Omega) \ : \ f(X) = \{0\}\}.$$

Proof: We have to show that any f from $\mathcal{N}_\Phi(\Omega)$ with $f(X) = \{0\}$ is of the form $f = g - \pi_X(g)$ with some $g \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$. To this end, we define $g := f - \pi_P(f) \in \mathcal{R}_{\Phi,\Omega}(\mathcal{L}_{\Phi,P}(\Omega))$ and look at

$$\begin{aligned} g - \pi_X(g) &= f - \pi_P(f) - \pi_X(f - \pi_P(f)) \\ &= f - \pi_P(f) - \pi_X(f) + \pi_X(\pi_P(f)) \\ &= f - \pi_P(f) - 0 + \pi_P(f) \\ &= f. \end{aligned}$$

□

Corollary 4.23 *If we define $S_X := \pi_X(\mathcal{N}_\Phi(\Omega))$ similarly to (2.34) and Theorem 3.28, the decomposition*

$$\mathcal{N}_\Phi(\Omega) = S_X + \mathcal{N}_{K_X}(\Omega \setminus X)$$

is orthogonal with respect to $(.,.)_{\mathcal{N}_\Phi(\Omega)}$.

Proof: See Theorem 3.28 and use Corollary 4.22. □

Proposition 4.24 *If Φ is an positive definite kernel, we have*

$$\|g\|_{K_X}^2 = \|g\|_\Phi^2 \textit{ for all } g \in \mathcal{N}_{K_X}(\Omega \setminus X).$$

Proof: We use the argument of (4.13) to see that $s_{g_Y - g_X, X}$ is in P. Hence its (semi-)norm is zero by definition. □

Note again that the proof shows that the assertion cannot be extended in general to the conditionally positive definite case, because $s_{g_Y - g_X, X}$ does not necessarily vanish.

We add a representation for the power function which is well-known in the literature:

Proposition 4.25 *The power function value $P_X(x)$ is given by*

$$P_X(x) = \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1, f(X) = \{0\}\} \text{ for all } x \in \Omega.$$

Recall that we always assume X to contain the P-unisolvent subset Z.

Proof: Since Proposition 4.11 yields the error bound

$$|f(x) - s_{f,X}(x)| \le P_X(x)\|f\|_{\mathcal{N}_\Phi(\Omega)}$$

for all $f \in \mathcal{N}_\Phi(\Omega)$, the *sup* exists. Then we define the function Q_X on Ω as

$$Q_X(x) = \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1, f(X) = \{0\}\} \ge 0.$$

Then, from

$$|f(x)| \le P_X(x)\|f\|_{\mathcal{N}_\Phi(\Omega)} + |\sum_{i=1}^{N} |u_j^X(x)||f(x_j)|$$

we get

$$Q_X(x) \le P_X(x).$$

If $P_X(x) = 0$, then $Q_X(x) = 0 = P_X(x)$. If $P_X(x) \ne 0$, then we define a function g_x on Ω as $g_x(y) = K_X(x,y)/P_X(x)$ such that $g_x \in R_{\Phi,\Omega}(L_P(\Omega))$. Then we get $g_x(x) = P_X(x)$ and its norm is bounded by

$$\begin{aligned} \|g_x\|^2_{\mathcal{N}_\Phi(\Omega)} &= \|K_X(x,.)\|^2_\Phi / P_X^2(x) \\ &= K_X(x,x)/P_X^2(x) \\ &= 1, \end{aligned}$$

where

$$g_x(x) = P_X(x) \le \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(X) = \{0\}\} = Q_X(x).$$

hence

$$P_X(x) \le Q_X(x).$$

Thus $P_X(x) = Q_X(x)$. □

We end this section by pointing out a few connections to the previous chapter.

Theorem 4.26 *In case of a conditionally positive definite kernel* Φ *with respect to some finite-dimensional space* P*, the power kernel* P_Z *is precisely the kernel* h *arising in the previous section.*

Proof: The Lagrange basis functions $p_1, \ldots, p_Q$ for P with respect to Z are exactly the Lagrange basis functions $u_1, \ldots, u_Q$ for interpolation on $X = Z$, and the interpolation is purely polynomial. Thus the kernels h and P_Z coincide, as follows from their representations in Proposition 3.32 and Theorem 4.8. □

Theorem 4.27 *The power kernel* $K_X^{\Phi_P}$ *associated to the extended kernel* Φ_P *according to Definition 3.38 coincides with the power kernel* K_X^{Φ} *associated to* Φ*, provided that the data set* $X = \{x_1, \ldots, x_N\}$ *for interpolation contains the set* Z *which is unisolvent for* P*.*

Proof: By Corollary 3.42 we know that the Lagrangian bases of interpolation with respect to the two kernels Φ and Φ_P coincide. Thus the power kernels coincide. □

Corollary 4.28 *Under the above assumptions, the transition from* Φ *to the power kernel* K_X *does not need the bypass via the kernels* h *or* Φ_P *of the previous chapter.*

Figures 4.1, 4.2, and 4.3 show the plots of the power function $P_X(x)$, the power kernel $K_X(x, y)$ and its contours as a function of x with a fixed point $y = (0, 0)^T$, when the data points in X are given via the MATLAB command

```
meshgrid (-3:2:3 , -3:2:3).
```

We chose as radial basis functions the Gaussian (Figure 4.1), the inverse multiquadric (Figure 4.2) and Wendland's C^2 function (Figure 4.3). The scales c in the sense $\phi_c(r) := \phi(r/c)$ were 1, 1, and 5, respectively.

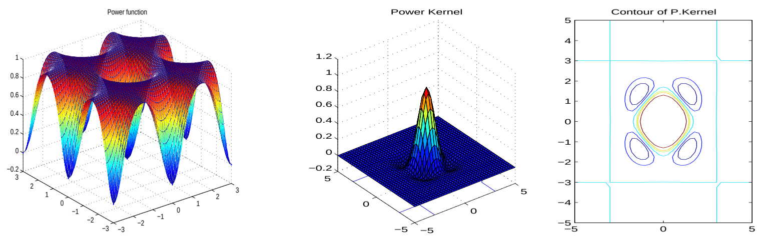

Figure 4.1: Power function, power kernel and its contours for the Gaussian.

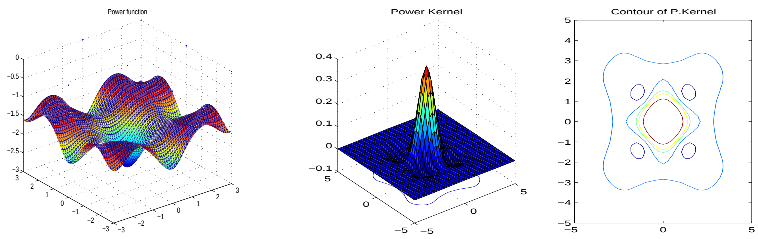

Figure 4.2: Power function, power kernel and its contours for the inverse multiquadric

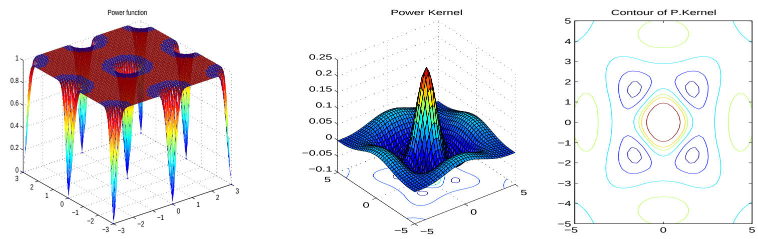

Figure 4.3: Power function, power kernel and its contours for the Wendland C^2 function

Chapter 5

Applications

5.1 Multistage interpolation

We are given a set $X = \{x_1, \ldots, x_N\}$ of pairwise distinct points $x_1, \ldots, x_N$ in a set $\Omega \subseteq \mathbb{R}^d$, and a real-valued function f with $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$. We take a conditionally positive definite continuous kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$ with respect to a finite-dimensional subspace P. To avoid complications as in Theorem 3.41, we shall always assume $Z \subseteq X$ in the rest of this chapter.

Then we denote the resulting interpolant to f by $s_{f,X,\Phi}$, making the dependence on f, X, and Φ transparent. For all functions $f \in \mathcal{N}_\Phi(\Omega)$ we define the *residual function* or *error function* g_f on Ω by

$$g_f : x \mapsto f(x) - s_{f,X,\Phi}(x).$$

We now interpolate the function g_f on a new finite set Y of points from $\Omega \setminus X$ using K_X, and denote the interpolant to g_f on Y associated to K_X by s_{g_f,Y,K_X}. We remark that for all $x_j \in X$ with $1 \leq j \leq N$ we have $g_f(x_j) = 0$ and $s_{g_f,Y,K_X}(x_j) = 0$. Then we conclude that for all $x \in X \cup Y$

$$\Big(f - s_{f,X,\Phi} - s_{g_f,Y,K_X}\Big)(x) = 0$$

which means

$$\Big(g_f - s_{g_f,Y,K_X}\Big)(x) = 0.$$

We want to find a relation between the interpolants $s_{f,X,\Phi} + s_{g_f,Y,K_X}$ and $s_{f,X\cup Y,\Phi}$ to f at all points in $X \cup Y$. The uniqueness of the interpolant for data on $X \cup Y$ with centers in $X \cup Y$ proves the following

Proposition 5.1 *Under the preceding assumptions, the interpolant of* Φ *on* $X \cup Y$ *is representable as*

$$s_{f,X\cup Y,\Phi} = s_{f,X,\Phi} + s_{g_f,Y,K_X}. \ \Box$$

Note that this requires $Z \subseteq X$ to be correct, if no further arguments are there.

Proposition 5.2 *If* Φ *is a positive definite kernel, then the power function* $P_{X\cup Y,\Phi}$ *is given by*

$$P_{X\cup Y,\Phi} = P_{Y,K_X}.$$

Proof: We use the Proposition 4.25 for $P_{X\cup Y,\Phi}$ and for P_{Y,K_X}. Then, with Proposition 4.24, we get

$$P_{X\cup Y,\Phi} = \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(X\cup Y) = \{0\}\}$$

and

$$\begin{array}{rcl} P_{Y,K_X} & = & \sup\{f(x) : f \in \mathcal{N}_{K_X},\ \|f\|_{K_X} \le 1,\ \ f(Y) = \{0\}\} \\ & = & \sup\{f(x) : f \in \mathcal{N}_{K_X},\ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(Y) = \{0\}\} \\ & = & \sup\left\{(g-\pi_X(g))(x) : \begin{array}{l} g \in \mathcal{N}_\Phi(\Omega),\ \|g-\pi_X(g)\|_\Phi \le 1 \\ (g-\pi_X(g))(Y) = \{0\} \end{array}\right\} \\ & \le & \sup\{f(x) : f \in \mathcal{N}_\Phi(\Omega),\ \|f\|_{\mathcal{N}_\Phi(\Omega)} \le 1,\ \ f(X\cup Y) = \{0\}\} \\ & = & P_{X\cup Y,\Phi}. \end{array}$$

The inequality sign follows from the fact that every $g - \pi_X(g)$ of the third line is some f in the fourth line. The other inequality follows from Corollary 4.22 when we take any f from the fourth line and define $g := f$ with $\pi_X(f) = \pi_X(g) = 0$ for the third line. $\Box$

Corollary 5.3 *If* Φ *is a positive definite kernel, we have*

$$|f(x) - s_{f,X\cup Y,\Phi}(x)| \le P_{Y,K_X}\|f\|_\Phi \text{ for all } f \in \mathcal{N}_\Phi(\Omega).$$

Proposition 5.4 *If* Φ *is an unconditionally positive definite kernel, then for finite sets* X, Y *with* $X \cap Y = \emptyset$ *we have*

$$(K_{X,\Phi})_{Y,K_{X,\Phi}} = K_{X\cup Y,\Phi},$$

where we indicated the appropriate "mother" kernels in the notation.

Proof: The native space of the right-hand side is

$$\{f \in \mathcal{N}_\Phi(\Omega) \;:\; f(X \cup Y) = \{0\}\},$$

while the native space for the left-hand side is the same:

$$\{f \in \mathcal{N}_{K_{X,\Phi}} \;:\; f(Y) = \{0\}\} = \{f \in \mathcal{N}_\Phi \;:\; f(X \cup Y) = \{0\}\}.$$

If f is an element of that space, the two reproduction properties are

$$\begin{array}{rcl} f(x) & = & (f, K_{X\cup Y,\Phi}(x,\cdot))_\Phi \\ f(x) & = & (f, (K_{X,\Phi})_{Y,K_{X,\Phi}}(x,\cdot))_{K_{X,\Phi}} \end{array}$$

and by Proposition 4.24 we can use any of the inner products $(.,.)_\Phi$, $(.,.)_{K_{X,\Phi}}$ and $(.,.)_{K_{X\cup Y,\Phi}}$ here. Now by uniqueness of reproducing kernels, the assertion follows first on $\Omega \setminus (X \cup Y)$, but since both kernels vanish on $X \cup Y$, we are done. □

Corollary 5.5 *Let Φ be an unconditionally positive definite kernel, and let X, Y satisfy the general assumptions above. Then there are orthogonal decompositions*

$$\begin{array}{rcll} \mathcal{N}_\Phi(\Omega) & = & S_{X,\Phi} & + \;\; \mathcal{N}_{K_{X,\Phi}}(\Omega \setminus X) \\ & = & S_{X,\Phi} & + \;\; S_{Y,K_X} + \mathcal{N}_{(K_{X,\Omega})_{Y,K_{X,\Phi}}}((\Omega \setminus X) \setminus Y) \\ & = & S_{X\cup Y,\Phi} & + \;\; \mathcal{N}_{K_{X\cup Y,\Phi}}(\Omega \setminus (X \cup Y)) \end{array}$$

with respect to the inner product $(.,.)_\Phi$ of $\mathcal{N}_\Phi(\Omega)$.

Proof: We use Corollary 4.23 and Propositions 5.1 and 5.4. □

So far, we have made a step from X to $X \cup Y$. We now want to do a sequence of such steps.

Assumption 5.6 *We assume to have a kernel Φ on Ω which is conditionally positive definite with respect to some finite-dimensional space P of functions on Ω, and we want to interpolate a function $f \in \mathcal{N}_\Phi(\Omega)$. We start with a finite set X of data points which contains a P-unisolvent subset Z.*

1. *Start a recursion with*

$$\begin{array}{rcl} j & := & 0 \\ \Phi_0 & := & \Phi \\ X_0 & := & X \\ f_0 & := & f - s_{f,X_0,\Phi_0} \\ & = & f - s_{f,X,\Phi} \end{array}$$

and do the following for $j = 0, 1, 2, \ldots$*:*

2. *Let* $Y_j \subset \Omega_j := \Omega \setminus X_j$ *be a finite set. Define*

$$\begin{array}{rcll} \Phi_{j+1} & := & K_{X_j} & \textit{positive definite on } \Omega_j \\ X_{j+1} & := & X_j \cup Y_j & \supset X_j \\ \Omega_{j+1} & := & \Omega \setminus X_{j+1} & \subset \Omega_j \\ f_{j+1} & := & f_j - s_{f_j,Y_j,\Phi_{j+1}} & \\ j & := & j+1 \textit{ and repeat.} & \end{array}$$

Note that from the second step on we have unconditional positive definiteness, while the first step from $\Phi_0 := \Phi$ to $\Phi_1 := K_X$ has the complications we encountered around Proposition 4.24.

Proposition 5.7 *Using Proposition 4.24 we get for all* $j \geq 1$

$$\|f\|^2_{\Phi_j} = \|f\|^2_{\Phi_{j-1}} = \|f\|^2_{\Phi_{j-2}} = \ldots = \|f\|^2_{\Phi_1} = \|f\|^2_{\Phi} + \|s_{g_y - g_X, X, \Phi}\|^2_{\Phi}.$$

with g_Y *and* g_X *from (4.9).*

Corollary 5.8 *The native space* $\mathcal{N}_{\Phi_{j+1}}$ *of* Φ_{j+1} *can be expressed as*

$$\mathcal{N}_{\Phi_{j+1}} = \mathcal{N}_{\Phi_1} \cap \{f |\ f(X_j) = \{0\}\} \textit{ for all } j \geq 0.$$

Proof: This follows from Proposition 5.7 and Corollary 4.22. □

We need the recursive power kernel form of our kernels:

Proposition 5.9 *The kernels have the property*

$$\Phi_{j+1} = (\Phi_j)_{Y_{j-1},\Phi_j} \textit{ for all } j \geq 1.$$

Proof: From Proposition 5.4 we conclude

$$\begin{aligned}\Phi_{j+1} &= K_{X_j,\Phi}\\ &= K_{Y_{j-1}\cup X_{j-1},\Phi}\\ &= \left(K_{X_{j-1},\Phi}\right)_{Y_{j-1},K_{X_{j-1},\Phi}}\\ &= (\Phi_j)_{Y_{j-1},\Phi_j}\,.\end{aligned}$$

□

We can always write the function f_j in the form

$$f_j = f_j - s_{f_j,Y_j,\Phi_{j+1}} + s_{f_j,Y_j,\Phi_{j+1}} = f_{j+1} + s_{f_j,Y_j,\Phi_{j+1}} \text{ for all } j \geq 0. \qquad (5.1)$$

Proposition 4.24 together with the orthogonality property of the interpolation allows us to write

$$\begin{aligned}\|f_j\|^2_{\Phi_{j+1}} &= \|f_{j+1}\|^2_{\Phi_{j+1}} + \|s_{f_j,Y_j,\Phi_{j+1}}\|^2_{\Phi_{j+1}}\\ &= \|f_j\|^2_{\Phi_j} \text{ for all } j \geq 0.\end{aligned} \qquad (5.2)$$

Using (5.2), a summation gives

Corollary 5.10 *Under the hypotheses of 5.6, we have*

$$\|f_1\|^2_{\Phi_1} = \|f_{j+1}\|^2_{\Phi_{j+1}} + \sum_{r=1}^{j} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_{r+1}} \textit{ for all } j \geq 0.$$

Lemma 5.11 *Under the hypotheses of 5.6, we can generalize Proposition 5.1 to get*

$$s_{f,X_{j+1},\Phi} = s_{f,X_j,\Phi} + s_{f_j,Y_j,\Phi_{j+1}} \textit{ for all } j \geq 0.$$

Proof: Both sides interpolate f on $X_{j+1} = X_j \cup Y_j$ using these points as centers. Since we assumed $Z \subseteq X$, we can use the argument of the proof for Proposition 5.1 again to get the assertion. □

Corollary 5.12 *Under the hypotheses of 5.6, we get*

$$f_j = f - s_{f,X_j,\Phi}.$$

Proof: We will prove it by induction. The assertion is true for $j = 0$. If it holds for j, we use Lemma 5.11 to get

$$\begin{aligned} f_{j+1} &= f_j - s_{f_j,Y_j,\Phi_{j+1}} \\ &= f - s_{f,X_j,\Phi} - s_{f_j,Y_j,\Phi_{j+1}} \\ &= f - s_{f,X_{j+1},\Phi}. \end{aligned}$$

□

Corollary 5.13 *Under the hypotheses of 5.6, we get*

$$s_{f,X_j,\Phi} = s_{f,X,\Phi} + \sum_{r=0}^{j-1} s_{f_r,Y_r,\Phi_{r+1}} \text{ for all } j \geq 0.$$

Proof: It is true for $j = 0$. If it holds for j, we get by Lemma 5.11 the result

$$\begin{aligned} s_{f,X_{j+1},\Phi} &= s_{f,X_j,\Phi} + s_{f_j,Y_j,\Phi_{j+1}} \\ &= s_{f_j,Y_j,\Phi_{j+1}} + s_{f,X,\Phi} + \sum_{r=0}^{j-1} s_{f_r,Y_r,\Phi_{r+1}} \\ &= s_{f,X,\Phi} + \sum_{r=0}^{j} s_{f_r,Y_r,\Phi_{r+1}}. \end{aligned}$$

□

Proposition 5.14 *Under Assumption 5.6, we generalize the result of Proposition 5.2 to get*

$$P_{X_{j+1},\Phi_j}(x) = P_{Y_j,\Phi_{j+1}}(x) \text{ for all } j \geq 1.$$

Corollary 5.15 *Under Assumption 5.6, we have*

$$P_{X_{j+1},\Phi_j}(x) = P_{X_{j+1},\Phi_1}(x) \text{ for all } j \geq 1.$$

Proof: Proposition 5.2 implies that

$$P_{X_{j+1},\Phi_j} = P_{Y_j,K_{X_j}} = P_{Y_j,\Phi_{j+1}}, \quad \text{for all } j \geq 0.$$

□

Proposition 5.16 *Under the preceding assumption 5.6 and for all* $x \in \Omega_j$ *we have*

$$|f(x) - s_{f,X_j,\Phi}(x)| \leq P_{X_j,\Phi_1}(x)\|f_{j-1}\|_{\Phi_1}.$$

Proof: We start with the definition of f_j to get

$$\begin{aligned} |f_j(x)| &= |f_{j-1}(x) - s_{f_{j-1},X_{j-1},\Phi_j}(x)| \\ &\leq P_{Y_{j-1},\Phi_j}(x)\|f_{j-1}\|_{\Phi_j}. \end{aligned}$$

Using Proposition 5.14 and Corollary 5.15 we get

$$\begin{aligned} P_{Y_{j-1},\Phi_j}(x) &= P_{X_j,\Phi_{j-1}}(x) \\ &= P_{X_j,\Phi_1}(x). \end{aligned}$$

Then

$$|f_j(x)| \leq P_{X_j,\Phi_1}(x)\|f_{j-1}\|_{\Phi_1}.$$

□

Theorem 5.17 *If* Φ *is a positive definite kernel then*

$$|f(x) - s_{f,X_j,\Phi}(x)| \leq P_{X_j,\Phi_1}(x)\|f - s_{f,X,\Phi}\|_{\Phi}.$$

Proof: Using Corollary 5.10 until order $j-1$ we get the equation

$$\|f_{j-1}\|^2_{\Phi_1} = \|f_{j-1}\|^2_{\Phi_{j-1}} = \|f_1\|^2_{\Phi_1} - \sum_{r=1}^{j-2} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_r}$$

for all $j \geq 2$. We can go one step further, using

$$\begin{aligned} \|f_0\|^2_{\Phi_1} &= \|f_0 - s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} + \|s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} \\ &= \|f_1\|^2_{\Phi_1} + \|s_{f_0,Y_0,\Phi_1}\|^2_{\Phi_1} \end{aligned}$$

to get

$$\|f_{j-1}\|^2_{\Phi_1} = \|f_0\|^2_{\Phi_1} - \sum_{r=0}^{j-2} \|s_{f_r,Y_r,\Phi_{r+1}}\|^2_{\Phi_r} \leq \|f - s_{f,X,\Phi}\|^2_{\Phi_1}.$$

If Φ is unconditionally positive definite, we can replace the norm by $\|.\|_{\Phi}$. Thus the assertion is proven via Proposition 5.16. □

However, the proof shows that the error bound of the previous theorem is weaker than the result of Proposition 5.16.

Using Corollary 5.5, one can write down an orthogonal decomposition of the native space related to the above construction.

5.2 Recursive constructions

In this subsection, we will study the effect when we add a single data point to the recovery problem .
Let $X = \{x_1, \ldots, x_N\}$ be a set of $N \geq Q \geq 1$ pairwise distinct points in Ω, and we consider a "new" point $z \in \Omega \setminus X$. We define a system of the form (2.9) with the notation taken from (2.10) by

$$\begin{pmatrix} A_{X,\,\Phi} & \Phi_X(z) & P_X \\ \Phi_X^T(z) & \Phi(z,z) & p^T(z) \\ P_X^T & p(z) & 0 \end{pmatrix} \begin{pmatrix} U_X^*(x) \\ W_X^*(z) \\ V_X^*(x) \end{pmatrix} = \begin{pmatrix} \Phi_X(x) \\ \Phi(x,z) \\ p(x) \end{pmatrix} \tag{5.3}$$

with $\Phi_X(z) = \Big(\Phi(x_j, z)\Big)_{1\leq j\leq N}$, $p(z) = \Big(p_k(z)\Big)_{1\leq k\leq Q}$. We now subtract (5.3) and the system (2.10):

$$\begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} U_X(x) \\ V_X(x) \end{pmatrix} = \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}.$$

Then we get

$$\begin{aligned} & \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} U_X^*(x) - U_X(x) \\ V_X^*(x) - V_X(x) \end{pmatrix} \\ = \;& -W_X^*(z) \begin{pmatrix} \Phi_X(z) \\ p(z) \end{pmatrix} \\ = \;& -W_X^*(z) \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} U_X(z) \\ V_X(z) \end{pmatrix} \end{aligned}$$

and conclude

$$\begin{aligned} U_X^*(x) &= U_X(x) - W_X^*(x) U_X(z) \\ V_X^*(x) &= V_X(x) - W_X^*(x) V_X(z). \end{aligned}$$

Lemma 5.18 *The value of $W_X^*(z)$ is given by:*

$$W_X^*(z) = \frac{K_X(x,z)}{K_X(z,z)}.$$

Proof: First we calculate the value of $\Phi(x,z)$, using the system (5.3) and applying (5.4).

$$\begin{aligned}
\Phi(x,z) &= \Phi_X^T(z)U_X^*(x) + \Phi(z,z)W_X^*(z) + p^T(z)V_X^*(x) \\
&= \Phi_X^T(z)\Big(U_X(x) - W_X^*(z)U_X(z)\Big) + \Phi(z,z)W_X^*(z) \\
&\quad + p^T(z)\Big(V_X(x) - W_X^*(z)V_X(z)\Big) \\
&= \Big(\Phi_X^T(z)U_X(x) + p^T(z)V_X(x)\Big) \\
&\quad - W_X^*(z)\Big(\Phi_X^T(z)U_X(z) + p^T(z)V_X(z) - \Phi(z,z)\Big)
\end{aligned}$$

We conclude then :

$$\begin{aligned}
W_X^*(z) &= \frac{\Phi(x,z) - \Phi_X^T(z)U_X(x) - p^T(z)V_X(x)}{\Phi(z,z) - \Phi_X^T(z)U_X(z) - p^T(z)V_X(z)} \\
&= \frac{K_X(x,z)}{K_X(z,z)}
\end{aligned}$$

where we used Proposition 4.12. □

Theorem 5.19 *Adding a point z to the data set $X \subset \Omega$, the power kernel $K_{X\cup\{z\}}$ for the point set $X \cup \{z\}$ is given by*

$$K_{X\cup\{z\}}(x,y) = K_X(x,y) - \frac{K_X(x,z)K_X(z,y)}{K_X(z,z)}.$$

Proof: For the set $X \cup \{z\}$ and using (4.12) and (5.3), the value of $K_{X\cup\{z\}}$ is given by

$$\begin{aligned}
K_{X\cup\{z\}}(x,y) &= \Phi(x,y) - \Big((U_X^*)^T(x), W_X^*(z)\Big)\begin{pmatrix} \Phi_X(y) \\ \Phi(x,z) \end{pmatrix} \\
&\quad - (V_X^*)^T(x)p(y) \\
&= \Phi(x,y) - (U_X^*)^T(x)\Phi_X(y) - W_X^*(z)\Phi(x,z)
\end{aligned}$$

$$\begin{aligned}
& -(V_X^*)^T(x)p(y) \\
= \; & \Phi(x,y) - \Big(U_X(x) - W_X^*(z)U_Z(z)\Big)^T \Phi_x(y) - W_X^*(z)\Phi(x,z) \\
& -\Big(V_X(x) - W_X^*(x)V_Z(z)\Big)^T p(y) \\
= \; & \Big(\Phi(x,y) - U_X^T(x)\Phi_x(y) - V_X^T(x)p(y)\Big) \\
& -W_X^*(z)\Big(\Phi(x,z) - U_Z^T(z)\Phi_x(y) - V_X^T(z)p(y)\Big) \\
= \; & K_X(x,y) - W_X^*(z)K_X(z,y).
\end{aligned}$$

And from (5.18) we will have

$$K_{X\cup\{z\}}(x,y) = K_X(x,y) - \frac{K_X(x,z)K_X(z,y)}{K_X(z,z)}.$$

□

Corollary 5.20 *For a set $X = \{x_1, \ldots, x_N\}$ of points in Ω and $z \in \Omega \setminus X$, the power function for $X \cup \{z\}$ is given by*

$$P^2_{X\cup\{z\}}(x) = P^2_X(x) - \frac{K^2_X(x,z)}{P^2_X(z)}. \; \square$$

Corollary 5.21 *For all $x \in \Omega$ and for all $z \in \Omega \setminus X$ we have*

$$P^2_{X\cup\{z\}}(x) \leq P^2_X(x). \; \square$$

This also follows from Proposition 4.25.

Proposition 5.22 *For a set of points $X = \{x_1, \ldots, x_N\} \in \Omega$ and $Z = \{z_1, \ldots, z_M\} \in \Omega \setminus X$ we denote $X_0 := X$, $X_k := X \cup \{z_1, \ldots, z_k\}$ for $1 \leq k \leq M$. Then we have*

$$P^2_{X_M}(x) = P^2_{X_0}(x) - \sum_{k=0}^{M-1} \frac{K^2_{X_k}(x, z_{k+1})}{P^2_{X_k}(z_{k+1})}$$

Proof: We apply (5.20) for all sets $X_k = X \cup \{z_1, \ldots, z_k\} = X_{k-1} \cup \{z_k\}$ recursively. □

Proposition 5.23 *The sequence* $\left(P^2_{X_M}(x)\right)_{M\geq 1}$ *converges in* $\mathbb{R}$*, if we add infinitely many distinct points* $z_1, z_2, \ldots$ *from* $\Omega \setminus X$ *to* X.

Proof: By Proposition 5.22, the sequence is decreasing and bounded from below. □

5.3 Recursive greedy interpolation

In this section we want to to prove a fundamental theorem about the native space norms of interpolants. Let Φ be conditionally positive definite on Ω, and assume $X = \{x_1, \ldots, x_N\}$ to be a set in Ω which contains a P-unisolvent subset. Furthermore, let $x \in \Omega \setminus X$ be given, and let f be a fixed function in the native space $\mathcal{N}_\Phi(\Omega)$.

Theorem 5.24 *The native space norm of interpolants can be recursively written as*

$$\|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} = \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \left(\frac{f(x) - s_{f,X}(x)}{P_X(x)}\right)^2.$$

Proof: We will divide the proof in two parts. First, we prove the theorem for unconditionally positive definite kernels, and then for conditionally positive definite kernels.

Case 1: Φ unconditionally positive definite

Suppose that Φ is a positive definite kernel. Then we introduce the system

$$\begin{pmatrix} A_{X,\,\Phi} & \Phi_X(x) \\ \Phi_X^T(x) & \Phi(x,x) \end{pmatrix} \begin{pmatrix} a_X(x) \\ \alpha(x) \end{pmatrix} = \begin{pmatrix} f_X \\ f(x) \end{pmatrix} \tag{5.4}$$

where $\Phi_X^T(x)(x) = (\Phi(x, x_j))_{1\leq j\leq N}$, $a(x) \in \mathbb{R}^N$, and $f_X^T = (f(x_j))_{1\leq j\leq N}$. And we want to compare it with the system (2.9) in the special form

$$A_{X,\,\Phi}\beta_X = f_X. \tag{5.5}$$

Hence, from (5.4) and (5.5) we get

$$\begin{aligned} A_{X,\,\Phi}\beta_X &= f_X & (5.6)\\ A_{X,\,\Phi}a_X(x) + \Phi_X(x)\alpha(x) &= f_X & (5.7)\\ \Phi_X^T(x)a_X(x) + \Phi(x,x)\alpha(x) &= f(x). & (5.8) \end{aligned}$$

First we observe from (5.5)

$$\|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} = \|s_{f,X}\|^2_\Phi = \beta_X^T f_X.$$

Then we set (5.6) equal to (5.7) and see that

$$A_{X,\,\Phi}\beta_X = A_{X,\,\Phi} a_X(x) + \Phi_X(x)\alpha(x)$$

$$\beta_X = a_X(x) + \alpha(x) A^{-1}_{X,\,\Phi}\Phi_X(x)$$

where we get

$$a_X(x) = \beta_X - \alpha(x) A^{-1}_{X,\,\Phi}\Phi_X(x)$$

$$\Phi_X^T a_X(x) = s_{f,X}(x) - \alpha(x)\Phi_X^T A^{-1}_{X,\,\Phi}\Phi_X(x).$$

We now replace the value of $a_X(x)$ in (5.8) to get

$$\begin{aligned} f(x) &= \Phi_X^T(x) a_X(x) + \Phi(x,x)\alpha(x) \\ &= \Phi_X^T(x)\Big(\beta_X - \alpha(x) A^{-1}_{X,\,\Phi}\Phi_X(x)\Big) + \Phi(x,x)\alpha(x) \\ &= \Phi_X^T(x)\beta_X + \alpha(x)\Big(\Phi(x,x) - \Phi_X^T(x) A^{-1}_{X,\,\Phi}\Phi_X(x)\Big) \\ &= s_{f,X}(x) + \alpha(x) P_X^2(x) \end{aligned}$$

where we used Theorem 4.10 and Proposition 4.12. Thus we have

$$\alpha(x) = \frac{f(x) - s_{f,X}(x)}{P_X^2(x)}. \tag{5.9}$$

The native space norm of the interpolant for $\{x\} \cup X$ then is by (5.4)

$$\begin{aligned} \|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} &= \begin{pmatrix} a_X(x) \\ \alpha(x) \end{pmatrix}^T \begin{pmatrix} f_X \\ f(x) \end{pmatrix} \\ &= a_X(x) f_X + \alpha(x) f(x) \\ &= \Big(\beta_X^T - \alpha(x)\Phi_X^T(x) A^{-1}_{X,\Phi}\Big) f_X + \alpha(x) f(x) \\ &= \beta_X^T f_X + \alpha(x)\Big(f(x) - \Phi_X^T(x) A^{-1}_{X,\,\Phi} f_X\Big) \\ &= \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \alpha(x)\Big(f(x) - \Phi_X^T(x) A^{-1}_{X,\,\Phi} f_X\Big). \end{aligned}$$

If we insert the value of $\alpha(x)$ from (5.9), then we get

$$\|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} = \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \frac{f(x) - s_{f,X}(x)}{P_X^2(x)}\Big(f(x) - \Phi_X^T(x) A^{-1}_{X,\,\Phi} f_X\Big). \tag{5.10}$$

From (5.5) we find

$$s_{f,X}(x) = \beta_X^T \Phi_X(x) = \Phi_X^T(x) A_{X,\,\Phi}^{-1} f_X.$$

Then we get

$$\begin{aligned}\|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} &= \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \frac{f(x)-s_{f,X}(x)}{P_X^2(x)}\Big(f(x)-s_{f,X}(x)\Big)\\ &= \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \Big(\frac{f(x)-s_{f,X}(x)}{P_X(x)}\Big)^2\end{aligned}$$

and we can replace $\|.\|_{\mathcal{N}_\Phi(\Omega)}$ by $\|.\|_\Phi$ here.

Case 2 : Φ conditionally positive definite

For a conditionally positive definite kernel Φ we now introduce the system

$$\begin{pmatrix} A_{X,\,\Phi} & \Phi_X(x) & P_X \\ \Phi_X^T(x) & \Phi(x,x) & p^T(x) \\ P_X^T & p(x) & 0 \end{pmatrix} \begin{pmatrix} \tilde a_X(x) \\ \alpha(x) \\ \tilde b_X(x) \end{pmatrix} = \begin{pmatrix} f_X \\ f(x) \\ 0 \end{pmatrix} \tag{5.11}$$

with similar notation as above, and with notation from (2.9), but for general P of dimension Q, including $p(x) := (p_1(x), \ldots, p_Q(x))^T$ for a basis $p_1, \ldots, p_Q$ for P. We now do the subtraction of (5.11) and

$$\tilde A_{X,\,\Phi} \begin{pmatrix} a_X \\ b_X \end{pmatrix} = \begin{pmatrix} A_{X,\,\Phi} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} a_X \\ b_X \end{pmatrix} = \begin{pmatrix} f_X \\ 0 \end{pmatrix}$$

to get

$$\begin{aligned}\tilde A_{X,\,\Phi} \begin{pmatrix} \tilde a_X(x) - a_X \\ \tilde b_X(x) - b_X \end{pmatrix} &= -\alpha(x) \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}\\ \begin{pmatrix} \tilde a_X(x) - a_X \\ \tilde b_X(x) - b_X \end{pmatrix} &= -\alpha(x) \tilde A_{X,\,\Phi}^{-1} \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}.\end{aligned}$$

We conclude then

$$\begin{pmatrix} \tilde a_X(x) \\ \tilde b_X(x) \end{pmatrix} = \begin{pmatrix} a_X \\ b_X \end{pmatrix} - \alpha(x) \tilde A_{X,\,\Phi}^{-1} \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}. \tag{5.12}$$

For simplification we denote

$$\tilde{\gamma}_X(x) = \begin{pmatrix} \tilde{a}_X(x) \\ \tilde{b}_X(x) \end{pmatrix}, \quad \gamma_X = \begin{pmatrix} a_X \\ b_X \end{pmatrix}$$

$$\tilde{\Phi}_X(x) = \begin{pmatrix} \Phi_X(x) \\ p(x) \end{pmatrix}, \quad \tilde{f}_X = \begin{pmatrix} f_X \\ 0 \end{pmatrix}.$$

We now start from the system (5.11), and we use (5.12) to get

$$\begin{aligned} f(x) &= \Phi_X^T(x)\tilde{a}_X(x) + \Phi(x,x)\alpha(x) + p^T(x)\tilde{b}_X(x) \\ &= \Phi(x,x)\alpha(x) + \tilde{\Phi}_X^T(x)\tilde{\gamma}_X(x) \\ &= \Phi(x,x)\alpha(x) + \tilde{\Phi}_X^T(x)\Big[\gamma_X - \alpha(x)\tilde{A}_{X,\,\Phi}^{-1}\tilde{\Phi}_X(x)\Big] \\ &= \alpha(x)\Big(\Phi(x,x) - \tilde{\Phi}_X^T(x)\tilde{A}_{X,\,\Phi}^{-1}\tilde{\Phi}_X(x)\Big) + \tilde{\Phi}_X^T(x)\gamma_X. \end{aligned}$$

Then we get the value of $\alpha(x)$ as

$$\alpha(x) = \frac{f(x) - \tilde{\Phi}_X^T(x)\gamma_X}{\Phi(x,x) - \tilde{\Phi}_X^T(x)\tilde{A}_{X,\,\Phi}^{-1}\tilde{\Phi}_X(x)}.$$

With

$$\begin{aligned} s_{f,X}(x) &= \tilde{\Phi}_X^T(x)\gamma_X \\ P_X^2(x) &= \Phi(x,x) - \tilde{\Phi}_X^T(x)\tilde{A}_{X,\,\Phi}^{-1}\tilde{\Phi}_X(x) \end{aligned}$$

we again get

$$\alpha(x) = \frac{f(x) - s_{f,X}(x)}{P_X^2(x)}. \tag{5.13}$$

We evaluate the native space norm of the enlarged interpolant as

$$\begin{aligned} \|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} &= \begin{pmatrix} \tilde{a}_X(x) \\ \alpha(x) \\ \tilde{b}_X(x) \end{pmatrix}^T \begin{pmatrix} f_X \\ f(x) \\ 0 \end{pmatrix} \\ &= \alpha(x)f(x) + \tilde{\gamma}_X(x)^T\tilde{f}_X \\ &= \alpha(x)f(x) + \Big(\gamma_X^T - \alpha(x)\tilde{\Phi}_X^T\tilde{A}_{X,\,\Phi}^{-1}\Big)\tilde{f}_X \\ &= a_X^T f_X + \alpha(x)f(x) - \alpha(x)\tilde{\Phi}_X^T\tilde{A}_{X,\,\Phi}^{-1}\tilde{f}_X \\ &= \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \alpha(x)\Big(f(x) - s_{f,X}(x)\Big). \end{aligned}$$

Then we insert the value of $\alpha(x)$ from (5.13) to get

$$\|s_{f,X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} = \|s_{f,X}\|^2_{\mathcal{N}_\Phi(\Omega)} + \frac{\Big(f(x) - s_{f,X}(x)\Big)^2}{P_X^2(x)}.$$

□

Corollary 5.25 *The power function P_X can be expressed in the form*

$$P_X^2(x) = \|s_{f_{x,X},X\cup\{x\}}\|^2_{\mathcal{N}_\Phi(\Omega)} - \|s_{f_{x,X},X}\|^2_{\mathcal{N}_\Phi(\Omega)}$$

Proof: We just apply the preceding Thoerem 5.24 for $f = f_{x,X} \in \mathcal{N}_\Phi(\Omega)$ and use 4.14 and 4.10. □

Proposition 5.26 *Assume that for a given interpolant $s_{f,X}$ to a function $f \in \mathcal{N}_\Phi(\Omega)$ at a point set $X = \{x_1, \ldots, x_N\}$ we calculate a new point $x_{N+1} \in \Omega \setminus X$ by*

$$|(f - s_{f,X})(x_{N+1})| = \|f - s_{f,X}\|_{\infty,\Omega}.$$

Then this "greedy method" has the convergence property

$$\lim_{N\to\infty} \|f - s_{f,X_N}\|^2_\infty = 0.$$

Proof: We have from the proposition (5.24)

$$\|s_{f,X_{N+1}}\|^2_{\mathcal{N}_\Phi(\Omega)} - \|s_{f,X_N}\|^2_{\mathcal{N}_\Phi(\Omega)} = \left(\frac{f(x_{N+1}) - s_{f,X_N}(x_{N+1})}{P_{X_N}(x_{N+1})}\right)^2.$$

Since uniform boundedness $P_{X_N}(x_{N+1}) \le \|P_{X_N}\|_\infty \le \|P_{X_Q}\|_\infty =: C$ follows from Proposition 4.25, we conclude

$$\begin{aligned}
|f(x_{N+1}) - s_{f,X_N}(x_{N+1})|^2 &\le C\left(\|s_{f,X_{N+1}}\|^2_{\mathcal{N}_\Phi(\Omega)} - \|s_{f,X_N}\|^2_{\mathcal{N}_\Phi(\Omega)}\right)\\
\|f - s_{f,X_N}\|^2_\infty &\le C\left(\|s_{f,X_{N+1}}\|^2_{\mathcal{N}_\Phi(\Omega)} - \|s_{f,X_N}\|^2_{\mathcal{N}_\Phi(\Omega)}\right)\\
\sum_{N=0}^{\infty} \|f - s_{f,X_N}\|^2_\infty &\le C\sum_{N=0}^{M}\left(\|s_{f,X_{N+1}}\|^2_{\mathcal{N}_\Phi(\Omega)} - \|s_{f,X_N}\|^2_{\mathcal{N}_\Phi(\Omega)}\right)\\
\sum_{N=0}^{\infty} \|f - s_{f,X_N}\|^2_\infty &\le C\|f\|_{\mathcal{N}_\Phi(\Omega)}
\end{aligned}$$

due to orthogonality. Thus we conclude

$$\lim_{N\to\infty} \|f - s_{f,X_N}\|_\infty^2 = 0.$$

□

Bibliography

[1] **R.K. BEATSON AND W. LIGHT**, *Fast solution of the radial basis functions: Interpolation equations:domain decomposation* , SIAM. Journal on Scientific Computing. Vol 22, pages 1717-1740, 2000.

[2] **M.J. BUHMANN** , *New developments in the theory of the radial basis function interpolation.* , CAGD pp. 35-75 . Mathematic-Numeric Departement Giessen, 1993.

[3] **M.J. BUHMANN** , *Radial basis functions* , ACTA-Numerica Press, pages 1-38, 2000.

[4] **M.J. BUHMANN** , *Approximation and interpolation with radial basis functions* , Cambridge University Press pp. 25-43 . Mathematic-Numeric Departement Giessen, 1993.

[5] **J. DUCHON** , *Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces.*, Rev. Francaise Automat. Informat. Rech. Opér. Anal. Numer. vol 10, pages 5-12, 1976.

[6] **J. DUCHON** , *Splines minimizing ratation-invariant semi-norms in sobolev spaces.*, In W. Schempp and K. Zeller, editors, *Constructive Theory of Functions of Several Variables*, pages:85-100 . Berlin, 1977. Springer.

[7] **J. DUCHON** , *Sur l'erreur d interpolation des fonctions de plusieures variables par les D^m-splines* ., Rev. Francaise Automat. Informat. Rech. Opér. Anal. Numer. vol 12, pages 325-334, 1978.

[8] **N. DYN** , *Interpolation of scattered data by radial basis functions* ., Academic Press pp. 47-61 . Boston, 1987.

[9] **N. DYN** , *Interpolation and approximation by radfial basis functions* ., Academic Press pp. 211-234 . New-York, 1989.

[10] **W. FREEDEN , R. FRANKE and SCHREINER** , *A survey on spherical spline approximation* , Surv. Math. Ind. vol 7, pages 29-85, 1997.

[11] **GUTZMER** , *Interpolation by positive definite functions on locally compact groups with application*, Department of Mathematics and Computer Science,

[12] **R.L. HARDY** , *Theory and applications of the multiquadric biharmonic method* .. 20 years of discovery 1968-1988. Comput. Math. Appl. Vol 19, pages 163-208, 1990

[13] **HARDY.R.L** , *Multiquadric equations of topography and anotherirregular surfaces* ., J. Geophys. Res. Vol 76, pages 1905-1915, 1971.

[14] **I.R.H. JACKSON** , *Radial basis functions: A survey and new results* ., Clarendon Press pp. 1971-1988 . Oxford 1989.

[15] **W. LIGHT AND H.S.J. WAYNE** , *Some aspects of radial basis function Approximation* Department of Mathematics and Computer Science, University of Leicester, Leicester. 1992

[16] **W. LIGHT AND H.S.J. WAYNE** , *Spaces of Distributions, interpolation by translates of a basis function and error estimtes*, Numer. Math. Vol 81, pages 415-450, 1999.

[17] **MADYCH AND NELSON** , *Multivariate interpolation and condionally positive definite functions II* , Math. Comput. Vol 54, pages 211-230, 1990.

[18] **MADYCH and NELSON** , *Bounds on multivariate polynomials and exponential error estimates for multiquadric interpolation* , J. Approx. Theory. Vol 70, pages 94-114, 1992.

[19] **MADYCH and NELSON** , *Multivariate interpolation : a variational theory* , Unpublished manuscript, 1983.

[20] **H. Meschkowski** , *Hilbertsche Räume mit Kernfunction*, Springer, Berlin, 1962.

[21] **J.D. POWELLM** , *The theory of the Radial Basis Function approximation in 1990* , In W.A. Light, editor, Advances in Numerical Analysis, Volume II: Wavelets, Subdivision Algorithms and Radial Basis Functions. Clarendon Press (Oxford),pages 105-210, 1992.

[22] **E.M STEIN and WEISS** , *Introduction to Fourrier analysis in eucludien space*, Princeton University Press, pages 297, 1971.

[23] **R.SCHABACK** , *Reconstruction of Multivariate Functions from Scattered Data*, Manuscript, available via ,http://www.num.math.uni-goettingen.de/schaback/research/group.html' , year 1997.

[24] **R.SCHABACK** , *Native Hilbert Space for Radial Basis Functions*, In M. W. Müler et al. editors, New developments in approximation theory. February 23-27, 1998, vol 132 of Int. Num. pages 255-282, Birkhäser Verlag, Basel, 1999.

[25] **R.SCHABACK** , *Improved Eroor Bounds for Scattered Data Interpolation by Radial Basis Functions*, Math. Cumpt. Vol 68, pages 201-216, 1999.

[26] **R.SCHABACK** , *Error Estimates and Condition Nunbers for radial Basis Functions Interpolation* , Adv. Comput. Math. Vol 3 pages 251-264, 1995.

[27] **R. SCHABACK** , *Creating surfaces from scattered data using Radial Basis Function* . In M. Dealhlen, T. Lyche and L.L> Schumaker, editors, Mathematical Methods for Curves and Surfaces. Vanderbilt University Press, pages 177-496, Nashville, 1995.

[28] **R. SCHABACK** , *Multivariate interpolation and approximation by translates of s basis function* , In C.K. Chui and L.L. Schumaker, edotors, Approximation Theory VIII, volume 1: Approximation and interpolation, World Scientific Publishing, pages 491-514, Singapore,1995.

[29] **R. SCHABACK** , *Local Error Estimates for radial basis function Interpolation of Scattered Data.*, IMAJNA . Vol 13, pages 13-27, 1993.

[30] **H. WENDLAND** , *Scattered Data Approximation*, Cambridge University Press, Cambbridge, UK, 2005.

[31] **H. WENDLAND and R. SCHABACK**, *Characterization and construction of radial basis functions* , Cambridge University Press, pages 1-24, 2001 .

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Von 01.10.1999 bis 30.11.1999 : Praktikant in der Buchhaltung der Gesellschaft ICONET. Beni-Mellal Marokko

Von 02.02.1999 bis 02.05.1999 : Praktikant in der Buchhaltung (ein Monat) und in der Produktionsabteilung (zwei Monate) am *Cabinet Marocaines des Assurances*, Casablanca Marokko.

Juli und August in den Jahren 1996, 1997, 1998 und 1999 : Praktikant in den Abteilungen Produktion, Buchhaltung und Informatik innerhalb der Niederlassungen der BANQUE POUPULAIRE. Beni-Mellal Marokko.

Die Jahre 96/97 und 97/98 : Abendkurse in *Informatik der Verwaltung*, zwei Jahre, mit Zertifikat in “Informatik der Verwaltung” am INSTITUT PASCAL D` INFORMATIQUE. Beni-Mellal Marokko.

IV- Sprache

Arabisch	$(les : 100\%, schr : 100\%, spr : 100\%, verst : 100\%)$.
Französisch	$(les : 100\%, schr : 100\%, spr : 100\%, verst : 100\%)$.
Deutsch	$(les : 100\%, schr : 100\%, spr : 100\%, verst : 100\%)$.
Englisch	$(les : 90\%, schr : 80\%, spr : 75\%, verst : 75\%)$.
Spanisch	$(les : 60\%, schr : 60\%, spr : 50\%, verst : 60\%)$.
Italienisch	$(les : 75\%, schr : 50\%, spr : 50\%, verst : 75\%)$.

(les=lesen, schr=schreiben, spr=sprechen, verst=verstehen).