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Integration of Routing and Timetabling in Public Transportation

Diplomarbeit

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Author:
Michael Siebert

Supervisor:
Prof. Dr. Anita Schöbel

Erklärung

Ich versichere hiermit, dass ich die vorliegende Diplomarbeit mit dem Titel

„Integration of Routing and Timetabling in Public Transportation“

selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Göttingen, 28. April 2011, *Michael Siebert*

List of Abbreviations

$[l^{\text{change}}, u^{\text{change}}]$ .... lower/upper change activity bounds $u^{\text{change}} = l^{\text{change}} + T - 1$

$[l^{\text{wait}}, u^{\text{wait}}]$ .... lower/upper wait activity bounds

$[l_a, u_a]$, $a \in \mathcal{A}$ ..... activity lower/upper bounds

$[l_e, u_e]$, $e \in \vec{E}$ ...... edge lower/upper bounds

ATT^{π}_{w}, ATT^{x}_{w} ..... Average Traveling Time w.r.t. a passenger distribution w and a feasible timetable π resp. feasible duration set x

$\boxed{\text{C}}$ $\boxed{\text{V}}$ $\boxed{\text{L}}$ $\boxed{\text{Q}}$ ...... Constant, Variable, Linear/Quadratic constraint/objective

$\text{EAN} = (\mathcal{E}, \mathcal{A})$ ..... Periodic Event Activity Network, $\mathcal{E}$ events, $\mathcal{A}$ activities

EPESP .... Extended Periodic Event Scheduling Problem

FA, FM .... Frequency as Attribute/Multiplicity indicator

$\text{LC} = (\vec{L}, F)$ ...... Line Concept, $\vec{L}$ lines, F frequencies

$\mathfrak{L} \supset \vec{L}$ .... line pool

$\text{obj}^{*}_{\text{P}}$ .... objective function value of an optimum of problem (P)

obj_{P} .... objective function of problem (P)

$\text{OD} = (w_{s_1 s_2})_{s_1, s_2 \in S}$ Origin Destination matrix

ODPESP .... Origin Destination aware PESP

PESP .... Periodic Event Scheduling Problem

$\text{PTN} = (S, \vec{E})$ ..... Public Tranportation Network, S stations, $\vec{E}$ edges

σ, $\varnothing$ .... standard deviation, average

f_ℓ, f_1, f_2 .... frequency of line $\ell, \ell_1, \ell_2 \in \vec{L}$, $f_\ell, f_1, f_2 | T$

h_e, $e \in \vec{E}$ .... edge headway

T .... period length

w_a, $a \in \mathcal{A}$ .... passenger distribution

Contents

Chapter 1

Introduction

1.1 Motivation

Public transportation affects the daily life of billions of people worldwide and besides ineffectiveness producing more costs and wasting more resources on the operators side, it wastes billions of hours of valueable time on the customers side and therefore is of global economical as well as ecological dimension.

Traditionally, the construction and maintainance of a public transporation network, like for busses, metro or intercity trains, consists of the following planning steps

1. **Network Design** Where to put the stations and infrastructure?

2. **Line Planning** How to layout the lines, i.e. the vehicle paths?

3. **Passenger Routing** Which paths will passengers take?

4. **Timetabling** At which times will lines arrive/depart at the stations?

5. **Vehicle Scheduling** How should the lines be served by vehicles?

6. **Crew Scheduling** How should the crew circulate within the vehicles?

7. **Delay Management** What to do in case of delays? How to prevent them?

Usually, the steps are done in the order above by hand and heuristics. But questions arise:

> Of what quality will our network be? How far is it away from the optimum? Can we do it better?

> These are difficult questions. In fact, politicians, transportation managers, customers, taxpayers, etc. frequently employ judgments such as "good" and "efficient", but nobody can give a definition what this exactly means. Since almost every public transportation system in the world is in the red, the cheapest system is no public transportation at all. On the other hand, the most convenient system for the passenger - a stop in front of every house with direct connections to everywhere – is much too expensive. What is the right compromise? Operations Research has no good answer either - so far. But OR can improve aspects of public transportation significantly [. . .]
>
> [BGP06]

There is evidence for the last statement:

> In December 2006, Netherlands Railways introduced a completely new timetable. Its objective was to facilitate the growth of passenger and freight transport on a highly utilized railway network and improve the robustness of the timetable, thus resulting in fewer operational train delays. Modifications to the existing timetable, which was constructed in 1970, were not an option; additional growth would require significant investments in the rail infrastructure. Constructing a railway timetable from scratch for about 5,500 daily trains was a complex problem. To support this process, we generated several timetables using sophisticated operations research techniques. Furthermore, because rolling-stock and crew costs are principal components of the costs of a passenger railway operator, we used innovative operations research tools to devise efficient schedules for these two resources.
>
> The new resource schedules and the increased number of passengers resulted in an additional annual profit of €40 million ($60 million); the additional revenues generated approximately €10 million of this profit. We expect this profit to increase to €70 million ($105 million) annually in the coming years. However, the benefits of the new timetable for the Dutch society as a whole are much greater: more trains are transporting more passengers on the same railway infrastructure, and these trains are arriving and departing on schedule more than they ever have in the past. In addition, the rail transport system will be able to handle future transportation demand growth and thus allow cities to remain accessible to more people. Therefore, we expect that many will switch from car transport to rail transport, thus reducing the emission of greenhouse gases.
>
> [KHA+09]

LinTim[1], a project by Prof. Schöbel, is a collection of methods to perform some of the planning steps above automatically. With LinTim we can e.g. evaluate the impact of different line planning methods on the average traveling time or the delay robustness.

> "Why can we not simply compute the optimum?"

There are two problems:

1. It is hard to define what that optimum should actually be. However a low average traveling time seems to be desireable when it comes to efficiency.

2. Computing space and time. Problems related to public transportation are usually $\mathcal{NP}$ hard and models that incorporate several levels of planning grow astromically in their sizes.

Therefore, traditional planning uses simple models. Let us have a look at the timetabling objective function:

$$\min \sum_{a \in \mathcal{A}} w_a x_a,$$

where w_a is a fixed number of passengers that take the activity a (drive, wait or change) and x_a its duration.

In that model, the number of passengers is *fixed* per activity. If we assume that passengers will take the shortest path in time to get from one station to another, lets say they looked it up at `reiseauskunft.bahn.de`, their number is actually *not fixed*. As expected: in general, the average traveling time decreases, if we reroute the passengers and recalculate the timetable. This holds for both tiny and gigantic networks and means that the traditional model only delivers an *approximation of unknown overall quality* to the possible optimum.

If we follow the traditional traffic planning workflow, another problem arises: some steps at the beginning actually depend on data we only get at the end, as we have seen for the timetabling where we need to perform some initial guess. But this goes down much further: At the line planning step we also made assumptions about how many passengers will use certain links within the network, which we only know after timetabling.

[1]`http://lintim.math.uni-goettingen.de`

1.2 Overview

In this Chapter we introduce the basic formalism used throughout the work, which is most basic notation of graph theory, a collection of later useful number theoretical lemmata and our notation of public transportation.

The next Chapter 2, Classical Models, builds on the definitions from this chapter, introduces the default methods of the LinTim framework, which should mostly correspond to what is used in research practice. Further it introduces the widely used periodic event scheduling problem PESP.

Chapter 3, Beyond Classical Models, the author introduces his extensions to LinTim, subject to comparison with classical models and base for later theoretical and computational results.

In Chapter 4, Planning Steps Lower Bounds we work on lower bounds for the average traveling time at different stages of planning.

Besides lower bounds we face the Worst Case Error in Chapter 5, which is a collection of various example networks, parametrized by the period length T as well as line frequencies and facing different common simplification techniques and sequential planning.

Finally, Chapter 6, Computational Results, compares the worst case findings with what happens in actual networks. Further, we introduce a scaleable, extensible and iterative heuristic method that for practice-relevant large scale networks can improve results more than an additional threefold in average and more than an additional eightfold if combined with a statistic framework compared to what had been possible before with state-of-the-art methods for timetabling [GS11].

1.3 Prerequisites

The reader should be familiar with optimization in public transportation, mathematical programming as well as elementary number, graph and complexity theory. For the former one, [Sch04] is our main source, for the latter one, the author proposes [GJ79] and for the others arbitrary lecture notes should do it.

To the taste of the author, this work could be less formal. However, one big issue is that most of the formalism is result of the work on LinTim, so that it is hard to judge to which amount definitions intersect with those of the reader, even if he or she is involved in the subject. On the other hand, many proofs require precise, if not pedantic formulations of the actual problem, since by itsself it usually spans several levels of planning in public transportation and often involves an aspect changed in between.

In this section we introduce the most basic preliminaries for this work.

1.3.1 Number Theory

This section is for reference purposes but should be read as a whole. Without reading the main text however it is most likely meaningless to the reader.

Definition 1.1 (Periodic Interval). *Let $a, b, T \in \mathbb{N} \setminus \{0\}$. A* periodic interval *$[a, b]_T$ is defined as*

$$[a, b]_T = \bigcup_{z \in \mathbb{Z}} [a + zT, b + zT] = [a, b] + \mathbb{Z}T \quad . \tag{1.1}$$

For example, $[1, 2]_{60} = \ldots [-59, -58] \cup [1, 2] \cup [61, 62] \cup \ldots$.

Lemma 1.2. *Let $a, b \in \mathbb{Z}$ with constraints $a, b \in [l, u]$, where $l, u \in \mathbb{Z}$ and $T \in \mathbb{N}$ with $l \leq u$ and $u - l \leq T - 1$. Then $a = b \mod T$ iff $a = b$.*

Proof. $a = b \mod T$ is defined by: there exist $z \in \mathbb{Z}$ such that $a = b + zT$.
"$\Rightarrow$": For z holds

$$\frac{a - b}{T} = z \quad . \tag{1.2}$$

Using the lower and upper bounds yields

$$0 \leq \left\lceil \frac{l - u}{T} \right\rceil \leq z \leq \left\lfloor \frac{u - l}{T} \right\rfloor = \left\lfloor \frac{T - 1}{T} \right\rfloor = 0 \quad , \tag{1.3}$$

therefore $z = 0$
"$\Leftarrow$": Chose $z = 0$. □

Note that Lemma 1.2 is not true if e.g. $l - u > T - 1$. Therefore consider $a, b \in [0, 60]$. Then both $a = 0$ and 60 yield $a - b = 0 \mod T$ with $b = 0$.

Theorem 1.3 (Bézout's Identity). *Let $a, b \in \mathbb{N} \setminus \{0\}$. Then there exist $x, y \in \mathbb{Z}$:*

$$xa + yb = \gcd(a, b) \quad . \tag{1.4}$$

Proof from Wikipedia. Let $S = \{na + mb > 0 : n, m \in \mathbb{Z}\}$. Since S is not empty and $S \subset \mathbb{N}$, there exists a minimal element $d = xa + yb$ with $d \leq s$, for all $s \in S$.

By division with remainder $a = qd + r$ with $r \in \{0, \ldots, d-1\}$. Solving for r yields $r = a - qd = a - q(xa + yb) = a(1 - qx) + b(-yq)$. If $r > 0$ it must be in S, which would contradict the fact that d is minimal in S. Therefore $r = 0$ and $d|a$. The same argument yields $d|b$. If c is another common divisior of a and b, it divides $xa + yb = d$, since it divides every summand and thus $c|d$ and d must be the greated common divisor. □

Lemma 1.4. *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. It holds that*

$$\gcd\left(\frac{T}{f_1}, \frac{T}{f_2}\right) = \frac{T}{\operatorname{lcm}(f_1, f_2)} \quad . \tag{1.5}$$

Proof. By the fundamental theorem of arithmetic, every positive integer has a unique decomposition into a product of prime powers. Let $p_1, \ldots, p_{n_p}$ be prime divisors of T. Since $f_1|T$ and $f_2|T$, f_1 and f_2 can be represented as a product of prime powers of T

$$T = \prod_{i=1}^{n_p} p_i^{T_i} \quad , \qquad\qquad T_i \in \mathbb{N} \setminus \{0\}, \tag{1.6}$$

$$f_1 = \prod_{i=1}^{n_p} p_i^{f_i^1} \quad , \qquad f_2 = \prod_{i=1}^{n_p} p_i^{f_i^2} \quad , \qquad f_i^1, f_i^2 \in \{0, \ldots, T_i\}, \tag{1.7}$$

and thus

$$\frac{T}{f_1} = \prod_{i=1}^{n_p} p_i^{T_i - f_i^1} \quad , \qquad \frac{T}{f_2} = \prod_{i=1}^{n_p} p_i^{T_i - f_i^2} \quad . \tag{1.8}$$

Further, lcm can be expressed by prime power products

$$\operatorname{lcm}(f_1, f_2) = \prod_{i=1}^{n_p} p_i^{\min(f_i^1, f_i^2)} \quad , \tag{1.9}$$

and gcd as well

$$\gcd\left(\frac{T}{f_1}, \frac{T}{f_2}\right) = \prod_{i=1}^{n_p} p_i^{\max(T_i - f_i^1, T_i - f_i^2)} = \prod_{i=1}^{n_p} p_i^{T_i - \min(f_i^1, f_i^2)} = \frac{T}{\operatorname{lcm}(f_1, f_2)} \quad . \tag{1.10}$$

□

Lemma 1.5 (Division With Negative Remainder)**.** *Let $x \in \mathbb{Z}$, $T \in \mathbb{N} \setminus \{0\}$. Then there are unique $k_1, k_2 \in \mathbb{Z}$ and $x_1, x_2 \in \{0, \ldots, T-1\}$ with*

$$x = k_1 T + x_1 = k_2 T - x_2 \quad . \tag{1.11}$$

Proof. Division with remainder yields $x = k_1T + x_1$ with unique $k_1 \in \mathbb{Z}$ and $x_1 \in \{0, \ldots, T-1\}$. If $x_1 = 0$, then select $k_2 = k_1$ and $x_2 = 0$. Otherwise select $k_2 = k_1 + 1$ and obtain

$$x = k_2T - x_2 = k_1T + T - x_2 \quad . \tag{1.12}$$

The lemma follows with $x_2 = T - x_1 \in \{1, \ldots, T-1\}$. □

Lemma 1.6 (Modulus Reducibility)**.** *Let $\tau, T \in \mathbb{N} \setminus \{0\}$ with $\tau | T$ and $a \in \mathbb{Z}$.*

$$\exists\ k \in \{0, \ldots, \tfrac{T}{\tau} - 1\} : a + k\tau = 0 \mod T \quad \Leftrightarrow \quad a = 0 \mod \tau \quad . \tag{1.13}$$

Proof. Equivalent are

$$a + k\tau = 0 \mod T \tag{1.14}$$

$$\exists\ z \in \mathbb{Z} : \qquad a + k\tau = zT \tag{1.15}$$

$$a = \left(\frac{zT}{\tau} - k\right)\tau \quad . \tag{1.16}$$

It remains to show that for all $z' \in \mathbb{Z}$ there exists $z \in \mathbb{Z}$ and $k \in \{0, \ldots, \frac{T}{\tau} - 1\}$ such that

$$z' = z\frac{T}{\tau} - k \quad . \tag{1.17}$$

Division with negative remainder as in Lemma 1.5 yields

$$z' = \overline{z}\frac{T}{\tau} - \overline{k}, \qquad \overline{z} \in \mathbb{Z},\ \overline{k} \in \{0, \ldots, \tfrac{T}{\tau} - 1\} \quad . \tag{1.18}$$

Choose $k = \overline{k}$ and $z = \overline{z}$ to obtain the lemma. □

Theorem 1.7 (A Periodic lcm Representation)**.** *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1 | T$, $f_2 | T$. Then there exist $\xi_1 \in \{0, \ldots, f_1 - 1\}$, $\xi_2 \in \{0, \ldots, f_2 - 1\}$ with*

$$\xi_1\frac{T}{f_1} - \xi_2\frac{T}{f_2} = \frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.19}$$

Proof. From Bézout's Identity (Theorem 1.3) with $a = \frac{T}{f_1}$, $b = \frac{T}{f_2}$ follows that there exist $x, y \in \mathbb{Z}$ with

$$x\frac{T}{f_1} + y\frac{T}{f_2} = \gcd\left(\frac{T}{f_1}, \frac{T}{f_2}\right) \quad . \tag{1.20}$$

Devision with (negative) remainder yields

$$x = k_1f_1 + \xi_1 \quad , \qquad k_1 \in \mathbb{Z},\ \xi_1 \in \{0, \ldots, f_1 - 1\}, \tag{1.21}$$

$$y = k_2f_2 - \xi_2 \quad , \qquad k_2 \in \mathbb{Z},\ \xi_2 \in \{0, \ldots, f_2 - 1\} \tag{1.22}$$

and hence

$$(k_1 f_1 + \xi_1)\frac{T}{f_1} + (k_2 f_2 - \xi_2)\frac{T}{f_2} = \gcd\left(\frac{T}{f_1}, \frac{T}{f_2}\right) \tag{1.23}$$

which is equivalent to

$$\begin{aligned} \xi_1 \frac{T}{f_1} - \xi_2 \frac{T}{f_2} &= \gcd\left(\frac{T}{f_1}, \frac{T}{f_2}\right) - T(k_1 + k_2) && (1.24) \\ &= \frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad , && (1.25) \end{aligned}$$

by Lemma 1.4. □

Lemma 1.8. *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. For all $\tilde{k} \in \mathbb{N}$ there are $i \in \{0, \dots, f_1 - 1\}$ and $j \in \{0, \dots, f_2 - 1\}$ with*

$$i\frac{T}{f_1} - j\frac{T}{f_2} = \tilde{k}\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.26}$$

Proof. Let $\ell := \operatorname{lcm}(f_1, f_2)$. As per Theorem 1.7 there are $\xi_1 \in \{0, \dots, f_1 - 1\}$, $\xi_2 \in \{0, \dots, f_2 - 1\}$ with

$$\xi_1 \frac{T}{f_1} - \xi_2 \frac{T}{f_2} = \frac{T}{\ell} \mod T \quad . \tag{1.27}$$

Thus, for a given $\tilde{k} \in \mathbb{N}$

$$\tilde{k}\xi_1 \frac{T}{f_1} - \tilde{k}\xi_2 \frac{T}{f_2} = \frac{\tilde{k}T}{\ell} \mod T \quad . \tag{1.28}$$

By devision with remainder

$$\begin{aligned} \tilde{k}\xi_1 &= k_1 f_1 + i \quad , && k_1 \in \mathbb{Z},\ i \in \{0, \dots, f_1 - 1\}, && (1.29) \\ \tilde{k}\xi_2 &= k_2 f_2 + j \quad , && k_2 \in \mathbb{Z},\ j \in \{0, \dots, f_2 - 1\}. && (1.30) \end{aligned}$$

It follows that

$$i\frac{T}{f_1} - j\frac{T}{f_2} = \frac{\tilde{k}T}{\ell} + T(k_2 - k_1) = \tilde{k}\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.31}$$

□

Lemma 1.9 (Representation of lcm)**.** *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. For each $\tilde{k} \in \mathbb{N}$ there exists $k \in K := \{0, \dots, \operatorname{lcm}(f_1, f_2) - 1\}$ so that*

$$\tilde{k}\frac{T}{\operatorname{lcm}(f_1, f_2)} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.32}$$

K is a smallest set with the property from equation (1.32)*.*

Proof. Let $\ell := \operatorname{lcm}(f_1, f_2)$. Devision with remainder yields

$$\tilde{k} = k_1\ell + k \quad , \qquad\qquad k_1 \in \mathbb{Z},\ k \in \{0, \ldots, \ell - 1\} \tag{1.33}$$

and thus

$$\tilde{k}\frac{T}{\ell} = \frac{k_1\ell T + kT}{\ell} = k_1 T + \frac{kT}{\ell} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.34}$$

The greatest multiple of $\frac{T}{\ell}$ which is still in $[0, T)$ is $\frac{(\ell-1)T}{\ell}$. Therefore, for given $\kappa_1, \kappa_2 \in \{0, \ldots, \ell - 1\}$ with $\kappa_1 \neq \kappa_2$ also $\frac{\kappa_1 T}{\ell} \neq \frac{\kappa_2 T}{\ell} \mod T$ is satisfied. It follows that K is a smallest set with the property from equation (1.32). □

Theorem 1.10 (Compact Representation of lcm). *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. For all $k \in \{0, \ldots, \operatorname{lcm}(f_1, f_2) - 1\}$ there are $i \in \{0, \ldots, f_1 - 1\}$ and $j \in \{0, \ldots, f_2 - 1\}$ with*

$$i\frac{T}{f_1} - j\frac{T}{f_2} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.35}$$

Proof. Combine Lemma 1.8 and 1.9. □

Theorem 1.11 (Compact Representation by lcm). *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. For all $i \in \{0, \ldots, f_1 - 1\}$ and $j \in \{0, \ldots, f_2 - 1\}$ there exists $k \in \{0, \ldots, \operatorname{lcm}(f_1, f_2) - 1\}$ with*

$$i\frac{T}{f_1} - j\frac{T}{f_2} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad . \tag{1.36}$$

Proof. Let $\ell := \operatorname{lcm}(f_1, f_2)$. Then there are $k_1 = \ell/f_1$, $k_2 = \ell/f_2$ so that

$$i\frac{T}{f_1} - j\frac{T}{f_2} = (ik_1 - jk_2)\frac{T}{\ell} = \tilde{k}\frac{T}{\ell} \mod T \quad , \tag{1.37}$$

where $\tilde{k} = ik_1 - jk_2 \in \mathbb{Z}$. With help of Lemma 1.9 follows that

$$i\frac{T}{f_1} - j\frac{T}{f_2} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \quad , \tag{1.38}$$

with $k \in \{0, \ldots, \operatorname{lcm}(f_1, f_2) - 1\}$. □

Corollary 1.12 (lcm Representation Map). *Let $T, f_1, f_2 \in \mathbb{N} \setminus \{0\}$, $f_1|T$, $f_2|T$. The* lcm Representation Map

$$\begin{aligned} k : \{0, \ldots, f_1 - 1\} \times \{0, \ldots, f_2 - 1\} &\to \{0, \ldots, \operatorname{lcm}(f_1, f_2) - 1\} \\ (i, j) &\mapsto k : i\frac{T}{f_1} - j\frac{T}{f_2} = k\frac{T}{\operatorname{lcm}(f_1, f_2)} \mod T \end{aligned}$$

is well-defined and surjective. Further, i and j together with f_1 and f_2 may be swapped.

Proof. Well-definedness follows from theorem 1.11 and surjectivity from Lemma 1.9 and Theorem 1.11. Swapping i and i together with f_1 and f_2 is possible due to the symmetry of lcm. □

Corollary 1.13. *With $\ell = \operatorname{lcm}(f_1, f_2)$, the* lcm *representation map from Corollary 1.12 may be represented as*

$$k = i\frac{\ell}{f_1} - j\frac{\ell}{f_2} \mod \ell \quad . \tag{1.39}$$

Proof. It holds

$$\exists\, z : \mathbb{Z} : \quad i\frac{T}{f_1} - j\frac{T}{f_2} = k\frac{T}{\ell} + zT \quad , \tag{1.40}$$

which is equivalent to

$$\exists\, z : \mathbb{Z} : \quad i\frac{\ell}{f_1} - j\frac{\ell}{f_2} = k + z\ell \quad , \tag{1.41}$$

from which the corollary follows. □

1.3.2 Graph Theory

Definition 1.14 (Graph)**.** *An* directed $G = (V, \vec{E})$ *is a tuple of a finite set V called* vertices *and $\vec{E}$, a finite subset of $V \times V \times \mathbb{N}$, called* edges. *An element $(v, v', i) \in E$ is called* edge from v to v' *resp.* edge from vertex v and to vertex v' *and the notation $(v, v') = e$ means that the tuple (v, v') contains the from and to vertices of e. For $v, v' \in V$ the set $E_{v,v'} = \{(v, v', i) \in E\}$ is called the* set of edges between nodes v and v'. *$G = (V, E)$ is an* undirected graph, *if the edges are sets $\{v, v'\}$ and thus* not oriented, *but for simplicity reasons still denoted by (v, v').*

Definition 1.15 (Subgraph)**.** *A* subgraph *$G' = (V', E')$ of $G = (V, E)$ has the property $V' \subset V$, $E' \subset E$ and is a graph by itsself, i.e. for all $(v, v') \in E'$ it holds $v, v' \in V'$.*

Definition 1.16 (Connected Edge Sequence, Path, Cycle)**.** *Let $G = (V, \vec{E})$ be a directed graph. A* connected edge sequence *of length $k - 1$ is a finite sequence of edges*

$$S = \big[(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\big] \tag{1.42}$$

with mutually distinct edges $(v_j, v_{j+1}) \in \vec{E}$ or $(v_{j+1}, v_j) \in \vec{E}$ for all $j \in \{1, \ldots, k-1\}$ with the sets $S^+ = \{(v_j, v_{j+1}) : (v_j, v_{j+1}) \in \vec{E}\}$ and $S^- = \{(v_{j+1}, v_j) : (v_j, v_{j+1}) \in \vec{E}\}$ being forward arcs *resp.* backward arcs *of P.*

If all nodes in S are mutually distinct, then S is denoted by P and called undirected path from v_1 to v_k. *If $P^- = \varnothing$, then P is called* directed path. *If for S holds $v_k = v_1$ it is called a* cycle, *denoted by C instead of S and if additionally*

$C^- = \varnothing$ it is called circle. *A cycle resp. circle for which all nodes besides the first and the last one are mutually distinct is called* simple.

For an undirected graph $G = (V, E)$ a structure P as above is just called path. *Connected edge sequences and cycles keep their names.*

Definition 1.17 (Connected). *Let $G = (V, \vec{E})$ be a directed graph. G is called* weakly connected *resp.* strongly connected *if for all $v_1, v_2 \in V$, $v_1 \neq v_2$ there is a undirected resp. directed path from v_1 to v_2.*

An undirected graph $G = (V, E)$ is called connected *if for every $v_1, v_2 \in V$ there is a path from v_1 to v_2.*

Definition 1.18 (Connected Component). *Let $G = (V, \vec{E})$ be a directed graph and $v_0 \in V$. The* weakly *resp.* strongly connected component of v_0 *is a subgraph $G_0 = (V_0, \vec{E}_0)$ of G that consists of the maximal set of vertices $v \in V$ so that there is an undirected resp. directed path from v_0 to v together with all edges $e \in E$ that are contained in any path from v_0.*

For undirected graphs analogously despite there is no destinction between undirected and directed paths.

For all graphs we work with we assume that they are (strongly) connected. If they are not, our results still hold for every connected component.

Definition 1.19 (Tree). *A* tree *is a (directed) graph $G = (V, E)$ that satisfies one of the equivalent conditions*

- *G is (weakly) connected and has no cycles.*
- *G is (weakly) connected and if one edge is removed from E, it is not (weakly) connected anymore.*
- *For any two $v_1, v_2 \in V$ there is a unique (undirected) path from v_1 to v_2.*

Definition 1.20 (Spanning Tree). *For a graph $G = (V, E)$ a* spanning tree *is a subgraph $G' = (V', E')$ that is a tree and satisfies $V' = V$, i.e. it is spanning.*

Definition 1.21 (Shortest Path). *Let $G = (V, \vec{E})$ be a (directed) graph, $w : E \to$* $\mathbb{R}_{\geq 0}$ edge weights *and v_1, $v_2 \in V$. A* shortest path P from v_1 to v_2 w.r.t. w *is a (directed) path from v_1 to v_2 minimizes*

$$\sum_{e \in P} w_e. \tag{1.43}$$

Definition 1.22 (Shortest Path Tree). *Let $G = (V, E)$ be a directed graph and $w : E \to \mathbb{R}_{\geq 0}$ edge weights and $v_0 \in V$. A* (directed) shortest path tree from v_0 w.r.t. w *is a spanning tree that contains v_0, called* root *and every (directed) path to other $v_1 \in V$, $v_1 \neq v_0$ is a shortest (directed) path from v_0 to v_1.*

Note that in this context, a directed tree refers to the underlying graph being directed and strong connectivity.

For nonnegative edge weights as above shortest paths trees and therefore shortest paths as well may be computed in polynomial time, e.g. $\mathcal{O}(E + V \log V)$ with Dijkstras algorithm with Fibonacci Heaps.

1.3.3 Public Transportation

Public Transportation is ubiquitous and there is no one who never used a bus or a train at least once in his or her lifetime. Therefore, the intention of this section is not to introduce anything new to anyone, but to link the perceived reality with a concrete formalism.

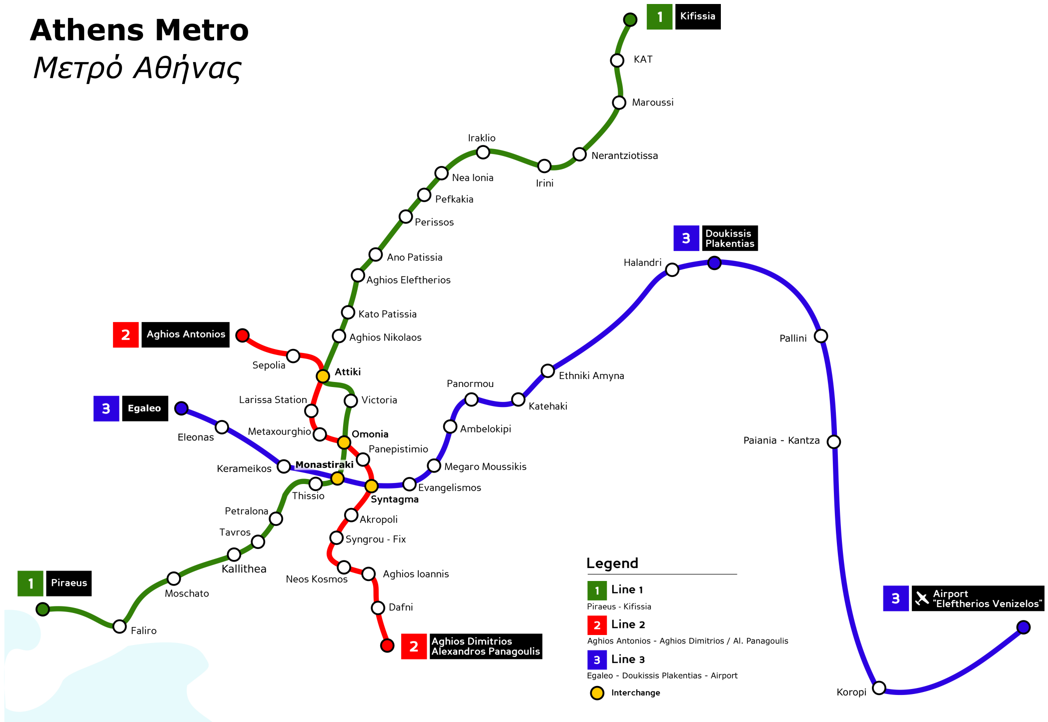


Figure 1.1: Athens Metro, a dataset in LinTim.
Image Source: http://en.wikipedia.org/wiki/Athens_Metro

This section are many repetitions that could easily be avoided but actually are indented, since they allow a quick look up.

Most of the defintions arise from the implementation in the LinTim project which is based on [Sch04].

Definition 1.23 (Public Transportation Network)**.** *A* Public Transportation Network PTN $= (S, E)$ *is an undirected resp.* PTN $= (S, \vec{E})$ *a directed graph that has a set of* stations S *as vertices and a set of* edges E *with possibly multiple edges.*

Edges represent physical connections like roads or rails between bus stops or train stations.

If the PTN $= (S, E)$ graph is undirected this means that all edges may be used in both directions, while a directed PTN $= (S, \vec{E})$ may have one-way roads, e.g. as they occur for bus networks in inner cities like it is the case in Göttingen.

For our purposes, we consider undirected edges as two identical directed edges, one pointing in one and one in the other direction, like the two lanes of a road or two parallel railway tracks. Therefore we effectively work with directed networks only.

Definition 1.24 (Time)**.** Time *is an integral number given in a* time unit $\mathfrak{t}$*.*

For example, $\mathfrak{t} = 1\,\mathrm{min}$ for intercity rail traffic like in Bahn Gross or $\mathfrak{t} = 6\,\mathrm{s}$ for rapid transit as in the Athens Metro dataset.

Definition 1.25 (Edge Lower and Upper Bounds)**.** *Given a* PTN $= (S, E)$*, a* lower bound map $l : E \to \mathbb{N}$ *assigns every edge* $e \in E$ *a* lower bound l_e*. An* upper bound map $u : E \to \mathbb{N}$ *assigns every edge* $e \in E$ *a* upper bound $u_e \geq l_e$*. Both are given in time units* $\mathfrak{t}$*.*

This implies that our networks are limited to only a single kind of vehicle. A lower bound l_e is engineerically given by the minimal time a vehicle needs to pass an edge e, whereas an upper bound u_e is a more or less arbitrary maximal time to pass e, with $l_e \leq u_e$.

The edge $e \in E$ could for example be a road of one kilometer length. If the speed limit on that road is 60km/h and the vehicle can go that fast, then $l_e = 1$. Since passengers can walk with around 6km/h, $u_e = 10$ would be a reasonable upper bound.

Definition 1.26 (Edge Headway)**.** *Given a* PTN $= (S, E)$*, a* headway map $h : E \to \mathbb{Z}$ *assigns every edge* $e \in E$ *a* headway h_e *in time units* $\mathfrak{t}$*.*

Roughly speaking, a headway h_e is a minimal safety time distance for every pair of vehicles that use e to make sure that vehicles do not crash. To be more precise, see headway activities in Definition 1.42.

A headway of $h_e = 2$ for some $e \in E$, means that if two vehicles v_1 and v_2 use h_e, that every departure of v_1 must be at least two time units later than that of v_2 and vice versa.

We defined lower bounds, upper bounds and headways on undirected public transportation networks, which works out that way for directed networks as well. Since we expect our edges to be identical in both directions, bounds and headways persist in the case of a switch to the directed representation.

Definition 1.27 (Vehicle Capacity)**.** *The* Vehicle Capacity *is an integral number* $\mathfrak{c}^{\text{Vehicle}}$ *given in passengers.*

In case of several vehicles, we would need a map, but since our networks are limited to one kind of vehicle, we can denote its capacity with $\mathfrak{c}^{\text{Vehicle}}$.

The vehicle capacity must not necessarily be the maximal number of passengers that fit into the vehicle and may be reduced for travel comfort reasons.

Definition 1.28 (Period Length)**.** *The* Period Length $T \in \mathbb{N}$ *is an integer given in time units* $\mathfrak{t}$.

Departure and arrival times of lines from Definition 1.32, i.e. the timetable (Section 2.5) repeats every period length.

For all our example datasets, the period length is one hour, i.e. $T = 60$ for Bahn Gross and $T = 600$ for Athens Metro.

Definition 1.29 (Origin Destination Matrix)**.** *Let* $\text{PTN} = (S, \vec{E})$ *be a public transportation network and* T *a period length. An* origin destination matrix $\text{OD} = (w_{s_1 s_2})_{s_1, s_2 \in S}$ *assigns an* origin destination pair $w_{s_1 s_2} \geq 0$*, for all* s_1*,* $s_2 \in S$*, which is the number of travelers from station* s_1 *to* s_2 *within* T*, given in passengers.*

In our datasets we interpolate $w_{s_1 s_2}$ from the number of passengers per day between s_1 and s_2, for all s_1, $s_2 \in S$ and thus we do not account for rush hours or idle times. In our model, the passenger distribution stays uniform throughout a period and thus throughout the day.

Definition 1.30 (PTN Passenger Route)**.** *Let* $\text{PTN} = (S, \vec{E})$ *be a public transportation network. For a given pair of stations* $s_1, s_2 \in S$*, a* PTN passenger route *is a path in the* PTN *from* s_1 *to* s_2*.*

Definition 1.31 (Passenger Load, Vehicle Demand, Maximal Vehicle Load)**.** *Given a* $\text{PTN} = (S, E)$ *and a period length* T*, a* passenger load map $p : E \to \mathbb{N}$ *assigns every edge* $e \in E$ *a* passenger load p_e*, given in passengers. A* vehicle demand map $f^{\text{low}} : \vec{E} \to \mathbb{N}$ *assigns every edge* $e \in E$ *an* edge vehicle demand $f_e^{\text{low}} = \lceil p_e / \mathfrak{c}^{\text{Vehicle}} \rceil$*. A* maximal vehicle load map $f^{\text{up}} : E \to \mathbb{N}$ *assigns every edge* $e \in E$ *a* maximal vehicle load $f_e^{\text{up}} \geq f_e^{\text{low}}$*.*

Since we do not have data on maximal vehicle loads, our f_e^{up} is arbitrary, for all $e \in E$.

Definition 1.32 (Line)**.** *Let* $\text{PTN} = (S, E)$ *be a public transportation network. A* line ℓ *is a path* $\ell = (e_1^\ell, \ldots, e_{n_\ell}^\ell)$ *in the* PTN*.*

A line can be thought as a bus or metro line. However, lines exist also for intercity rail traffic.

We cannot define a line by the stations it passes, since there can be multiple edges in the PTN.

The only reason why we limit lines to be paths is that our current formalism does not allow a station to be passed twice. This happens to some lines, especially in bus networks. Introducing an additional index to characterize departures and arrivals would fix that issue, but is not of conceptional interest for this work.

Definition 1.33 (Frequency)**.** *A* frequency $f_\ell \in \mathbb{N}$ *is the number of times a line occurs within a period length* T*. This implies that* $f_\ell | T$*.*

If $f_\ell | T$ is not explicitly stated, it still holds. However, for recalling purposes or formal clarity, we mention it occasionally.

The frequency $f_\ell \in F$, is roughly the number of vehicles used on ℓ equally distributed within the period length. Directed lines inherit f_ℓ from their undirected representation.

Definition 1.34 (Line Concept)**.** *A* line concept $\mathrm{LC} = (L, F)$ *is a set of lines* L *and a* frequencies map $F : L \to \mathbb{N} \cup \{0\}$ *that assigns a* frequency f_ℓ *to every line* $\ell \in L$*.*

We may derive line concepts from an initial set of lines, a so-called *line pool* $\mathfrak{L}$ or be generated in some other fashion. As for the PTN, although lines are undirected, we can represent LC in a directed manner: in $(\vec{L}, F)$ for each $\ell \in L$ we assign two lines; one that heads in one direction of the edge sequence, one in the other.

To do timetabling, we need to combine a public transportation network PTN with a line concept LC and an origin destination matrix OD. We give a quick overview about the most basic aspects. For a visualization of event activity networks and their construction, consult Sections 2.3 and 3.1.

Definition 1.35 (Event Activity Network)**.** *Let* $\mathrm{PTN} = (S, \vec{E})$ *be a public transportation network and* $\mathrm{LC} = (\vec{L}, F)$ *be a line concept. An associated* Event Activity Network $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ *is a directed graph that has a set of* events $\mathcal{E}$ *as vertices and a set of* activities $\mathcal{A}$ *as edges.* Events $\varepsilon \in \mathcal{E}$ *are either* departures (dep) *or* arrivals (arr) *and have a unique representation* $\varepsilon = (s, \ell, \mathrm{arr/dep}, i)$ *with* $s \in S$, $\ell \in \vec{L}$ *and a* frequency instance $i \in \{0, \ldots, f_\ell - 1\}$*. A* representative event *has a frequency instance* $i = 0$*.*

An event $\varepsilon = (s, \ell, \mathrm{arr/dep}, i)$ with $s \in S$, $\ell \in \vec{L}$ and $i \in \mathbb{Z}$ can be thought as an arrival/depature of the vehicle that serves the line ℓ. But strictly speaking, it only means that *some vehicle* serves it, which is to be determined by vehicle scheduling, which is not part of this work.

The term *event activity network* refers to *periodic event activity network*, since we work with the periodic version only anyway. Besides the pure graph, an EAN is assumed to have lower and upper bounds for activities as well as a period length attached, so we only mention them explicitly on usage.

Definition 1.36 (Timetable)**.** *Let* $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ *be an event activity network. A* timetable *is a map* $\pi : \mathcal{E} \to \mathbb{Z}$*.*

The only information at this point is: a timetable maps from the events to the integers. For more details on periodic timetabling see Section 2.5.

Definition 1.37 (Activitiy Lower and Upper Bounds). *Given an* EAN $= (\mathcal{E}, \mathcal{A})$*, a* lower bound map $l : \mathcal{A} \to \mathbb{N}$ *assigns every activity* $a \in \mathcal{A}$ *a* lower bound l_a*. An* upper bound map $u : \mathcal{A} \to \mathbb{N}$ *assigns every activity* $a \in \mathcal{A}$ *a* upper bound $u_a \geq l_a$*. Both are given in time units* $\mathfrak{t}$*.*

The diversity among activities $a = (\varepsilon, \varepsilon') \in \mathcal{A}$ is greater than that of events.

Definition 1.38 (Drive Activity). *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. A* drive activity *leads from a departure* $\varepsilon = (\text{dep}, \ell, s, i) \in \mathcal{E}$ *at some station* $s \in S$ *to an arrival* $\varepsilon' = (\text{arr}, \ell, s', i) \in \mathcal{E}$ *of the same line* $\ell \in \vec{L}$ *at a different station* $s' \neq s$ *but same frequency instance* i *and is called* drive from s to s'*. The* drive activity edge map $\text{edge} : \mathcal{A} \to \vec{E}$ *assigns every drive activity* a *an* $\text{edge}_a = (e, s, s') \in \vec{E}$*. To every event* $\varepsilon \in \mathcal{E}$ *there is only one unique* assigned drive activitiy $\text{drive}_\varepsilon \in \mathcal{A}$*, that is outgoing of* ε *if it is a* (dep) *or incoming, if* ε *is an* (arr)*. It inherits a lower bound* $l_a = l_{\text{edge}_a}$ *as well as an upper bound* $u_a = u_{\text{edge}_a}$ *from its assigned* $\text{edge}_a \in \vec{E}$ *and induces* $\text{edge}_\varepsilon := \text{edge}_{\varepsilon'} := \text{edge}_a$ *as the event's* assigned edge*. The* set of all drive activities *is denoted by* $\mathcal{A}_{\text{drive}}$*.*

Definition 1.39 (Wait Activity). *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. A* wait activity *leads from an arrival* $\varepsilon = (\text{arr}, \ell, s, i) \in \mathcal{E}$ *to a departure* $\varepsilon' = (\text{dep}, \ell, s, i) \in \mathcal{E}$ *of the same line* $\ell \in \vec{L}$ *at the same station* s *with same frequency instance* i*. The* set of all wait activities *is denoted by* $\mathcal{A}_{\text{wait}}$*.*

In our datasets, $l_a = l^{\text{wait}}$, $u_a = u^{\text{wait}}$ for all $a \in \mathcal{A}$, with $l^{\text{wait}} \leq u^{\text{wait}}$ being some arbitrary global constants, typically $l^{\text{wait}} = 1$ and $u^{\text{wait}} = 3$.

Definition 1.40 (Change Activity). *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. A* change activity *leads from an arrival* $\varepsilon = (\text{arr}, \ell, s, i) \in \mathcal{E}$ *to a departure* $\varepsilon' = (\text{dep}, \ell', s, j) \in \mathcal{E}$ *of different lines* $\ell, \ell' \in \vec{L}$*,* $\ell \neq \ell'$ *at the same station* $s \in S$ *but arbitrary frequency instances* $i \in \{0, \ldots, f_1 - 1\}, j \in \{0, \ldots, f_2 - 1\}$*. If for* $(\varepsilon_1, \varepsilon) := \text{drive}_\varepsilon$*,* $(s_1, \ell, \text{dep}, i) := \varepsilon$*,* $(\varepsilon', \varepsilon_2) := \text{drive}_{\varepsilon'}$*,* $(s_2, \ell', \text{arr}, j) := \varepsilon_2$ *holds* $s_1 = s_2$*, then the* change a *is called* local station loop*. The* set of all change activities *is denoted by* $\mathcal{A}_{\text{change}}$*.*

In our datasets, $l_a = l^{\text{change}}$, $u_a = u^{\text{change}}$ for all $a \in \mathcal{A}_{\text{change}}$, with $l^{\text{change}} \leq u^{\text{change}}$ being some arbitrary global constants, e.g. $l^{\text{wait}} = 4$ and $u^{\text{wait}} = T + 3$, where T is the period length.

To take a local station loop basically means that that one travels from some station s_1 to s and changes into some line that brings one back to station s_1. Since we consider traveling time only, such changes may be ignored[2].

[2]From a comfort point of view, this may be an issue if vehicles are air-conditioned and stations are not.

Definition 1.41 (Passenger Usable Activities)**.** *For an* EAN $= (\mathcal{E}, \mathcal{A})$ *the union of all drive, wait and change activities* $\mathcal{A}_{\text{drive}} \cup \mathcal{A}_{\text{wait}} \cup \mathcal{A}_{\text{change}}$ *is denoted by* $\mathcal{A}_p$ *and called the* set of passenger usable activities*.*

But there are also activities passengers can not utilize.

Definition 1.42 (Headway Activity)**.** *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. For every ordered pair of departures* $\varepsilon_1 = (\text{dep}, \ell, s, i)$*,* $\varepsilon_2 = (\text{dep}, \ell', s, j) \in \mathcal{E}$ *at the same station* $s \in S$ *of different lines* $\ell, \ell' \in \vec{L}$ *there is a* headway activity *a if the associated drive activities share the same edge, i.e.* $\text{edge}_{\text{drive}_{\varepsilon_1}} = \text{edge}_{\text{drive}_{\varepsilon_2}}$*. It inherits a lower bound* $l_a = h_{\text{edge}_a}$ *from its assigned* $\text{edge}_a \in \vec{E}$*. The* set of all headway activities *is denoted by* $\mathcal{A}_{\text{headway}}$*.*

Our headway definition is not the most general case, since we only require headways between departures. In practice, especially in train networks, there are also headways between arrivals and between departures and arrivals for single-way tracks.

To get more information about the influence of the frequency instances i and j, consult Sections 2.3.3 and 3.1.4.

Definition 1.43 (EAN Passenger Route)**.** *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. For a given pair of stations* $s_1, s_2 \in S$*, a* EAN passenger route *is a path from* s_1 *to* s_2 *in the* EAN *and can utilize passenger usable activities, i.e.* drive*,* wait *and* change*. Its first activity has to be a drive from* s_1 *and its last activity a drive to* s_2*.*

Definition 1.44 (PTN Passenger Route Trace)**.** *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept,* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network,* $s_1, s_2 \in S$ *a pair of stations and* $P \subset \mathcal{A}_p$ *an* EAN *passenger route. A* PTN passenger route trace *is the image* $P^{\text{PTN}} = \text{edge}(P \cap \mathcal{A}_{\text{drive}})$*, i.e. the set of edges in the* PTN *that the path* P *uses.*

Lemma 1.45. P^{PTN} *from Definition 1.44 contains a path from* s_1 *to* s_2*.*

Proof. Every passenger path in an EAN can either follow a line with drive and wait activities, of which the trace yield a path in the PTN or take a change and since both departure and arrival of every change must be at the same station, connectivity between s_1 and s_2 can not be broken. $\square$

Chapter 2

Classical Models

In this chapter we introduce actual models to get from a line concept to an event activity network, mostly relying on [Sch04] and perform timetable optimization, for which we use [Lie02]. The term *classical* refers to the popularity in teaching and research rather than to whether a model is state-of-the-art.

2.1 Passenger Load

Let a PTN $= (S, E)$ with lower time bounds l_e for all $e \in E$. For line planning as in the subsequent Section 2.2, for every edge $e \in E$ we need a to calculate the edge vehicle demand $f_e^{\text{low}} = \lceil p_e / \mathfrak{c}^{\text{Vehicle}} \rceil$ from Definition 1.31. Therefore, we need passenger loads p_e for all $e \in E$, as in [Sch04]. Without any line concept available, a first estimation is that passengers travel along the edges E of PTN on shortest paths with l_e being the weight for all $e \in E$.

2.2 Line Planning

In [Sch04] the author introduces a cost minimizing line concept linear formulation.

Linear Program 2.1 (Cost Minimizing Line Concept)**.** *Let a* PTN $= (S, E)$*, a line pool* $\mathfrak{L}$ *as well as costs* c_ℓ *for all* $\ell \in \mathfrak{L}$ *and a period length* T *be given.*

We use an inclusion representation

$$\boxed{\text{C}} \qquad \mathfrak{L} = (\mathfrak{l}_{e\ell})_{e \in E, \ell \in \mathfrak{L}} \ ,$$

$$\mathfrak{l}_{e\ell} = \begin{cases} 1 & \textit{line } \ell \textit{ contains link } e \\ 0 & \textit{otherwise} \end{cases} \ , \qquad \forall\, \ell \in \mathfrak{L},\ e \in E, \tag{2.1}$$

as well vehicle demands and maximal vehicle loads from definition 1.31

$$\boxed{\text{C}} \qquad f_e^{low},\ f_e^{up} \in [0, T] \cap \mathbb{Z} \tag{2.2}$$

and we want to determine the frequency

$$\boxed{\mathrm{V}} \qquad f_\ell \in [0, T] \cap \mathbb{Z} \quad , \qquad \forall\ \ell \in \mathfrak{L}. \tag{2.3}$$

Our objective is to minimize the cost

$$\boxed{\mathrm{L}} \qquad \min \sum_{\ell \in \mathfrak{L}} f_\ell c_\ell \tag{2.4}$$

while satisfying the edge bounds

$$\boxed{\mathrm{L}} \qquad f_e^{low} \leq \sum_{\ell \in \mathfrak{L}} \mathfrak{l}_{e\ell} f_\ell \leq f_e^{up} \qquad \forall\ e \in E. \tag{2.5}$$

Finding a feasible line concept for the Linear Program 2.1 is generally $\mathcal{NP}$-complete, since the exact cover by 3-sets problem may be reduced to it as shown in [Sch04]. However, without upper bounds feasibility can easily be checked by simply using all lines from $\mathfrak{L}$ with some global upper bound frequency.

2.3 Event Activity Network

Let a PTN $= (S, \vec{E})$ and a LC $= (\vec{L}, F)$ be given. Our *Event Activity Network Construction* consists of the following steps:

1. Roll Out Lines,
2. Generate Change Activities,
3. Generate Headways.

2.3.1 Lines Roll Out

In this step we construct an initial base for an event activity network: departures, arrivals, drive and wait activities for all lines in the line concept. Note that our public transportation network PTN $= (S, \vec{E})$ as well as line concept LC $= (\vec{L}, F)$ are supposed to be *directed* or in case they still are undirected need to be made directed.

The classical lines roll out model we call `frequency_as_attribute`, which iterates through every line and creates *one* arrival and *one* departure per edge, no matter what the frequency of that line is, see Algorithm 1 and Figure 2.1 for illustration.

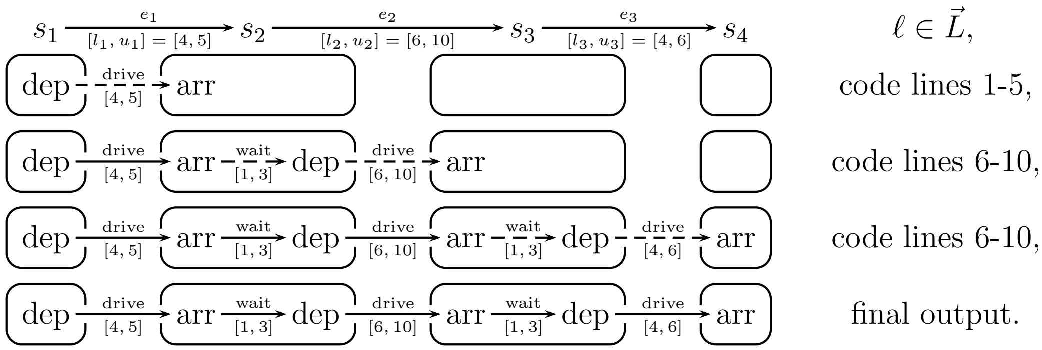


Figure 2.1: Illustration of `frequency_as_attribute` LINES ROLL OUT: line ℓ consists of three edges: e_1, e_2 and e_3 with bounds ranged in $[4,5]$, $[6,10]$ resp. $[4,6]$. The minimal and maximal waiting times are $l^{\text{wait}} = 1$ resp. $u^{\text{wait}} = 3$.

Algorithm 1 LINES ROLL OUT, `frequency_as_attribute`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- bounds $[l_e, u_e]$ for all $e \in \vec{E}$,
- bounds for wait activities $[l^{\text{wait}}, u^{\text{wait}}]$.

Output:

- EAN $= (\mathcal{E}, \mathcal{A}_{\text{drive}} \cup \mathcal{A}_{\text{wait}})$,
- bounds $[l_a, u_a]$ for all $a \in \mathcal{A}$.

1: **for all** $\ell = (e_1, \ldots, e_{n_\ell}) \in \vec{L}$ **do**
2:     **if** $f_\ell \neq 0$ **then**
3:         $(u, v) := e_1$
4:         Add $\varepsilon_1 := (u, \ell, \text{dep}, 0)$, $\varepsilon_2 := (v, \ell, \text{arr}, 0)$ to $\mathcal{E}$
5:         Add $a := (\varepsilon_1, \varepsilon_2, \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
6:         **for all** $e = (u, v) \in (e_2, \ldots, e_{n_\ell})$ **do**
7:             Add $\varepsilon_1 := (u, \ell, \text{dep}, 0)$ to $\mathcal{E}$
8:             Add $a := (\varepsilon_2, \varepsilon_1, \text{wait})$ to $\mathcal{A}$ with $[l_a, u_a] := [l^{\text{wait}}, u^{\text{wait}}]$
9:             Add $\varepsilon_2 := (v, \ell, \text{arr}, 0)$ to $\mathcal{E}$
10:            Add $a := (\varepsilon_1, \varepsilon_2, \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
11:        **end for**
12:    **end if**
13: **end for**

2.3.2 Change Activities

When two lines $\ell_1, \ell_2 \in \vec{L}$ cross at a station $s \in S$, passengers may use *change activities* to get from one line into another. Basically, we could allow all changes, but some do not make sense, e.g. the local station loops from Definition 1.40, therefore we skip them.

Note that the `simple` Generate Changes model inserts change activities also between different frequency instances. For the `frequency_as_attribute` Lines Roll Out model, there is only one frequency instance visible, however, for the `frequency_as_multiplicity` Lines Roll Out model to be introduced in Section 3.1.1, all are visible and thus `simple` has a different outcome then.

Algorithm 2 Generate Changes, `simple`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- EAN $= (\mathcal{E}, \mathcal{A})$ without changes,
- bounds for change activities $[l^{\text{change}}, u^{\text{change}}]$ (typically: $u^{\text{change}} = T - 1 + l^{\text{change}}$).

Output:

- EAN $= (\mathcal{E}, \mathcal{A})$ with changes.

for all $\varepsilon_1 = (s, \ell_1, \text{dep}, i), \varepsilon_2 = (s, \ell_2, \text{arr}, j) \in \mathcal{E}$ **do**
  $(\tilde{\varepsilon}_1, \varepsilon_1) := \text{drive}_{\varepsilon_1}$, $(s_1, \ell_1, \text{dep}, i) := \tilde{\varepsilon}_1$
  $(\varepsilon_2, \tilde{\varepsilon}_2) := \text{drive}_{\varepsilon_2}$, $(s_2, \ell_2, \text{arr}, j) := \tilde{\varepsilon}_2$
  if $s_1 \neq s_2$ (i.e. no local station loop changes) **then**
    Add $a := (\varepsilon_1, \varepsilon_2, \text{change})$ to $\mathcal{A}$ with $[l_a, u_a] := [l^{\text{change}}, u^{\text{change}}]$
  end if
end for

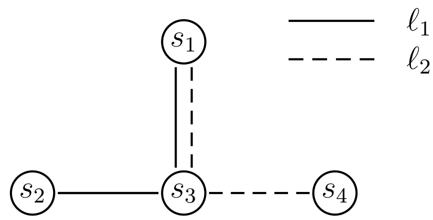
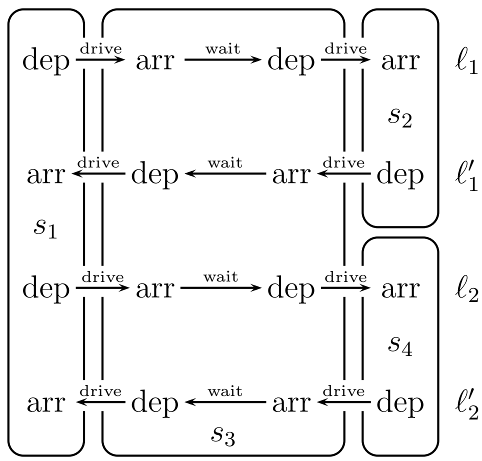
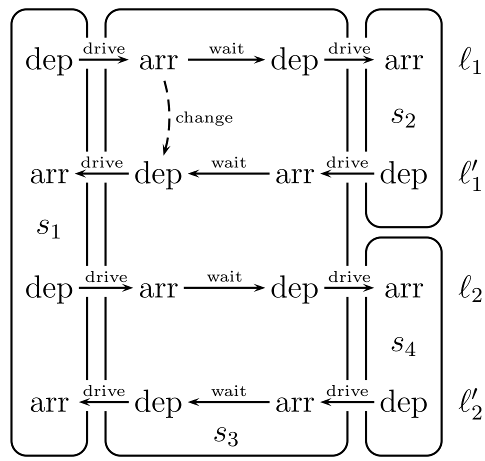


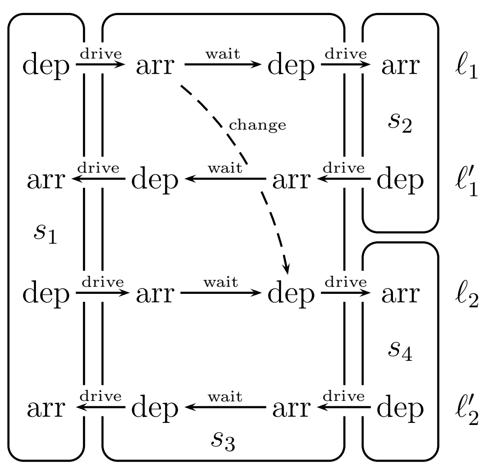
Figure 2.2: A public transportation network PTN $= (V, E)$ that has four stations $V = \{s_1, s_2, s_3, s_4\}$ and four edges $E = \{\overline{e_{13}}, \overline{e_{23}}, \overline{e}_{1\overline{3}}, \overline{e}_{3\overline{4}}\}$.



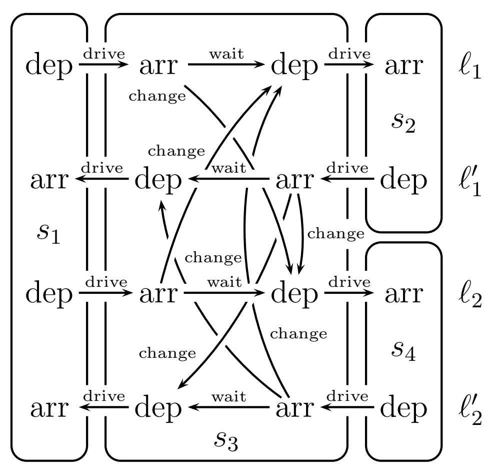
(a) After Lines Roll Out.



(b) Induces local loop to s_1.



(c) Code line 4 satisfied, no local loops induced.



(d) Final output after iteration trough all arrivals.

Figure 2.3: Illustration of Generate Changes for the PTN from Figure 2.2

2.3.3 Headways

Let PTN $= (S, \vec{E})$, T be a period length and h_e headways, for all $e \in \vec{E}$. We want to construct an EAN $= (\mathcal{E}, \mathcal{A})$ and because of periodicity and us working on representants, events cannot be ordered in time. Therefore we must ensure that between *every* pair of departures $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ which use the same edge $e = \text{edge}_{\varepsilon_1} = \text{edge}_{\varepsilon_2}$ headways are enforced: A timetable $\pi : \mathcal{E} \to \{0, \dots, T-1\}$ must thus satisfy

$$\pi_{\varepsilon_1} - \pi_{\varepsilon_2} \in [h_e, T-1]_T \qquad \text{and} \qquad \pi_{\varepsilon_2} - \pi_{\varepsilon_1} \in [h_e, T-1]_T \quad , \tag{2.6}$$

which is equivalent to define two durations $x, x' \in \{h, \dots, T-1\}$ as depicted in Figure 2.4. They form a single cycle (cycles in timetabling introduced in Section 2.5) that yields the constraint $x + x' = 0 \mod T$.

$$\begin{tikzcd} \mathrm{dep} \arrow[r, bend left, "{x \in [h_e, T-1]}"] & \mathrm{dep} \arrow[l, bend left, "{x' \in [h_e, T-1]}"] \end{tikzcd} \quad \mapsto \quad \mathrm{dep} \xrightarrow{x \in [h_e, T-h_e]} \mathrm{dep}$$

Figure 2.4: A Headway Reduction.

However, one duration does the trick as well:

Lemma 2.2. *Let $h, T \in \mathbb{N}$, $h < T$ and $x, x' \in \{h, \dots, T-1\}$ then the constraint $x + x' = 0 \mod T$ is equivalent to $x \in \{h, \dots, T-h\}$ and x' may be omitted.*

Proof. As a linear constraint $x + x' = 0 \mod T$ writes

$$x + x' = kT \quad , \qquad\qquad x, x' \in \{h, \dots, T-1\},\ k \in \mathbb{Z}. \tag{2.7}$$

Solving for k yields

$$\left\lceil \frac{2h}{T} \right\rceil = 1 \leq k \leq \left\lfloor 2 - \frac{2}{T} \right\rfloor = 1 \tag{2.8}$$

and thus $k = 1$. We further obtain

$$x = T - x' \quad , \qquad x' = T - x \tag{2.9}$$

and therefore, for each $x \in \{h, \dots, T-h\}$ there is an $x' \in \{h, \dots, T-h\}$ that satisfies $x + x' = 0 \mod T$ and x' may be omitted, since it passengers cannot use them and they thus do not occur in the timetabling objective function from equation (2.20) in section 2.5. □

It follows that algorithm 3 constructs the headways required, if all lines have frequency one and when using `frequency_as_attribute` for Lines Roll Out. We call it the Generate Headways model `simple`. In Section 3.1.4 we see that for frequencies greater than one `frequency_as_multiplicity` may be needed.

Algorithm 3 Generate Headways, `simple`

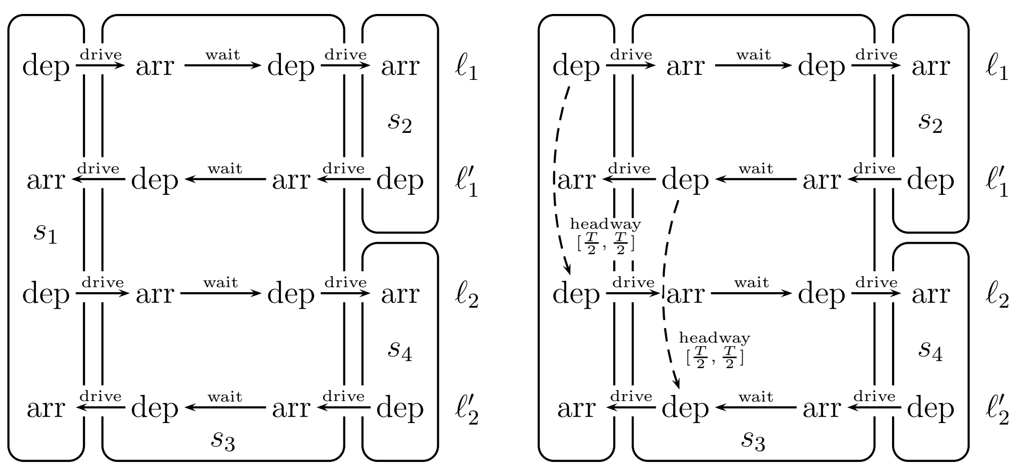
Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- EAN $= (\mathcal{E}, \mathcal{A})$ without headways,
- headways h_e for all $e \in \vec{E}$,
- period length T.

Output:

- EAN $= (\mathcal{E}, \mathcal{A})$ with headways.

for all $\varepsilon_1 = (s, \ell_1, \text{dep}, i), \varepsilon_2 = (s, \ell_2, \text{dep}, j) \in \mathcal{E}$ **do**
  edge := $\text{edge}_{\varepsilon_1}$
  if edge $= \text{edge}_{\varepsilon_2}$ **then**
    Add $a := (\varepsilon_1, \varepsilon_2, \text{headway})$ to $\mathcal{A}$ with $[l_a, u_a] := [h_{\text{edge}}, T - h_{\text{edge}}]$
  end if
end for



(a) After Lines Roll Out (b) After Generate Headways

Figure 2.5: Illustration of Generate Headways for the PTN from Figure 2.2.

2.4 Passenger Distribution

In this section we introduce both the concept of a passenger distribution as well as a method to compute one.

Definition 2.3 (Passenger Distribution). *Let* $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ *be an event activity network with* $\mathrm{PTN} = (S, E)$ *as an underlying public transportation network. A* passenger distribution $w : \mathcal{A} \to \mathbb{R}_{\geq 0}$*, assigns to every activity* $a \in \mathcal{A}$ *a* passenger weight $w_a \geq 0$ *that statisfy the property that it is the sum of coefficents from a linear combination of* EAN *passenger routes (Def. 1.43) from arbitrary tuples* $(s_1, s_2) \in S \times S$*. If for a given origin destination matrix* $\mathrm{OD} = (w_{s_1 s_2})_{s_1, s_2 \in S}$ *coefficients of the linear combination for every tuple* (s_1, s_2) *are* $w_{s_1 s_2}$*, then the passenger distribution is* derived from OD.

Indeed, we use the same letter w for both passenger distribution and origin destination matrix entry. However since former uses an activity and latter two stations as indices, we can always distinguish between them.

The requirement that the w_a should be linear combinations of EAN passenger routes in Definition 2.3 actually poses limitations to the possible passenger distributions, since every path must start and end with a drive activity and thus not all passenger weights yield passenger distributions as can be seen in Figure 2.6. Especially, $w_a = 0$ for all $\mathcal{A} \setminus \mathcal{A}_p$, i.e. those activities that are not passenger usable, since by Definition 1.43 these activities may not be passed.

$$(s_1, \ell, \mathrm{dep}) \xrightarrow[\text{drive}]{0} (s_2, \ell, \mathrm{arr}) \xrightarrow[\text{wait}]{1} (s_2, \ell, \mathrm{dep}) \xrightarrow[\text{drive}]{0} (s_3, \ell, \mathrm{arr})$$

Figure 2.6: Not a passenger distribution.

An evident way to compute an OD derived passenger distribution is to distribute passengers along shortest paths. Therefore let $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ be an event activity network derived from $\mathrm{PTN} = (S, \vec{E})$ and LC with passenger usable activities $\mathcal{A}_p = \mathcal{A}_{\text{drive}} \cup \mathcal{A}_{\text{wait}} \cup \mathcal{A}_{\text{change}} \subset \mathcal{A}$ and further $\mathrm{OD} = (w_{s_1 s_2})_{s_1 s_2 \in S}$. We introduce source and sink events as well as enter and leave activities

$$\mathcal{E}_{\text{source}} = \{\varepsilon^s_{\text{source}} : s \in S\} \quad , \tag{2.10}$$
$$\mathcal{E}_{\text{sink}} = \{\varepsilon^s_{\text{sink}} : s \in S\} \quad , \tag{2.11}$$
$$\mathcal{A}_{\text{enter}} = \{a = (\varepsilon^s_{\text{source}}, \varepsilon) : \varepsilon = (s, \ell, \mathrm{dep}),\ \ell \in L,\ s \in S\} \quad , \tag{2.12}$$
$$\mathcal{A}_{\text{leave}} = \{a = (\varepsilon, \varepsilon^s_{\text{sink}}) : \varepsilon = (s, \ell, \mathrm{arr}),\ \ell \in L,\ s \in S\} \quad , \tag{2.13}$$

the *passenger usable origin destination closure* $\overline{\mathrm{EAN}} = (\overline{\mathcal{E}}, \overline{\mathcal{A}})$

$$\overline{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_{\text{source}} \cup \mathcal{E}_{\text{sink}} \quad , \tag{2.14}$$

$$\overline{\mathcal{A}} = \mathcal{A}_p \cup \mathcal{A}_{\text{enter}} \cup \mathcal{A}_{\text{leave}} \tag{2.15}$$

and furthermore an *initial duration assumption*

$$x^{\text{init}} : \overline{\mathcal{A}} \to \mathbb{R}_{\geq 0} \quad . \tag{2.16}$$

We set $x_a^{\text{init}} = 0$, for all $a \in \mathcal{A}_{\text{enter}} \cup \mathcal{A}_{\text{leave}}$, since we discard enter and leave activities for timetabling anyway. We can obtain x_a^{init} from an arbitrary choice or some feasible timetable from Definition 2.5 and the passenger distribution by Algorithm 4.

Algorithm 4 PASSENGER DISTRIBUTION

Input:
- $\mathrm{PTN} = (S, E)$, $\mathrm{OD} = (w_{s_1 s_2})_{s_1, s_2 \in S}$, $\overline{\mathrm{EAN}} = (\overline{\mathcal{E}}, \overline{\mathcal{A}})$,
- initial duration assumption x_a^{init}, for all $a \in \overline{\mathcal{A}}$.

Output:
- passenger distribution $w_a \geq 0$, for all $a \in \overline{\mathcal{A}}$.

for all $a \in \mathcal{A} \cup \mathcal{A}_{\text{enter}} \cup \mathcal{A}_{\text{leave}}$ **do**
    $w_a := 0$
end for
for all $s_1 \in S$ **do**
    Compute shortest path tree t from $\varepsilon_{s_1}^{\text{source}}$ w.r.t. x_a^{init} on $\overline{\mathcal{A}}$
    for all $s_2 \in S$ **do**
        if $w_{s_1 s_2} > 0$ **then**
            Compute path p in t from $\varepsilon_{s_1}^{\text{source}}$ to $\varepsilon_{s_2}^{\text{sink}}$
            for all $a \in \overline{\mathcal{A}}$ **do**
                $w_a := w_a + p_a w_{s_1 s_2}$
            end for
        end if
    end for
end for

To save computing time, we reuse the shortest paths tree t instead of calculating the shortest paths for every pair $s_1, s_2 \in S$, since widely used methods compute that tree anyway or simply stop once s_2 is reached. Further, we do not need to calculate shortest paths for all pairs of events; passengers only travel between stations, shortest paths between source and sink events suffice. For large scale networks Bellman-Ford based methods like Floyd-Warshall need hours while Dijkstra based methods with Fibonacci Heaps only seconds, which should be considered in an actual implementation.

2.5 Timetabling

In this section, we introduce the linear program of the *Periodic Event Scheduling Problem* PESP.

In its original formulation by [Ser89] the PESP is about finding a feasible periodic timetable for all lines occuring within the period T exactly once and widely used according to [Lie02], from which wide parts of this section are taken. We adapt to our notation and add some small results we make use of in the chapters later. For different line frequencies, [Ser89] proposed the *Extended Periodic Event Scheduling Problem* EPESP, which we have a look at later in this section.

Throughout the section we assume that any event activity networks as graphs are connected. If they are not, our results still hold for connected components.

For our purposes, we add the objective to minimize the *average traveling time* which is basically the sum over activity duration times passengers that uses the activity[1] and when we refer to PESP we not only mean constraints, as some sources do, but include the average traveling time objective.

Linear Program 2.4 (Periodic Event Scheduling Problem PESP)**.** *Let T be a period length and* EAN $= (\mathcal{E}, \mathcal{A})$ *an event activity network with passenger distribution $w_a \geq 0$, $a \in \mathcal{A}$.*

We introduce times

$$\boxed{\mathsf{V}} \qquad \pi_\varepsilon \in [0, T-1] \cap \mathbb{Z} \quad , \qquad \forall\ \varepsilon \in \mathcal{E}, \tag{2.17}$$

and modulo parameters

$$\boxed{\mathsf{V}} \qquad z_a \in \mathbb{Z} \quad , \qquad \forall\ a \in \mathcal{A}, \tag{2.18}$$

want periodic interval constraints *or* time windows *to be satisfied*

$$\boxed{\mathsf{L}} \qquad l_a \leq \pi_{\varepsilon'} - \pi_\varepsilon + z_a T \leq u_a \qquad \forall\ a \in \mathcal{A} \tag{2.19}$$

and to minimize the average traveling time

$$\boxed{\mathsf{L}} \qquad \min \sum_{\substack{a \in \mathcal{A} \\ (\varepsilon, \varepsilon') = a}} w_a(\pi_{\varepsilon'} - \pi_\varepsilon + z_a T) \quad . \tag{2.20}$$

---

[1]Actually, the term *average* would imply that we take the average over some domain, e.g. divide by the total number of passengers in an additionally given OD matrix or the average in time over a longer term. However, we skip it since in our model the number of passengers is constant and the timetable keeps repeating.

Definition 2.5 (Feasible Timetable)**.** *Let T be a period length and* EAN $= (\mathcal{E}, \mathcal{A})$ *an event activity network. A timetable π_ε, $\varepsilon \in \mathcal{E}$ is called* feasible *if it is feasible for Linear Program 2.4 resp. the equivalent Linear Program 2.15.*

To illustrate what the PESP is about, take some activity $(\varepsilon, \varepsilon') = a \in \mathcal{A}$ and let us have a look at Inequation (2.19). Without the $z_a T$ term it writes

$$l_a \leq \pi_{\varepsilon'} - \pi_\varepsilon \leq u_a \quad , \tag{2.21}$$

which basically states that event ε' should take at least l_a and at most u_a time units after ε, e.g. $\pi_\varepsilon = 5$ and $\pi_{\varepsilon'} = 10$ would mean that ε' happens 5 time units after ε and which would be feasible for $l_a = 3$, but not for $l_a = 7$, analogously for upper bounds. However, in a periodic timetable ε and ε' represent *infinitely many* events that keep repeating every T and the interval constraints have to be satisfied for *just one tuple* of actual events being represented by ε and ε', e.g. if $T = 60$, then $\pi_\varepsilon = 57$ and $\pi_{\varepsilon'} = 2$ still satisfy the lower bound $l_a = 3$, since for $z = 1$ holds $2 - 57 + 60 = 5$.

In Equation (2.19) it looks like as if bounds like $[l_a, u_a] = [2, T + 1]$ do not make sense, since the constraint is always satisfied. However, note that it makes a difference in the objective function. Let $T = 60$, then for $(\varepsilon, \varepsilon') = a \in \mathcal{A}$ with $\pi_\varepsilon = 0$ and $\pi_{\varepsilon'} = 1$ and $w_a = 1$ it holds that

$$2 \leq 1 - 0 + z_a T \leq T + 1 \tag{2.22}$$

implies $z_a = 1$ and thus the objective gets an additional $T + 1$ while for $[l_a, u_a] = [1, T]$ we obtain

$$1 \leq 1 - 0 + z_a T \leq T \quad , \tag{2.23}$$

thus $z_a = 0$ and just an extra summand of 1. If a had been a change activity, then $l_a = 2$ means that passengers need at least two time units to change and if the time difference is 1, they cannot take the transfer and have to wait a whole period.

Note that (2.19) is actually equivalent to

$$\begin{gathered}\text{There must be some representant } x_a \text{ in} \\ x_a = \pi_{\varepsilon'} - \pi_\varepsilon \mod T \\ \text{for which holds} \\ x_a \in [l_a, u_a] \quad , \qquad \forall\, a \in \mathcal{A}.\end{gathered}$$

Theorem 2.6. *The* PESP *is $\mathcal{NP}$-complete.*

In their paper [Ser89] showed $\mathcal{NP}$-completeness for even finding a feasible solution to the PESP by reducing the Hamiltonian Circuit Problem to it. Although being $\mathcal{NP}$-complete, advances in solving constraint satisfaction problems allow to find feasible periodic timetables for large networks (80k activities) within seconds, e.g. with [Lec08]. This thus does not ease the task to find an optimal timetable. However, if one fixes the modulo parameters z_a, for all $a \in \mathcal{A}$ the problem transforms into an aperiodic event scheduling problem and is polynomially solveable, since its constraint matrix is totally unimodular [Sch04].

There are some meaningless time windows, i.e. if $u_a - l_a \geq T$, since then $u_a - l_a - \lfloor (u_a - l_a)/T \rfloor T$ would still be in $u_a - l_a$, thus the constraint is still satisfied and since $w_a \geq 0$ for all $a \in \mathcal{A}$ the simplication $u_a - l_a \leq T - 1$ does not worsen the objective function.

If $\mathcal{A}$ forms an undirected tree, PESP has a trivial solution by simply setting $z_a = 0$, for all $(\varepsilon, \varepsilon') = a \in \mathcal{A}$, $\pi_{\varepsilon_0} = 0$ for an arbitrary $\varepsilon_0 \in \mathcal{E}$ and determining the remaining $\pi_\varepsilon \in \mathcal{E}$ by summing up l_a along undirected paths, addition if the path has the same orientation as a, substraction otherwise. Therefore, cycles pose the actual challange, from which arises the next formulation. A function that maps from the vertices of a graph into the reals is often referred to as *potential*, like our timetable π. For a potential, one usually defines a *tension* as well.

Definition 2.7 (Tension, Feasible Durations). *A function $x : \mathcal{A} \to \mathbb{Z}$ is a* periodic tension with period T *if there is a potential π_ε, $\varepsilon \in \mathcal{E}$ and there are modulo parameters z_a, $a \in \mathcal{A}$ so that it holds*

$$x_a = \pi_{\varepsilon'} - \pi_\varepsilon + z_a T \quad , \qquad \forall \ (\varepsilon, \varepsilon') = a \in \mathcal{A}. \tag{2.24}$$

If the potential π is a timetable, then x_a, $a \in \mathcal{A}$ is called duration of activity a. *If π is feasible and for x_a holds $l_a \leq x_a \leq u_a$, for all $a \in \mathcal{A}$ then x_a, $a \in \mathcal{A}$ are* feasible durations *or* feasible duration set.

Lemma 2.8 (Modulo Parameter Uniqueness). *For a given feasible timetable π modulo parameters and durations are unique iff for every activity $a \in \mathcal{A}$ it holds $u_a - l_a \leq T - 1$.*

Proof. Apply Lemma 1.2, where a and b are two durations for an activity. □

Therefore, assuming $u_a - l_a \leq T - 1$ for all $a \in \mathcal{A}$ is not only a lossless simplification w.r.t. the PESP objective, but also allows us to reconstruct modulo parameters and durations from a feasible timetable π[2].

Definition 2.9 (Derived Timetable). *Let x_a, $a \in \mathcal{A}$ be feasible durations. A timetable that is a potential π for x is called* periodic tension derived timetable *or* derived timetable *for short.*

A derived timetable is not unique since all times may be shifted by T and may be obtained by setting $\pi_\varepsilon = 0$ for some $\varepsilon \in \mathcal{E}$ and summing up durations along a spanning tree, as in the proof of Theorem 2.14.

[2] Note that this is not possible if for some $(\varepsilon, \varepsilon') = a \in \mathcal{A}$ it holds $u_a - l_a \geq T$, e.g. $\pi_\varepsilon = 0$ and $\pi_{\varepsilon'} = 1$ could then either mean $x_a = 1$ or $x_a = T + 1$.

Definition 2.10 (Derived Modulo Parameters)**.** *Let π be as in Lemma 2.8. The unique modulo parameters that belong to π are called* derived modulo parameters*. If $u_a - l_a \leq T - 1$ for all $a \in \mathcal{A}$ is not explicitly stated, then it is assumed to hold.*

Definition 2.11 (Derived Durations)**.** *Let π be as in Definition 2.10 with derived modulo parameters z_a, $a \in \mathcal{A}$. Then $x_a = \pi_{\varepsilon'} - \pi_\varepsilon + z_a T$, $a \in \mathcal{A}$ are called* durations derived from π*.*

Note that if $u_a - l_a \leq T - 1$ for all $a \in \mathcal{A}$ does not hold derived durations from a derived timetable may differ from the original durations[3].

This allows us to formally define an average traveling time that depends only on the timetable and passenger distribution but not on the modulo parameters.

Definition 2.12 (Average Traveling Time)**.** *Let T be a period length,* EAN $=$ $(\mathcal{E}, \mathcal{A})$ *and π_ε, $\varepsilon \in \mathcal{E}$ be a feasible (derived) timetable with derived durations x_a, as well as $w_a \geq 0$, $a \in \mathcal{A}$ a passenger distribution. Then*

$$\mathrm{ATT}_w^\pi := \mathrm{ATT}_w^x := \sum_{a \in \mathcal{A}} w_a x_a \tag{2.25}$$

is called the average traveling time w.r.t. π and w *resp.* w.r.t. x and w.

Corollary 2.13 (PESP Average Traveling Time)**.** *The* PESP *minimizes the average traveling time* ATT_w^π *for a fixed passenger distribution w.*

Proof. As per definition of the PESP in Linear Program 2.4. □

Let us get back to the potentials and tensions: Every solution of the PESP is a potential and therefore yields a periodic tension. However, to really profit from it we also have to be able to do the reverse, i.e. checking whether a function is a periodic tension without having to solve the PESP, for which we formulate a necessary and sufficient condition.

A cycle $C \in \mathfrak{C}$, where $\mathfrak{C}$ denotes the set of all cycles in $\mathcal{A}$ is said to have the *cycle periodicity property* if it satisfies

$$\exists\, q_C \in \mathbb{Z} : \quad \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = T q_C \quad . \tag{2.26}$$

Theorem 2.14. *Let T be a period length and* EAN $= (\mathcal{E}, \mathcal{A})$ *and $\mathfrak{C}$ the set of cycles in $\mathcal{A}$. A function $\pi : \mathcal{A} \to \mathbb{R}$ is a periodic tension with period T if and only if each cycle $C = C^+ \cup C^- \in \mathfrak{C}$ has the cycle periodicity property.*

[3] Same reason as for modulo parameters being ambiguous: e.g. if $x_a = 61$, $T = 60$, $x_a \in [1, 61]$. A derived timetable π could look like $\pi_\varepsilon = 0$, $\pi_{\varepsilon'} = 1$ for $(\varepsilon, \varepsilon') = a$. However, after deriving x from π it may hold $x_a = 1$.

Proof from [Nac94]. "$\Rightarrow$": Let $x : \mathcal{A} \to \mathbb{R}$ be a periodic tension. Taking the sum of values $x_a = \pi_{\varepsilon'} - \pi_\varepsilon + T z_a$ along a cycle according to orientation, the π_ε cancel out, which yields

$$\sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = \sum_{a \in C^+} z_a T - \sum_{a \in C^-} z_a T \tag{2.27}$$

$$= T\left(\sum_{a \in C^+} z_a - \sum_{a \in C^-} z_a\right) =: T q_C \quad , \tag{2.28}$$

where q_C is called the *modulo parameter of* C and since $T q_C$ is an integral multiple of T, one way of the equivalence follows.

"$\Leftarrow$": Suppose x_a, $a \in \mathcal{A}$ satisfies

$$\sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = T q_C, \qquad \forall\ C \in \mathcal{A}. \tag{2.29}$$

To construct a corresponding solution (π, z) for the PESP, choose an arbitrary spanning tree $H \subset \mathcal{A}$ and set $z_a = 0$, for all $a \in H$ and some arbitrary $\varepsilon_0 \in \mathcal{E}$, for which $\pi_{\varepsilon_0} := 0$. For all other events $\varepsilon \in \mathcal{E}$, $\varepsilon \neq \varepsilon_0$ set

$$\pi_\varepsilon = \sum_{a \in P^+_{\varepsilon_0 \varepsilon}} x_a - \sum_{a \in P^-_{\varepsilon_0 \varepsilon}} x_a \quad , \tag{2.30}$$

with $P_{\varepsilon_0 \varepsilon}$ being the path from ε_0 to ε w.r.t. H. A tree edge $(\varepsilon, \varepsilon') = a \in H$, π thus satisfies

$$\pi_{\varepsilon'} - \pi_\varepsilon = \sum_{a' \in P^+_{\varepsilon_0 \varepsilon'}} x_{a'} - \sum_{a' \in P^-_{\varepsilon_0 \varepsilon'}} x_{a'} - \left(\sum_{a' \in P^+_{\varepsilon_0 \varepsilon}} x_{a'} + \sum_{a' \in P^-_{\varepsilon_0 \varepsilon}} x_{a'}\right) = x_a \quad , \tag{2.31}$$

since both paths $P_{\varepsilon_0 \varepsilon'}$ and $P_{\varepsilon_0 \varepsilon}$ just differ in a. For a non tree edge $(\varepsilon, \varepsilon') = a \in \mathcal{A} \setminus H$, adding a to H creates a cycle C. If C contains ε_0

$$x_a + \left(\sum_{a' \in P^+_{\varepsilon_0 \varepsilon}} x_{a'} - \sum_{a' \in P^-_{\varepsilon_0 \varepsilon}} x_{a'}\right) - \left(\sum_{a' \in P^+_{\varepsilon_0 \varepsilon'}} x_{a'} - \sum_{a' \in P^-_{\varepsilon_0 \varepsilon'}} x_{a'}\right) = T q_C \quad , \tag{2.32}$$

since C consists of $P_{\varepsilon_0 \varepsilon}$, $a = (\varepsilon, \varepsilon')$, $P_{\varepsilon' \varepsilon_0}$. If C does not contain ε_0, then the common part of $P_{\varepsilon_0 \varepsilon}$ and $P_{\varepsilon' \varepsilon_0}$ cancels out in the above expression. Therefore, it holds $x_a - \pi_\varepsilon + \pi_{\varepsilon'} = T q_C$ and setting $p_a = q_C$ therefore yields $x_a = \pi_{\varepsilon'} - \pi_\varepsilon = T q_C$. $\square$

The method with the spanning tree from Theorem 2.14 allows us to obtain a feasible timetable π from durations, given they are a periodic tension.

As a consequence of Theorem 2.14 we can introduce the *Cyclic Periodicity Formulation* CPF as an equivalent alternative to PESP.

Linear Program 2.15 (Cyclic Periodicity Formulation CPF)**.** *Let a* PTN $= (S, E)$ *and an* EAN $= (\mathcal{E}, \mathcal{A})$ *be given, as well as* $\mathfrak{C}$ *the set of all cycles in* $\mathcal{A}$ *and* $w_a \geq 0$ *a passenger distribution,* $a \in \mathcal{A}$*.*

We introduce durations

$$\boxed{\text{V}} \qquad x_a \in [l_a, u_a] \cap \mathbb{Z} \quad , \qquad \forall\, a \in \mathcal{A}. \tag{2.33}$$

We want to minimize the average traveling time

$$\boxed{\text{L}} \qquad \min \sum_{a \in \mathcal{A}} w_a x_a \tag{2.34}$$

subject to periodicity being satisfied

$$\boxed{\text{L}} \qquad \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = z_C T \quad , \qquad \forall\, C \in \mathfrak{C}. \tag{2.35}$$

The PESP has an obvious lower bound, which is $\sum_{a \in \mathcal{A}} w_a l_a$ and can be obtained by setting all durations to their lower bounds.

Definition 2.16. *Let* $x_a \in [l_a, u_a]$*,* $a \in \mathcal{A}$ *be durations and* w_a *a passenger distribution,* $a \in \mathcal{A}$*. The* slack of a *is defined by* $x_a - l_a$*. The* weighted slack sum *is*

$$\sum_{a \in \mathcal{A}} w_a (x_a - l_a) \quad . \tag{2.36}$$

We call the weighted slack sum simply *slack* as well. It cannot be mixed up since one refers to an activity and the other to a whole PESP.

Lemma 2.17. *Minimizing the weighted slack sum is equivalent to minimize the* PESP *objective.*

Proof. Since the CPF is equivalent to PESP, take (2.34) as objective. It holds

$$\sum_{a \in \mathcal{A}} w_a x_a = \sum_{a \in \mathcal{A}} w_a (x_a - l_a) + \sum_{a \in \mathcal{A}} w_a l_a \quad , \tag{2.37}$$

where $\sum_{a \in \mathcal{A}} w_a l_a$ is constant and thus does not affect minimization. ☐

Lemma 2.18 (PESP Lower Bound)**.** *For the* PESP*,*

$$l_{\text{PESP}} = \sum_{a \in \mathcal{A}} w_a l_a \tag{2.38}$$

is a lower bound.

Proof. Since every term in $\sum_{a\in\mathcal{A}} w_a(x_a - l_a)$ is nonnegative, the PESP objective cannot be less than l_{PESP}. □

In the PESP, if the weighted slack sum is zero, it is impossible to further improve the average traveling time, but for variable passenger distributions this is delusive as can be seen in Section 5.6.

However, there is still a significant drawback in the CPF: for nontrivial graphs, the number of cycles in a graph grows exponentially with the number of activities, e.g. the complete graph on n nodes has $\sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2}$ simple cycles and therefore we get exponentially many variables q_C. For comparison, the PESP only has $m = |\mathcal{A}|$ time windows and modulo parameters. From the set of all cycles, some may always be removed.

Definition 2.19 (Period Spanning Activity)**.** *An activity $a \in \mathcal{A}$* spans a period *if $u_a - l_a \geq T - 1$.*

Period spanning activities may still play a role when it comes to objective functions, but may be ignored when searching for a feasible timetable.

Lemma 2.20. *Let* EAN $= (\mathcal{E}, \mathcal{A})$ *be an event activity network, T a period length and $a \in \mathcal{A}$ a period spanning activity as well as π a feasible timetable for* EAN$' = (\mathcal{E}, \mathcal{A} \setminus \{a\})$*. Then π is also feasible for* EAN*.*

Proof. With $u_a = T - 1$ the the constraint induced by $(\varepsilon, \varepsilon') = a$ is the tightest

$$\exists\, k \in \mathbb{Z}: \qquad \pi_\varepsilon - \pi_{\varepsilon'} + kT \in [l_a, l_a + T - 1] \cap \mathbb{Z} \quad , \tag{2.39}$$

which is equivalent to

$$\exists\, k \in \mathbb{Z}: \qquad \pi_\varepsilon - \pi_{\varepsilon'} + kT - l_a \in [0, T - 1] \cap \mathbb{Z} \quad , \tag{2.40}$$

with $u_a \geq l_a + T - 1$. Let $\Delta = \pi_\varepsilon - \pi_{\varepsilon'} - l_a \in \mathbb{Z}$. By division with remainder $\Delta = \overline{k}T + \delta$, where $\overline{k} \in \mathbb{Z}$ and $\delta \in \{0, \ldots, T - 1\}$. Thus

$$\exists\, k \in \mathbb{Z}: \qquad \pi_\varepsilon - \pi_{\varepsilon'} + kT = (k + \overline{k})T + \delta \in [0, T - 1] \cap \mathbb{Z} \tag{2.41}$$

may be satisfied with $k = -\overline{k}$. □

For example, change activities a with $l_a = 4$ and $u_a = 63$ are periodically unbounded for $T = 60$.

Definition 2.21. *A* trivial cycle *is a cycle that contains a period spanning activity $a \in \mathcal{A}$ as in Definition 2.19 with no passengers using it, i.e. $w_a = 0$ in the objective function. All other cycles are called* nontrivial cycles*.*

Lemma 2.22. *Trivial cycles may be ignored in* CPF*.*

Proof. Since $w_a = 0$ the activity a has only influence on feasibility, but not on the objective. However, with Lemma 2.20 it has even no influence on feasibility as well. Therefore, it may be removed from $\mathcal{A}$ and every cycle that contains a gets broken that way, so it does not occur in the constraints (2.35). □

For some small networks we consider in Chapter 5 there is usually only one nontrivial cycle due to the passenger distribution, so that we can apply the CPF directly.

Let us establish an estimation for q_C.

Theorem 2.23 (Theorem of Odijk [Odi96])**.** *A* PESP *instance defined by a given* EAN $= (\mathcal{E}, \mathcal{A})$ *and period length T is feasible if and only if there exists an integer vector z_a, $a \in \mathcal{A}$ that satisfies the cycle inequalities*

$$a_C \leq \sum_{a \in C^+} z_a - \sum_{a \in C^-} z_a \leq b_C \tag{2.42}$$

for all (simple) cycles $C \in \mathfrak{C}$, where $\mathfrak{C}$ denotes the set of all cycles in $\mathcal{A}$ and a_C and b_C are defined by

$$a_C = \left\lceil \frac{1}{T} \left(\sum_{a \in C^+} l_a - \sum_{a \in C^-} u_a \right) \right\rceil \quad , \quad b_C = \left\lfloor \frac{1}{T} \left(\sum_{a \in C^+} u_a - \sum_{a \in C^-} l_a \right) \right\rfloor \quad . \tag{2.43}$$

Lemma 2.24 (Lemma of Odijk)**.** *Let a_C and b_C be from Theorem 2.23. In equation* (2.35) *the cycle periodicity variables q_C are bounded by $a_C \leq q_C \leq b_C$.*

Proof. In the constraint (2.35) divide by T. The feasible time windows $[l_a, u_a]$ of x_a may be used to get a lower and upper bounds for z_C. Due to the integrality of z_C, floor and ceil may be applied to the bounds and yield a_C and b_C. □

We also use Lemma 2.24 extensively in Chapter 5, but even for the big issue with exponentially many variables there is a solution: instead of taking all cycles, one can use an *integral cycle base* that reduces the number of cycle to $k = |E| - |V| + 1$ for an directed, connected graph $G = (V, E)$. However, although the implementation makes use of it, we do not use it directly in this work and therefore skip it. For a brief review see [Lie02] and for a broader survey see [KLM$^+$09].

Let us have a look at the EPESP announced at the beginning of the section. Introduced by [Ser89], especially [Nac96] is cited by different authors. As for PESP, the EPESP is only about feasibility in its original form, but when we refer to it we think of minimizing some sort of average traveling time as objective, which we detailed out in Section 3.1.

Linear Program 2.25 (Extended Periodic Event Scheduling Problem EPESP)**.** *Let* EAN $= (\mathcal{E}, \mathcal{A})$ *be an event activity network with* activity periods T_a *and passenger distribution* $w_a \geq 0$*,* $a \in \mathcal{A}$*.*

We introduce times

$$\boxed{\mathsf{V}} \qquad \pi_e \in [0, T-1] \cap \mathbb{Z} \quad , \qquad \forall\ e \in \mathcal{E}, \tag{2.44}$$

and modulo parameters

$$\boxed{\mathsf{V}} \qquad z_a \in \mathbb{Z} \quad , \qquad \forall\ a \in \mathcal{A}, \tag{2.45}$$

want periodic interval constraints to be satisfied

$$\boxed{\mathsf{L}} \qquad l_a \leq \pi_{\varepsilon'} - \pi_\varepsilon + z_a T_a \leq u_a \qquad \forall\ a \in \mathcal{A} \tag{2.46}$$

and to minimize the average traveling time

$$\boxed{\mathsf{L}} \qquad \min \sum_{\substack{a \in \mathcal{A} \\ (\varepsilon, \varepsilon') = a}} w_a(\pi_{\varepsilon'} - \pi_\varepsilon + z_a T_a) \quad . \tag{2.47}$$

If $T_a = T$, for all $a \in \mathcal{A}$ for some T, then the EPESP obviously transforms into PESP. However, the question arisies what different T_a may even mean. Again answer can be found in Section 3.1. The intention of the EPESP is to circumvent the necessity of the `frequency_as_multiplicity` Lines Roll Out model to be introduced in Section 3.1.1, therefore it only make sense with the already introduced `frequency_as_attribute` model.

Chapter 3

Beyond Classical Models

In this chapter we mainly deal with how frequencies may be considered in event activity networks and how passenger routing affects timetabling. Therefore, we not only introduce new models but also compare with the EPESP, which as a classical model is intended to account for frequencies.

The term *beyond* refers to extensions the author made during his work on LinTim, which is based on classical models from [Sch04].

Some results the author has been able to find in literature as well, e.g. in the widely cited [Nac96], from which he took the results for 3.1.3 and wants to thank Marie Schmidt for providing him [Kin08] and [Lue09], who worked on integrating timetabling and rerouting as well and introduced models similar to ours. However, they did neither take different line frequencies nor headways into account. The author aquired [Nac98] when Section 3.1.4 had already been written, which he does not consider as a disadvantage, since he could use the number theoretic methods he deployed for Chapter 5 as well.

3.1 Event Activity Network

This section introduces the `frequency_as_multiplicity` Lines Roll Out model, the Periodic Rollout transformation that allows to map timetables from the classical `frequency_as_attribute` model, an estimation for the best change between two lines given a feasible timetable as well as three headway models to fix a feasibility issue.

3.1.1 Frequency as Multiplicity

We introduce a new kind of activity.

Definition 3.1 (Sync Activity). *Let* PTN $= (S, \vec{E})$ *be a public transportation network,* LC $= (\vec{L}, F)$ *be a line concept and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network. A* sync activity *connects two departures* $\varepsilon_1 = (\mathrm{dep}, \ell, s, i), \varepsilon_2 = (\mathrm{dep}, \ell, s, i+1) \in \mathcal{E}$ *at the same station* s *that use the same line* ℓ *and belong to* *consecutive frequency instances, for all $i \in \{0, \dots, f_\ell - 2\}$ and has a lower and upper bound $l_a = u_a = T/f_\ell$.*

The sync activities ensures that departures of the same line are equally distributed throughout the period. Note that it is not necessary to connect instances $f_\ell - 1$ and 0, since $f_\ell | T$ by Definition 1.33.

In the `frequency_as_attribute` model from Section 2.3.1 we did not account for frequencies. Now, we do so by iterating through every line $\ell \in \vec{L}$ and create f_ℓ arrivals and f_ℓ departures per edge, i.e. as many as the frequency f_ℓ and add sync activities in between, see algorithm 5 and Figure 3.1 for illustration[1].

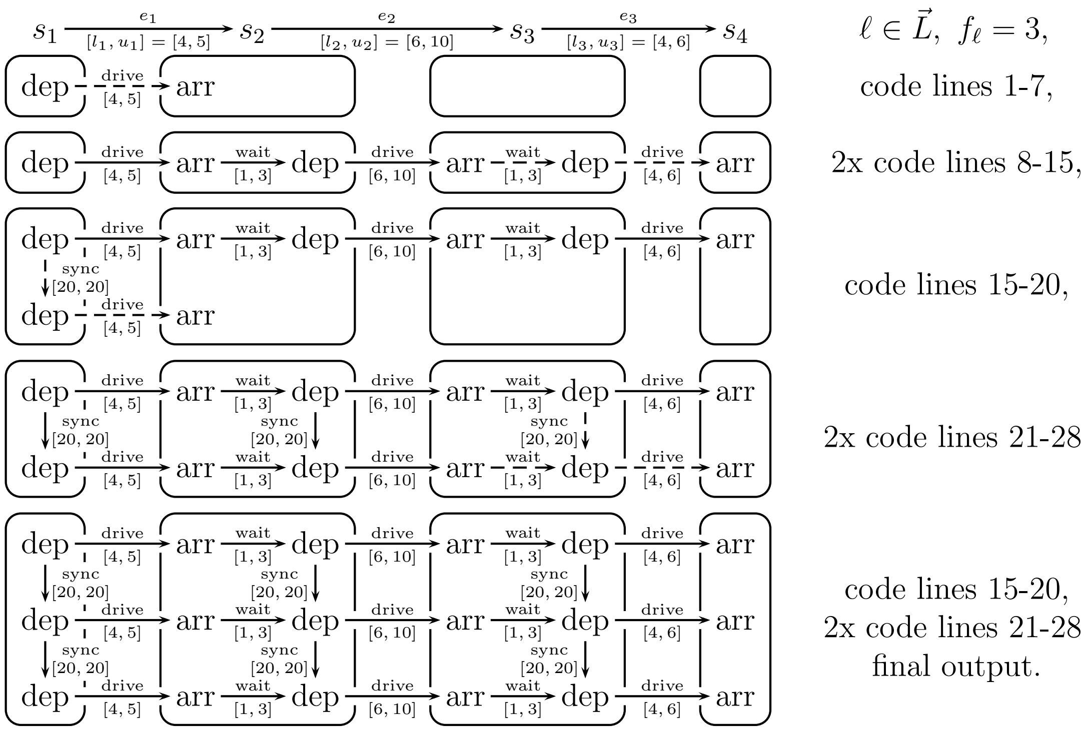


Figure 3.1: Illustration of `frequency_as_multiplicity` LINES ROLL OUT.

[1]The `frequency_as_attribute` model is not to be confused with the X-PESP model from [Kin08]. In his model, the network size grows with the period length while our network size grows with the line frequencies.

Algorithm 5 LINES ROLL OUT, `frequency_as_multiplicity`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- bounds $[l_e, u_e]$ for all $e \in E$,
- bounds for wait activities $[l^{\text{wait}}, u^{\text{wait}}]$,
- period length T.

Output:

- EAN $= (\mathcal{E}, \mathcal{A} := \mathcal{A}_{\text{drive}} \cup \mathcal{A}_{\text{wait}} \cup \mathcal{A}_{\text{sync}})$,
- bounds $[l_a, u_a]$ for all $a \in \mathcal{A}$.

for all $\ell = (e_1, \ldots, e_{n_\ell}) \in \vec{L}$ **do**
  if $f_\ell \neq 0$ **then**
    $m : \{1, \ldots, n_\ell\} \to \mathcal{E}$
    $(s, s') := e_1$
    Add $\varepsilon := (s, \ell, \text{dep}, 0)$, $\varepsilon' := (s', \ell, \text{arr}, 0)$ to $\mathcal{E}$
    $m_1 := \varepsilon_1$
    Add $a := (\varepsilon, \varepsilon', \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
    for all $e_i = (s, s') \in (e_2, \ldots, e_{n(\ell)})$ **do**
      Add $\varepsilon := (s, \ell, \text{dep}, 0)$ to $\mathcal{E}$
      Add $a := (\varepsilon', \varepsilon, \text{wait})$ to $\mathcal{A}$ with $[l_a, u_a] := [l^{\text{wait}}, u^{\text{wait}}]$
      $m_i := \varepsilon$
      Add $\varepsilon' := (s', \ell, \text{arr}, 0)$ to $\mathcal{E}$
      Add $a := (\varepsilon, \varepsilon', \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
    end for
    for all $j \in (1, \ldots, f_\ell - 1)$ **do**
      $(s, s') := e_1$
      Add $\varepsilon := (s, \ell, \text{dep}, j)$ $\varepsilon' := (s', \ell, \text{arr}, j)$ to $\mathcal{E}$
      Add $a := (m_1, \varepsilon, \text{sync})$ to $\mathcal{A}$ with $[l_a, u_a] := [T/f_\ell, T/f_\ell]$
      $m_1 := \varepsilon$
      Add $a := (\varepsilon, \varepsilon', \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
      for all $e_i = (s, s') \in (e_2, \ldots, e_{n(\ell)})$ **do**
        Add $\varepsilon := (s, \ell, \text{dep}, j)$ to $\mathcal{E}$
        Add $a := (\varepsilon', \varepsilon, \text{wait})$ to $\mathcal{A}$ with $[l_a, u_a] := [l^{\text{wait}}, u^{\text{wait}}]$
        Add $a := (m_i, \varepsilon, \text{sync})$ to $\mathcal{A}$ with $[l_a, u_a] := [T/f_\ell, T/f_\ell]$
        $m_i := \varepsilon$
        Add $\varepsilon' := (s', \ell, \text{arr}, j)$ to $\mathcal{E}$
        Add $a := (\varepsilon, \varepsilon', \text{drive})$ to $\mathcal{A}$ with $[l_a, u_a] := [l_e, u_e]$
      end for
    end for
  end if
end for

3.1.2 Periodic Rollout

An $\mathrm{EAN}_{\mathrm{FA}} = (\mathcal{E}_{\mathrm{FA}}, \mathcal{A}_{\mathrm{FA}})$ constructed with the `frequency_as_attribute` frequency model seems more attractive for periodic timetabling since it yields less variables and constraints than an event activity network with Lines Roll Out model `frequency_as_multiplicity`. However, the former is less complete and thus there are some pitfalls as can be seen in Chapter 5, where we extensively use the concept of a *periodic rollout* or just *rollout* for short, which constructs an $\mathrm{EAN}_{\mathrm{FM}} = (\mathcal{E}_{\mathrm{FM}}, \mathcal{A}_{\mathrm{FM}})$ and especially maps the timetable π^{FA} into π^{FM} with Algorithm 6.

Algorithm 6 Periodic Rollout

Input:
- $\mathrm{PTN} = (S, \vec{E})$, $\mathrm{LC} = (\vec{L}, F)$, $\mathrm{EAN}_{\mathrm{FA}}(\mathcal{E}_{\mathrm{FA}}, \mathcal{A})$,
- bounds $[l_e, u_e]$ for all $e \in E$, wait activity bounds $[l^{\mathrm{wait}}, u^{\mathrm{wait}}]$,
- period length T, timetable π^{FA}.

Output:
- $\mathrm{EAN}_{\mathrm{FM}} = (\mathcal{E}_{\mathrm{FM}}, \mathcal{A}')$, timetable π^{FM},
- bounds $[l_a, u_a]$ for all $a \in \mathcal{A}$.

```
run LINES ROLL OUT, frequency_as_multiplicity (Algorithm 5)
for all (s, ℓ, arr / dep, i) = ε ∈ E_FM do
    π^FM_ε := π^FM_(s,ℓ,arr / dep,i) + i T/f_ℓ
end for
run GENERATE CHANGES (Algorithm 2)
run GENERATE HEADWAYS (Algorithm 3)
```

We may take over an old passenger distribution w by *embedding* it as $\overline{w}$ into $\mathrm{EAN}_{\mathrm{FM}}$

$$\overline{w}_a = \begin{cases} w_a & a \in \mathcal{A}_{\mathrm{FA}} \\ 0 & \text{otherwise} \end{cases}, \qquad \forall\, a \in \mathcal{A}_{\mathrm{FM}}, \tag{3.1}$$

where $(\varepsilon, \varepsilon') = a \in \mathcal{A}_{\mathrm{FA}} \cap \mathcal{A}_{\mathrm{FM}}$ if both ε and ε' have a frequency instance of zero.

However, redistributing passengers in the rolled out network generally decreases the average traveling time as can be observed in Chapter 6 or at least never increases it.

Theorem 3.2 (Periodic Rollout ATT)**.** *Let* $\mathrm{EAN}_{\mathrm{FA}} = (\mathcal{E}_{\mathrm{FA}}, \mathcal{A}_{\mathrm{FA}})$ *be an event activity network on* $\mathrm{PTN} = (S, \vec{E})$ *with* Lines Roll Out *model* ***frequency_as_attribute*** *and feasible timetable* π_{FA}*,* $\mathrm{OD} = (w_{s_1 s_2})_{s_1 s_2 \in S}$ *an origin destination matrix and* w_a*,* $a \in \mathcal{A}$ *an* OD *derived passenger distribution as well as* $\mathrm{EAN}_{\mathrm{FM}} = (\mathcal{E}_{\mathrm{FM}}, \mathcal{A}_{\mathrm{FM}})$ *the* Periodic Rollout *of* $\mathrm{EAN}_{\mathrm{FA}}$ *with timetable* π_{FM} *and* w'_a *obtained by* Passenger Distribution *w.r.t.* π_{FM} *derived durations as shortest paths weights. Then it holds*

$$\mathrm{ATT}_{w'}^{\pi_{\mathrm{FM}}} \leq \mathrm{ATT}_{w}^{\pi_{\mathrm{FA}}} \quad . \tag{3.2}$$

Proof. Since $\mathrm{EAN}_{\mathrm{FM}}$ with π_{FM} derived durations x_{FM} contains $\mathrm{EAN}_{\mathrm{FA}}$ with π_{FA} derived durations x_{FA}, for the embedded distribution $\overline{w}$ it holds

$$\mathrm{ATT}_{\overline{w}}^{\pi_{\mathrm{FM}}} = \mathrm{ATT}_{w}^{\pi_{\mathrm{FA}}} \quad , \tag{3.3}$$

where ATT is the average traveling time from Definition 2.12. By the upcoming Corollary 3.18, replacing w by w' does not increase the average traveling time and thus

$$\mathrm{ATT}_{w'}^{\pi_{\mathrm{FM}}} \leq \mathrm{ATT}_{\overline{w}}^{\pi_{\mathrm{FM}}} = \mathrm{ATT}_{w}^{\pi_{\mathrm{FA}}} \quad . \tag{3.4}$$

□

3.1.3 Change Activities

After a Periodic Rollout not only departures, but also arrivals are synchronized by construction of the rolled out timetable. This allows us to estimate the duration of the newly introduced inter frequency changes, since they form a nice pattern as can be seen in Figure 3.2.

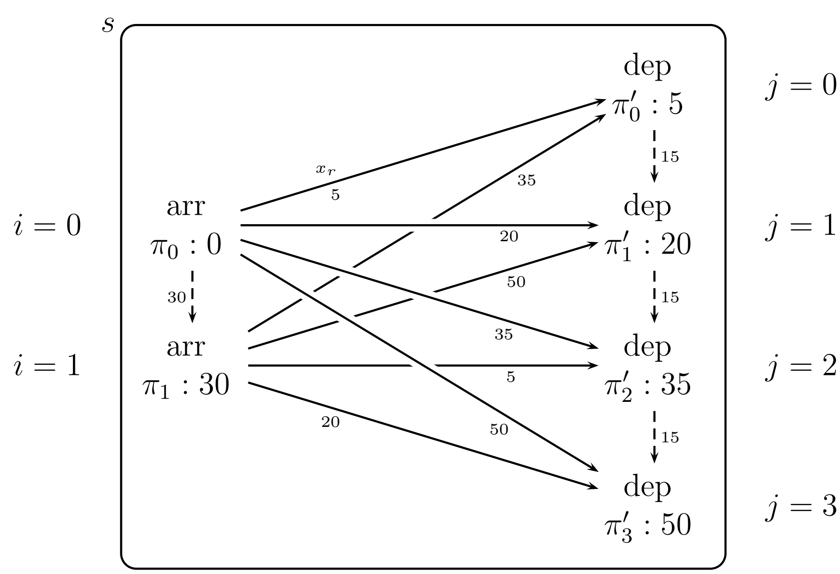


Figure 3.2: Detail of an $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ with period length $T = 60$ associated to a $\mathrm{PTN} = (S, \vec{E})$ and a $\mathrm{LC} = (\vec{L}, F)$, two lines $\ell_1, \ell_2 \in \vec{L}$ that cross at station $s \in S$, have frequencies $(f_1, f_2) = (2, 4)$ and a timetable π. Note that the durations are five plus a multiple of fifteen.

The question arises whether or not there is some simple way to obtain and characterize the durations of all changes between two lines only knowing a *reference duration* x_r from the `frequency_as_attribute` model. And indeed, it is possible.

Lemma 3.3 (Change Duration Pattern)**.** *Let T be a period length,* PTN $= (S, \vec{E})$*,* LC $= (\vec{L}, F)$ *and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network, further $\varepsilon_i = (\mathrm{arr}, \ell_1, s, i), \varepsilon'_j = (\mathrm{dep}, \ell_2, s, j) \in \mathcal{E}$, with $\ell_1 \neq \ell_2$, $a_{ij} \in \mathcal{A}$ a* change *activity between ε_i and ε'_j, x_{ij} be the duration of a_{ij} for a feasible timetable π, with $x_r := x_{00}$ defined as the reference duration and further l^{change} the lower bound as well as $u^{\mathrm{change}} = l^{\mathrm{change}} + T - 1$ the upper bound for all change activities, f_1 and f_2 the frequencies of ℓ_1 resp. ℓ_2 as well as $\tau = T/\operatorname{lcm}(f_1, f_2)$. The possible change durations x_{ij} satisfy the property*

$$x_{ij} = x_r - k(i,j)\tau \mod T \quad , \tag{3.5}$$

where $k(i,j)$ is the lcm *representation map from Corollary 1.12.*

Proof. By the Periodic Rollout durations between events $\varepsilon_0, \ldots, \varepsilon_{f_1-1}$ as well as $\varepsilon'_0, \ldots, \varepsilon'_{f_2-1}$ are fixed as if there had been sync activities, not only for departures but also for arrivals. The duration of x_{ij} may be obtained by the cycle it shares with x_r

$$i\frac{T}{f_1} + x_{ij} - j\frac{T}{f_2} - x_r = 0 \mod T \tag{3.6}$$

which, with the lcm representation map is equivalent to

$$x_{ij} = x_r - k(i,j)\tau \mod T \tag{3.7}$$

and thus the lemma follows. □

From Lemma 3.3 we already see what is the shortest change activity.

Lemma 3.4 (Best Change Activity)**.** *The shortest change or* best change activity *an activity between ε_i and ε_j from Lemma 3.3 has a duration that satisfies*

$$\overline{x}_r = x_r \mod \tau \quad , \tag{3.8}$$

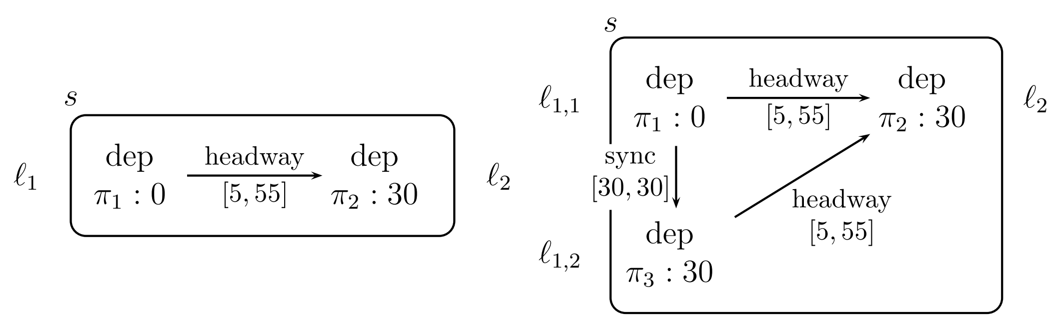
where $\overline{x}_r \in \{l^{\mathrm{change}}, \ldots, l^{\mathrm{change}} + \tau - 1\}$.

Proof. In Equation (3.5), apply Lemma 1.6. □

For change activities $a \in \mathcal{A}_{\mathrm{change}}$ between lines of frequencies f_1 and f_2, setting $T_a = \tau$ yields the best change activity duration for a in the objective of EPESP from Section 2.5, which is stated in [Nac96] as well. Further, the upper bound may be reduced to $u_a = \tau + l^{\mathrm{change}}$ in that case. However, although a passenger may be able to take the best change at some station, in general it is not possible to take all best change activities on the way from one station to some other, which is shown in Section 5.7.

3.1.4 Headways

The `simple` Headway Generation model does not necessarily yield a Periodic Rollout feasible timetable, when we perform the Lines Roll Out with `frequency_as_attribute`[2], as we can see in Figure 3.3.



(a) Event Activity Network Detail, `frequency_as_attribute`

(b) Event Activity Network Detail, `frequency_as_multiplicity`, after Periodic Rollout

Figure 3.3: Detail of an EAN $= (\mathcal{E}, \mathcal{A})$ with period length $T = 60$ associated to a PTN $= (S, \vec{E})$ and a LC $= (\vec{L}, F)$, two lines $\ell_1, \ell_2 \in \vec{L}$ that cross at station $s \in S$, have frequencies $(f_1, f_2) = (2, 1)$ and a timetable π. In (a), i.e. when using the `frequency_as_attribute` model, $\pi_1 = 0$ and $\pi_2 = 30$ satisfy the only headway constraint. However, when extrapolating the EAN in (b) with a Periodic Rollout, a second headway arises and is violated, i.e. $\pi_3 - \pi_2 = 0 \notin [5, 55]_T$ and therefore π becomes infeasible.

However, there is a solution: We copy the headways from the rolled-out network[3]. Instead of one headway per departure pair with same associated edge the `product_of_frequencies` Headway Generation model adds, as the name says, product $f_1 f_2$ many headways, where f_1 and f_2 are the frequencies of the lines.

[2]When headways were introduced to LinTim, Michael Schachtebeck was working on aperiodically rolled out periodic event activity networks, which were constructed with the `frequency_as_attribute` model and wondered he why his delay management did not work anymore, thus he discovered that the initial periodic timetable was already infeasible.

[3]Marc Goerigk was working on periodic timetabling and wanted to create aperiodic rollout feasible timetables without the effort to solve the periodically rolled out problem. Therefore the `product_of_frequencies` Headway Generation model is one of his many contributions to LinTim.

Algorithm 7 HEADWAY GENERATION, `product_of_frequencies`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- EAN $= (\mathcal{E}, \mathcal{A})$ without headways,
- headways h_e for all $e \in \vec{E}$,
- period length T.

Output:

- EAN $= (\mathcal{E}, \mathcal{A})$ with headways.

for all $\varepsilon_1 = (s, \ell_1, \text{dep}, 0), \varepsilon_2 = (s, \ell_2, \text{dep}, 0) \in \mathcal{E}$ **do**
  $e := \text{edge}_{\varepsilon_1}$
  if $e = \text{edge}_{\varepsilon_2}$ **then**
    for all $(i, j) \in \{0, \dots, f_{\ell_1}\} \times \{0, \dots, f_{\ell_2}\}$ **do**
      $\delta := jT/f_{\ell_2} - iT/f_{\ell_1}$
      Add $a := (\varepsilon_1, \varepsilon_2, \text{headway})$ to $\mathcal{A}$ with $[l_a, u_a] := [h_e + \delta, T - h_e + \delta]$
    end for
  end if
end for

Theorem 3.5 (`product_of_frequencies` Feasibility)**.** *Let π_{FA} be a timetable for the event activity network $\text{EAN}_{\text{FA}} = (\mathcal{E}_{\text{FA}}, \mathcal{A}_{\text{FA}})$ with* LINES ROLL OUT *model* ***`frequency_as_attribute`*** *and as* HEADWAY GENERATION *model* ***`product_of_frequencies`***. *Let $\text{EAN}_{\text{FM}} = (\mathcal{E}_{\text{FM}}, \mathcal{A}_{\text{FM}})$ be the* PERIODIC ROLLOUT *of EAN_{FA} with timetable π_{FM} and* ***`simple`*** *as* HEADWAY GENERATION *model. Then π_{FA} is feasible for EAN_{FA} iff π_{FM} is feasible for EAN_{FM}.*

Proof. Let T be the period length, $\varepsilon_i = (\text{dep}, \ell_1, s, i), \varepsilon'_j = (\text{dep}, \ell_2, s, j) \in \mathcal{E}_{\text{FM}}$, $\text{edge}_{\varepsilon_i} = \text{edge}_{\varepsilon'_j}$, f_1, f_2 frequencies of ℓ_1 and ℓ_2, $l_1 \neq l_2$, $\pi_i := \pi_{\text{FM}}(\varepsilon_i)$ and $\pi'_j := \pi_{\text{FM}}(\varepsilon'_j)$ for all $i \in \{0, \dots, f_1 - 1\}$, $j \in \{0, \dots, f_2 - 1\}$. Then, by PERIODIC ROLLOUT

$$\pi_0 = \pi_{\text{FM}}(\varepsilon_0) = \pi_{\text{FA}}(\varepsilon_0) \quad , \qquad \pi'_0 = \pi_{\text{FM}}(\varepsilon'_0) = \pi_{\text{FA}}(\varepsilon'_0) \quad , \tag{3.9}$$

and

$$\pi_i = \pi_0 + i\frac{T}{f_1} \quad , \qquad \pi'_j = \pi'_0 + j\frac{T}{f_2}. \tag{3.10}$$

Therefore EAN_{FM} poses the constraints

$$\begin{aligned} &\exists\, x_{ij} \in \{h, \dots, T - h\} : \\ &\qquad \pi'_0 - \pi_0 = \pi'_j - \pi_i + j\frac{T}{f_2} - i\frac{T}{f_1} = x_{ij} - i\frac{T}{f_1} + j\frac{T}{f_2} \mod T \quad , \\ &\qquad\qquad \forall\, i \in \{0, \dots, f_1 - 1\}, j \in \{0, \dots, f_2 - 1\}, \end{aligned} \tag{3.11}$$

thus (3.11) is equivalent to

$$\exists\; x_{ij} \in \{h + \delta_{ij}, \ldots, T - h + \delta_{ij}\} : \qquad \pi_0' - \pi_0 = x_{ij} \mod T \;\;,$$
$$\delta_{ij} := j\frac{T}{f_2} - i\frac{T}{f_1},\; \forall\; i \in \{0, \ldots, f_1 - 1\}, j \in \{0, \ldots, f_2 - 1\}, \quad (3.12)$$

as stated in Algorithm 7, lines 5 and 6. □

Generally however, fewer constraints as generated by Algorithm 8 already ensure feasibility.

Algorithm 8 Headway Generation, `lcm_of_frequencies`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- EAN $= (\mathcal{E}, \mathcal{A})$ without headways,
- headways h_e for all $e \in \vec{E}$,
- period length T.

Output:

- EAN $= (\mathcal{E}, \mathcal{A})$ with headways.

for all $\varepsilon_1 = (s, \ell_1, \mathrm{dep}, 0), \varepsilon_2 = (s, \ell_2, \mathrm{dep}, 0) \in \mathcal{E}$ **do**
  $e := \mathrm{edge}_{\varepsilon_1}$
  if $e = \mathrm{edge}_{\varepsilon_2}$ **then**
    for all $k \in \{0, \ldots, l - 1\}$ **do**
      $\delta := kT/\operatorname{lcm}(f_1, f_2)$
      Add $a := (\varepsilon_1, \varepsilon_2, \mathrm{headway})$ to $\mathcal{A}$ with $[l_a, u_a] := [h_e + \delta, T - h_e + \delta]$
    end for
  end if
end for

Theorem 3.6 (`lcm_of_frequencies` Equivalence)**.** *Let π be a timetable for the event activity networks* EAN $= (\mathcal{E}, \mathcal{A})$ *and* EAN$'$ $= (\mathcal{E}, \mathcal{A}')$ *with* Lines Roll Out *model* ***`frequency_as_attribute`*** *and* Generate Headways *models* ***`product_of_frequencies`*** *resp.* ***`lcm_of_frequencies`***. *Then π is feasible for* EAN$'$ *iff π is feasible for* EAN.

Proof. Let T be the period length, $\varepsilon = (\mathrm{dep}, l_1, s, 0), \varepsilon' = (\mathrm{dep}, l_2, s, 0) \in \mathcal{E}$, $\mathrm{edge}_\varepsilon = \mathrm{edge}_{\varepsilon'}$, f_1, f_2 frequencies of l_1 and l_2, $\ell = \operatorname{lcm}(f_1, f_2)$, $\pi_0 := \pi(\varepsilon)$ and $\pi_0' := \pi(\varepsilon')$. In the `product_of_frequencies` Headway Generation model holds

$$\exists\; x_{ij} \in \{h, \ldots, T - h\} : \quad \pi_0' - \pi_0 = \tilde{x}_{ij} = x_{ij} - i\frac{T}{f_1} + j\frac{T}{f_2} \mod T \;\;,$$
$$\forall\; i \in \{0, \ldots, f_1 - 1\}, j \in \{0, \ldots, f_2 - 1\}, \quad (3.13)$$

where $\tilde{x}_{ij} \in \{h + \delta_{ij}, \ldots, T - h + \delta_{ij}\}$ with $\delta_{ij} = jT/f_2 - iT/f_1$ are the headway activities constructed by Algorithm 7. Let

$$\begin{aligned} k : \{0, \ldots, f_1 - 1\} \times \{0, \ldots, f_2 - 1\} &\to \{0, \ldots, \ell - 1\} \\ (i, j) &\mapsto k : j\frac{T}{f_2} - i\frac{T}{f_1} = k\frac{T}{\ell} \mod T \end{aligned} \tag{3.14}$$

be the swapped lcm representation map, which, as per Corollary 1.12 is well-defined and surjective, $i, i' \in \{0, \ldots, f_1 - 1\}$ and $j, j' \in \{0, \ldots, f_2 - 1\}$ with $k(i, j) = k(i', j')$. Then

$$\begin{aligned} x_{ij} - i\frac{T}{f_1} + j\frac{T}{f_2} = x_{ij} + k(i,j)\frac{T}{\ell} = \pi_0' - \pi_0 = x_{i'j'} - i'\frac{T}{f_1} + j'\frac{T}{f_2} \\ = x_{i'j'} + k(i',j')\frac{T}{\ell} = x_{i'j'} + k(i,j)\frac{T}{\ell} \mod T, \end{aligned} \tag{3.15}$$

therefore $x_{ij} = x_{i'j'} \mod T$ and since $x_{ij}, x_{i'j'}$ are both in $\{h, \ldots, T - h\}$ Lemma 1.2 yields $x_{ij} = x_{i'j'}$. It follows that if $i \neq i'$ or $j \neq j'$ the variable $x_{i'j'}$ together with its constraint are redundant and, since k is surjective, the constraints from equation (3.13) are equivalent to

$$\begin{aligned} \exists\, x_k \in \{h, \ldots, T - h\} : \quad \pi_0' - \pi_0 = x_k + k\frac{T}{\mathrm{lcm}(f_1, f_2)} \mod T \quad , \\ \forall\, k \in \{0, \ldots, \mathrm{lcm}(f_1, f_2) - 1\}, \end{aligned} \tag{3.16}$$

which just is the `lcm_of_frequencies` Headway Generation model from Algorithm 8. ☐

Corollary 3.7 (`lcm_of_frequencies` Feasibility)**.** *Let π_{FA} be a timetable for the event activity network* $\mathrm{EAN}_{\mathrm{FA}} = (\mathcal{E}_{\mathrm{FA}}, \mathcal{A}_{\mathrm{FA}})$ *with* Lines Roll Out *model* ***frequency_as_attribute*** *and* ***lcm_of_frequencies*** *as* Generate Headways *model. Let* $\mathrm{EAN}_{\mathrm{FM}} = (\mathcal{E}_{\mathrm{FM}}, \mathcal{A}_{\mathrm{FM}})$ *be the* Periodic Rollout *of* $\mathrm{EAN}_{\mathrm{FA}}$ *with timetable* π_{FM} *and* ***simple*** *as* Headway Generation *model. Then* π_{FA} *is feasible for* $\mathrm{EAN}_{\mathrm{FA}}$ *iff* π_{FM} *is feasible for* $\mathrm{EAN}_{\mathrm{FM}}$.

Proof. Direct consequence from theorem 3.5 and 3.6. ☐

t	area	t	area	t	area
0	X	10	X	20	X
1		11		21	
2		12		22	
3		13		23	
4		14		24	
5		15		25	
6		16		26	
7		17		27	
8		18		28	
9		19		29	

(a) Infeasible Area Induced by ℓ_1, ...

t	area	t	area	t	area
0	X	10	X	20	X
1		11		21	
2		12		22	
3		13		23	
4		14		24	
5	X	15		25	
6		16		26	
7		17		27	
8		18		28	
9		19		29	

(b) ... by 3$^{\text{rd}}$ Frequency Instance of ℓ_1 and 1$^{\text{st}}$ of ℓ_2, ...

t	area	t	area	t	area
0	X	10	X	20	X
1		11		21	
2		12		22	
3		13		23	
4		14		24	
5	X	15	X	25	
6		16		26	
7		17		27	
8		18		28	
9		19		29	

(c) ... by 1$^{\text{st}}$ of ℓ_1 and 1$^{\text{st}}$ of ℓ_2 ...

t	area	t	area	t	area
0	X	10	X	20	X
1		11		21	
2		12		22	
3		13		23	
4		14		24	
5	X	15	X	25	X
6		16		26	
7		17		27	
8		18		28	
9		19		29	

(d) ... and by 2$^{\text{nd}}$ of ℓ_1 and 1$^{\text{st}}$ of ℓ_2

Figure 3.4: Lines $\ell_1 \in \vec{L}$ and $\ell_2 \in \vec{L}$ from LC $= (\vec{L}, F)$ with frequencies $f_1 = 3$ resp. $f_2 = 2$ using the same edge $e \in E$ from PTN $= (S, \vec{E})$ with headway $h_e = 2$, and $T = 30$. The time of the representative departure event of ℓ_1 w.l.o.g. is zero. Gray areas show the infeasible area to place any departure of ℓ_2 and X denotes that either there is a departure of ℓ_1 or a time which may not be used for ℓ_2 because of another X. Note that in (d) the white area may be parametrized by $t \in [2, 3] + 5k$ with $k \in \{0, \dots, 5\}$.

Theorem 3.6 and Figure 3.4 let us already suggest that periodic headways between lines of different frequencies f_1 and f_2 have a regular structure. And indeed, this is the case: regardless what frequencies f_1 and f_2 are, we only need one activity and two variables to ensure rollout feasibility.

Algorithm 9 HEADWAY GENERATION, `lcm_representation`

Input:

- PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$,
- EAN $= (\mathcal{E}, \mathcal{A})$ without headways,
- headways h_e for all $e \in \vec{E}$,
- period length T.

Output:

- EAN $= (\mathcal{E}, \mathcal{A})$ with headways.

for all $\varepsilon_1 = (s, \ell_1, \mathrm{dep}, 0), \varepsilon_2 = (s, \ell_2, \mathrm{dep}, 0) \in \mathcal{E}$ **do**
  $e := \mathrm{edge}_{\varepsilon_1}$
  if $e = \mathrm{edge}_{\varepsilon_2}$ **then**
    $l := \mathrm{lcm}(f_1, f_2)$, $\tau := T/l$
    Add $a := (\varepsilon_1, \varepsilon_2, \mathrm{headway})$ to $\mathcal{A}$ with $[l_a, u_a] := [h_e, T - h_e]$
    In the periodic timetabling step, state $x_a = \overline{x}_a + \kappa\tau$,
    where $\overline{x}_a \in \{h_e, \dots, \tau - h_e\}$, $\kappa \in \{0, \dots, l-1\}$
  end if
end for

Theorem 3.8 (`lcm_representation` Equivalence)**.** *Let π be a timetable for the event activity networks* EAN $= (\mathcal{E}, \mathcal{A})$ *and* EAN$'$ $= (\mathcal{E}, \mathcal{A}')$ *with* LINES ROLL OUT *model* ***`frequency_as_attribute`*** *and* GENERATE HEADWAYS *models* ***`lcm_of_frequencies`*** *resp.* ***`lcm_representation`***. *Then π is feasible for* EAN$'$ *iff π is feasible for* EAN.

Proof. Let $\varepsilon = (\mathrm{dep}, l_1, s, 0), \varepsilon' = (\mathrm{dep}, l_2, s, 0) \in \mathcal{E}$, $\mathrm{edge}_\varepsilon = \mathrm{edge}_{\varepsilon'}$, f_1, f_2 frequencies of l_1 and l_2, $\ell = \mathrm{lcm}(f_1, f_2)$, $\tau = T/\ell$, $\pi_0 := \pi(\varepsilon)$ and $\pi_0' := \pi(\varepsilon')$.

The `lcm_of_frequencies` model yields

$$\exists\, x_k \in \{h, \dots, T-h\} : \quad x_r := \pi_0' - \pi_0 = \tilde{x}_k = x_k + k\tau \mod T \;, \quad \forall\, k \in \{0, \dots, \ell - 1\}, \tag{3.17}$$

where $\tilde{x}_k \in \{h + \delta_k, \dots, T - h + \delta_k\}$, $\delta_k = k\tau$ and $x_r \in \{0, \dots, T-1\}$ is the *reference duration*. From division with remainder follows

$$x_r = x + \kappa\tau \;, \qquad x \in \{0, \dots, \tau - 1\},\ \kappa \in \{0, \dots, \ell - 1\} \tag{3.18}$$

and

$$\forall\, k \in \{0, \dots, \ell - 1\} : \exists\, x_k \in \{h, \dots, T - h\},\ x \in \{0, \dots, \tau - 1\} : \quad x + \kappa\tau = x_k + k\tau \mod T \;. \tag{3.19}$$

With substituting $k \mapsto \kappa - k$, equations from (3.17) write as

$$\forall\, k \in \{\kappa - \ell + 1, \dots, \kappa\} : \exists\, x_k \in \{h, \dots, T - h\},\ x \in \{0, \dots, \tau - 1\} : \\ x_k = x + (\kappa - (\kappa - k))\tau = x + k\tau \mod T \quad , \tag{3.20}$$

and since $\ell\tau = T = 0 \mod T$ the range $k \in \{\kappa - \ell + 1, \dots, \kappa\}$ is equivalent to $k \in \{0, \dots, \ell - 1\}$

$$\forall\, k \in \{0, \dots, \ell - 1\} : \exists\, x_k \in \{h, \dots, T - h\},\ x \in \{0, \dots, \tau - 1\} : \\ x_k = x + k\tau \mod T \quad . \tag{3.21}$$

It must further hold that $h \leq x \leq \tau - h$: Assume that $x < h$, then $x_k < h$ for $k = 0$. For $x > \tau - h$, i.e. $x = \tau - h + a$, $a > 0$ with $k = \ell - 1$ follows

$$x_k = \tau - h + a + (\ell - 1)\tau = \ell\frac{T}{\ell} - h + a = T - h + a > T - h \quad . \tag{3.22}$$

Thus both assumptions contradict $h \leq x_k \leq T - h$ and therefore $h \leq x \leq \tau - h$, which also yields

$$2h \leq \tau \quad , \tag{3.23}$$

a necessary condition for a feasible timetable to exist at all, so let h and τ be such that (3.23) is valid. Yet was shown that every reference duration x_r must have the `lcm_representation` representation

$$x_r = x + \kappa\tau \quad , \qquad x \in \{h, \dots, \tau - h\} \quad , \qquad \kappa \in \{0, \dots, \ell - 1\}. \tag{3.24}$$

to satisfy the `lcm_of_frequencies` model, i.e. necessity. To show sufficiency, consider

$$\forall\, k \in \{0, \dots, \ell - 1\} : \exists\, x_k \in \{h, \dots, T - h\},\ x \in \{h, \dots, \tau - h\}, z \in \mathbb{Z} : \\ x_k - x - k\tau = zT \quad , \tag{3.25}$$

equivalent to `lcm_of_frequencies` constraints (3.21) with

$$\left\lceil \frac{h - \tau + h - k\tau}{T} \right\rceil \leq z \leq \left\lfloor \frac{T - h - h - k\tau}{T} \right\rfloor \quad . \tag{3.26}$$

For the lower bound, $k = 0$ poses the greatest restriction, for the upper bound $k = \ell - 1$. Further

$$\left\lceil \frac{2h - \tau}{T} \right\rceil \overset{(3.23)}{=} 0 \leq z \leq \left\lfloor \frac{T - 2h - (\ell - 1)\frac{T}{\ell}}{T} \right\rfloor = \left\lfloor \frac{-2h + \tau}{T} \right\rfloor \overset{(3.23)}{=} 0 \quad , \tag{3.27}$$

thus the `lcm_representation` representation (3.24) always satisfies the model `lcm_of_frequencies`, i.e. it is sufficient and therefore the theorem follows. □

Corollary 3.9 (`lcm_representation` Feasibility). *Let π_{FA} be a timetable for the event activity network $\mathrm{EAN}_{\mathrm{FA}} = (\mathcal{E}_{\mathrm{FA}}, \mathcal{A}_{\mathrm{FA}})$ with* Lines Roll Out *model* ***`frequency_as_attribute`*** *and* ***`lcm_representation`*** *as* Headway Generation *model. Let $\mathrm{EAN}_{\mathrm{FM}} = (\mathcal{E}_{\mathrm{FM}}, \mathcal{A}_{\mathrm{FM}})$ be the* Periodic Rollout *of $\mathrm{EAN}_{\mathrm{FA}}$ with timetable π_{FM} and* ***`simple`*** *as* Headway Generation *model. Then π_{FA} is feasible for $\mathrm{EAN}_{\mathrm{FA}}$ iff π_{FM} is feasible for $\mathrm{EAN}_{\mathrm{FM}}$.*

Proof. Direct consequence from Theorem 3.8 and Corollary 3.7. □

In terms of the EPESP from Section 2.5, Corollary 3.9 states that if a headway $a \in \mathcal{A}_{\mathrm{headway}}$ between lines of frequencies f_1 and f_2 with minimum duration h is to be satisfied, then $T_a = \tau := \mathrm{lcm}(f_1, f_2)$ bounds $l_a = h$ and $u_a = \tau - h$ does the trick, which [Nac98] found out as well. Therefore, using the `lcm_representation` actually transforms the PESP into an EPESP, besides that by the representation $x_a = \overline{x}_a + \kappa\tau$ the advantages of the CPF may still be used. The `lcm_of_frequencies` model, since it reduces the problem size compared to the initial `product_of_frequencies` is interesting for heuristic methods that rely on the PESP model, like the modulo simplex from [GS11].

Corollary 3.10 (A Timetable Feasibility Criterion for Line Concepts). *Let T be a period length, $\mathrm{PTN} = (S, \vec{E})$ be a public transportation network, $\mathrm{LC} = (\vec{L}, F)$ a line concept, $\ell_1, \ell_2 \in \vec{L}$ two lines with frequencies $f_1|T$ resp. $f_2|T$, $h : \vec{E} \to \{0, \ldots, T-1\}$ a headway map, $\mathrm{EAN}_{\mathrm{FA}}$ an event activity network and $\mathrm{EAN}_{\mathrm{FM}}$ its* Periodic Rollout *with timetable π. Then*

$$\max_{e \in \ell_1 \cap \ell_2 \subset \vec{E}} h_e \leq \left\lfloor \frac{T}{2\,\mathrm{lcm}(f_1, f_2)} \right\rfloor \quad . \tag{3.28}$$

Proof. Let $\tau := T/\mathrm{lcm}(f_1, f_2)$. For every $e \in \ell_1 \cap \ell_2$, Headway Generation creates a headway activity with duration x_a for which in $\mathrm{EAN}_{\mathrm{FM}}$, trough the `lcm_representation` representation as per corollary 3.9 must hold

$$x_a = x + \kappa\tau \quad , \qquad x \in \{h_e, \ldots, \tau - h_e\} \quad , \qquad \kappa \in \{0, \ldots, \ell - 1\}, \tag{3.29}$$

where the existance of a valid x implies that the set $\{h_e, \ldots, \tau - h_e\}$ is nonempty and therefore $h_e \leq \tau - h_e$ or $h_e \leq \tau/2$ and since h_e is integral

$$h_e \leq \left\lfloor \frac{\tau}{2} \right\rfloor \tag{3.30}$$

and thus the corollary follows. □

3.2 Timetabling and Routing

In this section, we introduce both quadratic and linear models for an extension of the PESP, namely the *Origin Destination Aware Periodic Event Scheduling Problem* ODPESP[4].

Quadratic Program 3.11 (Quadratic ODPESP)**.** *Let a* PTN $= (S, E)$ *and an* EAN $= (\mathcal{E}, \mathcal{A})$ *be given,* $\mathcal{A}_p \subset \mathcal{A}$ *the passenger usable activities as well as an integral cycle basis or just the set of all cycles* $\mathcal{C}$*,* $a \in \mathcal{A}$ *and an origin destination matrix* OD $= (w_{s_1 s_2})_{s_1, s_2 \in S}$.

We use an incidence matrix

$$\boxed{\mathrm{C}} \qquad \mathcal{A} = (\mathfrak{a}_{a\varepsilon})_{a \in \mathcal{A}, \varepsilon \in \mathcal{E}} \quad , \qquad \mathfrak{a}_{a\varepsilon'} = \begin{cases} 1 & \begin{array}{l} \exists\ \varepsilon' \in \mathcal{E} : \\ a = (\varepsilon', \varepsilon) \end{array} \\ -1 & \begin{array}{l} \exists\ \varepsilon' \in \mathcal{E} : \\ a = (\varepsilon, \varepsilon') \end{array} \\ 0 & \textit{otherwise} \end{cases} \quad , \qquad \begin{array}{r} \forall\ a \in \mathcal{A}, \\ \varepsilon \in \mathcal{E}, \end{array} \tag{3.31}$$

introduce path decision variables

$$\boxed{\mathrm{V}} \qquad p_{a s_1 s_2} = \begin{cases} 1 & \begin{array}{l} \textit{path from } s_1 \textit{ to } s_2 \\ \textit{uses activity } a \end{array} \\ 0 & \textit{otherwise} \end{cases} \quad , \qquad \begin{array}{r} \forall\ a \in \mathcal{A}_p, \\ s_1, s_2 \in S, \\ w_{s_1 s_2} > 0, \end{array} \tag{3.32}$$

and durations

$$\boxed{\mathrm{V}} \qquad x_a \in [l_a, u_a] \cap \mathbb{Z} \quad , \qquad \forall\ a \in \mathcal{A}. \tag{3.33}$$

We want to minimize the average traveling time

$$\boxed{\mathrm{L}} \qquad \min \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0, \\ a \in \mathcal{A}_p}} w_{s_1 s_2} p_{a s_1 s_2} x_a \tag{3.34}$$

subject to periodicity being satisfied

$$\boxed{\mathrm{L}} \qquad \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = z_c T \quad , \qquad \forall\ C \in \mathcal{C}, \tag{3.35}$$

---

[4]Independent of our results, [Lue09] also worked out that model in his diploma thesis and called it IntMod.

and the path are connected

$$\boxed{\mathrm{L}} \qquad \sum_{\substack{a \in \mathcal{A}_p \setminus \\ (\mathrm{drive}^{\mathrm{dep}}_{s_1} \\ \cup \mathrm{drive}^{\mathrm{arr}}_{s_2})}} \mathfrak{a}_{a\varepsilon} p_{as_1s_2} = 0 \quad , \qquad \begin{array}{r} \forall\ \varepsilon \in \mathcal{E}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \end{array} \tag{3.36}$$

$$\boxed{\mathrm{L}} \qquad \sum_{a \in \mathrm{drive}^{\mathrm{dep}}_{s_1}} \mathfrak{a}_{a\varepsilon} p_{as_1s_2} = 1 \quad , \qquad \begin{array}{r} \forall\ \varepsilon \in \mathcal{E}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \end{array} \tag{3.37}$$

$$\boxed{\mathrm{L}} \qquad \sum_{a \in \mathrm{drive}^{\mathrm{arr}}_{s_2}} \mathfrak{a}_{a\varepsilon} p_{as_1s_2} = -1 \quad , \qquad \begin{array}{r} \forall\ \varepsilon \in \mathcal{E}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0. \end{array} \tag{3.38}$$

Quadratic Program 3.11 is based on the PESP cyclic periodicity formulation CPF. A feasible timetable thus may be derived from the durations x_a, $a \in \mathcal{A}$, proofs for feasibility and all kind of equivalences below. From a classification point of view, the ODPESP is a minimum cost multi-commodity flow problem[5].

Theorem 3.12 (ODPESP Shortest Paths Subproblem)**.** *For an* ODPESP *optimum the paths* $p_{as_1s_2}$*,* $a \in \mathcal{A}$ *are shortest paths w.r.t.* x_a *for all* $s_1, s_2 \in S$ *with* $w_{s_1s_2} > 0$*. Especially, replacing* p *with shortest paths w.r.t.* x *does not increase the objective function.*

Proof. Assume some path $p_{as'_1s'_2}$, $a \in \mathcal{A}_p$ for a tuple $(s'_1, s'_2) \in S^2$ with $w_{s'_1s'_2} > 0$ is not a shortest path w.r.t. x_a, i.e. there exists a path $\tilde{p}_{as'_1s'_2}$, $a \in \mathcal{A}_p$ with

$$w_{s'_1s'_2} \sum_{a \in \mathcal{A}_p} \tilde{p}_{as'_1s'_2} x_a < w_{s'_1s'_2} \sum_{a \in \mathcal{A}_p} p_{as'_1s'_2} x_a \quad . \tag{3.39}$$

The ODPESP objective function (3.34) then writes as

$$\sum_{\substack{s_1,s_2 \in S, \\ w_{s_1s_2} > 0, \\ a \in \mathcal{A}_p}} w_{s_1s_2} p_{as_1s_2} x_a = \left[\sum_{\substack{s_1,s_2 \in S, \\ w_{s_1s_2} > 0, \\ (s_1,s_2) \neq (s'_1,s'_2), \\ a \in \mathcal{A}_p}} w_{s_1s_2} p_{as_1s_2} x_a \right] + w_{s'_1s'_2} \sum_{a \in \mathcal{A}_p} p_{as'_1s'_2} x_a \tag{3.40}$$

$$> \left[\sum_{\substack{s_1,s_2 \in S, \\ w_{s_1s_2} > 0, \\ (s_1,s_2) \neq (s'_1,s'_2), \\ a \in \mathcal{A}_p}} w_{s_1s_2} p_{as_1s_2} x_a \right] + w_{s'_1s'_2} \sum_{a \in \mathcal{A}_p} \tilde{p}_{as'_1s'_2} x_a \tag{3.41}$$

[5] http://en.wikipedia.org/wiki/Multi-commodity_flow_problem

and therefore contradicts optimality. Iterating over all $(s_1', s_2') \in S \times S$ with $w_{s_1 s_2} > 0$ yields that every non-shortest path may be replaced by a shortest path, which reduces the objective function and in case all paths are already shortest, (other) shortest paths do not change the objective. Therefore, replacing paths by shortest paths cannot increase the ODPESP objective function. □

Thus, finding shortest paths w.r.t. a given timetable may be considered as a subproblem of ODPESP.

Theorem 3.13 (ODPESP Feasible Timetable)**.** *A timetable π from* ODPESP *is feasible iff it is feasible by Definition 2.5.*

Proof. Constraint (3.35) is the same as (2.35) in the CPF and the x_a is defined with the same bounds as well. Since no other constraints limit the choice of x_a in neither ODPESP nor the CPF, the theorem follows. □

Theorem 3.14 (ODPESP $\mathcal{NP}$-completeness)**.** *The* ODPESP *is $\mathcal{NP}$-complete.*

Proof. By Theorem 3.13 a timetable π from ODPESP is feasible iff it is feasible by Definition 2.5, i.e. iff it is feasible for PESP. Since finding feasible solutions for the PESP is $\mathcal{NP}$-complete, so is ODPESP. □

ODPESP and PESP have in common than just timetable feasbility.

Theorem 3.15 (PESP Subproblem of ODPESP)**.** *Let* EAN $= (\mathcal{E}, \mathcal{A})$ *be an event activity network, w_a, $a \in \mathcal{A}$ be an* OD $= (w_{s_1 s_2})_{s_1, s_2 \in S}$ *derived passenger distribution with paths $\tilde{p}_{as_1 s_2}$, $a \in \mathcal{A}_p$ for all $s_1, s_2 \in S$ with $w_{s_1 s_2} > 0$ and* ODPESP *and* PESP *two problem instances for* EAN *and* OD *with fixed $p_{as_1 s_2} = \tilde{p}_{as_1 s_2}$ in Equation* (3.32) *resp. for* EAN *and w. Then the problems are equivalent.*

Proof. It holds

$$\min \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0, \\ a \in \mathcal{A}_p}} w_{s_1 s_2} p_{as_1 s_2} x_a = \sum_{a \in \mathcal{A}_p} \left(\sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} w_{s_1 s_2} p_{as_1 s_2} \right) x_a = \sum_{a \in \mathcal{A}_p} w_a x_a \tag{3.42}$$

with $w_a := \sum_{s_1, s_2 \in S} w_{s_1 s_2} p_{as_1 s_2}$, i.e. fixing the passenger paths not only removes variables, but converts the ODPESP it into a PESP with OD derived passenger distribution. □

Thus solving the PESP for an OD derived passenger distribution may be considered as a subproblem of ODPESP.

Corollary 3.16 (ODPESP Optimum Recoverability)**.** *If from an* ODPESP *optimum the durations x_a, $a \in \mathcal{A}$ are known, then the passenger paths can be obtained by pairwise shortest paths w.r.t. x. If on the other hand the* ODPESP *optimal passenger distribution w_a, $a \in \mathcal{A}$ is known, then solving the* PESP *for w_a recovers the durations from the* ODPESP *optimum.*

Proof. Combine Theorems 3.15 and 3.12. □

Therefore, the actual difference between ODPESP and PESP (compare Corollary 2.13) is the way the passengers are modeled in the objective function.

Corollary 3.17 (ODPESP Average Traveling Time)**.** *The* ODPESP *minimizes the average traveling time* ATT_w^x *from Definition 2.12 over the domain of feasible duration sets x (resp. timetables) and* OD *derived passenger distributions w.*

In other words: the ODPESP objective function actually is the average traveling time from Definition 2.12.

Proof. In the Quadratic Program 3.11 the passenger paths are arbitrary EAN passenger routes as in Definition 1.43 and the objective is to minimize the average traveling time over a derived passenger distribution as per Theorem 3.15. □

Corollary 3.18 (Shortest Paths Average Traveling Time)**.** *For a given feasible duration set x (resp. timetable) an* OD *derived passenger distribution w may be replaced by w' obtained from the* Passenger Distribution *Algorithm 4 with x as shortest path weights and it holds*

$$\mathrm{ATT}_{w'}^x \leq \mathrm{ATT}_w^x \quad . \tag{3.43}$$

Proof. By Corollary 3.17, the ODPESP objective is the average traveling time, which by Theorem 3.12 cannot be increased by replacing paths with shortest paths. □

Theorem 3.19 (Aperiodic ODPESP $\mathcal{NP}$-completeness)**.** *The* ODPESP *stays $\mathcal{NP}$-complete, even with fixed modulo parameters in equation* (3.35)*.*

A proof for Theorem 3.19 may be found in [SS10], where the authors reduce the Minimum Cover problem to aperiodic timetabling with even only one origin destination pair and thus show that it is strongly $\mathcal{NP}$-hard, which also covers $\mathcal{NP}$ completeness.

Compared to the PESP the ODPESP is gigantic in its dimensions and since it is already hard in practice to solve the PESP for large instances, fully solving the ODPESP seems to be utopic. However, there is an evident heuristic: *Retimetabling*[6].

Definition 3.20 (Retimetabling Step)**.** *Let* $\mathrm{OD} = (w_{ss'})_{s,s' \in S}$ *an origin destination matrix,* $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ *an event activity network and w_a, $a \in \mathcal{A}$ a passenger distribution derived from* OD*. Solving the* PESP *for w with timetable π and rerouting passengers along shortest paths w.r.t. derived durations from π is called* Retimetabling step w.r.t. EAN and OD *or* ReTim step *for short.*

[6] Independently of us, in his diploma thesis [Kin08] worked on this heuristic as well. We and [Lue09] use his naming.

Definition 3.21 (Retimetabling)**.** *A loop of* ReTim *steps, i.e. solving the* PESP *and distributing passengers along shortest paths w.r.t. the last timetable yields a sequence of passenger distributions and timetables* $w^0 = w, \pi^1, w^2, \pi^3, w^4, \ldots$ *and is called* Retimetabling *or* ReTim *for short. Let* ATT^n *be the average traveling time w.r.t.* π^{n-1} *and the passenger distribution* w^n *if* $2|n$ *and w.r.t.* π^n *and* w^{n-1} *otherwise. The* ReTim limit *is defined as*

$$\lim_{\mathrm{ReTim}} := \lim_{n\to\infty} \mathrm{ATT}^n \quad . \tag{3.44}$$

Theorem 3.22 (Retimetabling Convergence)**.** *Let* $\mathrm{EAN} = (\mathcal{E}, \mathcal{A})$ *be an event activity network,* $\mathrm{OD} = (w_{ss'})_{s,s'\in S}$ *an origin destination matrix. For* ReTim *on* EAN *and* OD *the average traveling time, already if never increasing by (suboptimally) solving the* PESP*, decreases monotonically and converges with* $\lim_{\mathrm{ReTim}} \geq \mathrm{obj}^*_{\mathrm{ODPESP}}$*, where* $\mathrm{obj}^*_{\mathrm{ODPESP}}$ *is the objective function of the* ODPESP *optimum for* EAN *and* OD*.*

Proof. By Corollary 3.16, fixing the passenger distribution transforms ODPESP into a PESP and fixing the durations into an all pairs shortest paths problem. Therefore, ReTim only solves subproblems and if it converges it holds $\lim_{\mathrm{ReTim}} \geq \mathrm{obj}^*_{\mathrm{ODPESP}}$. If an instance of ODPESP is feasible, there exist some initial feasible $p^0_{as_1s_2}, x^0_a$ with $s_1, s_2 \in S$, $a \in \mathcal{A}_p$ with modulo parameters z^0_c derived from x^0_a, $c \in \mathcal{C}$ for $\mathcal{C}$ and objective function value obj^0. Let p^0 w.l.o.g. be shortest paths and q^0_a, $a \in \mathcal{A}$ be the *derived passenger distribution*, i.e.

$$q^0_a = \sum_{\substack{s_1,s_2\in S\\ w_{s_1s_2}>0}} p^0_{as_1s_2} \quad , \qquad \forall\, a \in \mathcal{A}_p. \tag{3.45}$$

A timetabling iteration provides a new x^1 (and z^1) that by the requirements from the theorem has the property

$$\mathrm{ATT}^{x^1}_{q^0} \leq \mathrm{ATT}^{x^0}_{q^0} \quad . \tag{3.46}$$

By Corollary 3.18 the average traveling time cannot increase by rerouting passengers along shortest paths w.r.t. x^1 and since it did not increase by the preceding timetabling step, this makes ReTim monotonic decreasing. Since it has a lower bound as well, i.e. $\mathrm{obj}^*_{\mathrm{ODPESP}}$, it converges. □

Theorem 3.22 allows for heuristics for the timetabling step. The simpliest one could think of is just fixing the modulo parameters to those of the previous timetable or more elaborated methods like the modulo simplex [GS11].

Nowadays linear solvers like CPLEX are capable to find optimal solutions to quadratic programs. However, therefore the objective function must be convex, which is not the case for ODPESP, as shown in [Lue09]. Therefore, we introduce a linearization with integral auxiliary variables.

Linear Program 3.23 (Linear ODPESP)**.** *Let the Quadratic Program 3.11 of the* ODPESP *be given.*

We introduce auxiliary variables

$$\boxed{\text{V}} \qquad d_{as_1s_2} = \begin{cases} x_a & p_{as_1s_2} = 1 \\ 0 & \textit{otherwise} \end{cases} \quad , \qquad \begin{array}{r} \forall\ a \in \mathcal{A}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0 \end{array} \tag{3.47}$$

that have to satisfy their definition

$$\boxed{\text{L}} \qquad 0 \leq d_{as_1s_2} \leq x_a \quad , \qquad \begin{array}{r} \forall\ a \in \mathcal{A}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \end{array} \tag{3.48}$$

$$\boxed{\text{L}} \qquad x_a - u_a(1 - p_{as_1s_2}) \leq d_{as_1s_2} \leq p_{as_1s_2} u_a \quad , \qquad \begin{array}{r} \forall\ a \in \mathcal{A}, \\ s_1, s_2 \in S, \\ w_{s_1s_2} > 0 \end{array} \tag{3.49}$$

and linearize the formulation by modifying the objective function

$$\boxed{\text{L}} \qquad \min \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \\ a \in \mathcal{A}}} w_{s_1s_2} d_{as_1s_2} \quad . \tag{3.50}$$

Theorem 3.24. *Linear Formulation 3.23 is equivalent to 3.11.*

Proof. If the auxiliary variables satisfy their Definition 3.47, then the objective function

$$\sum_{\substack{s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \\ a \in \mathcal{A}}} w_{s_1s_2} d_{as_1s_2} = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1s_2} > 0, \\ a \in \mathcal{A}}} w_{s_1s_2} p_{as_1s_2} x_a \quad , \tag{3.51}$$

which would prove the theorem. Therefore it suffices to show that Equations (3.48) and (3.49) ensure it. Let $p_{as_1s_2} = 0$, then Inequation (3.49) states

$$x_a - u_a \leq d_{as_1s_2} \leq 0 \quad , \tag{3.52}$$

which is always possible since $x_a - u_a \leq 0$ and since $d_{as_1s_2} \geq 0$ by Inequation (3.48) it holds $d_{as_1s_2} = 0$. Let on the other hand $p_{as_1s_2} = 1$. Then (3.49) writes as

$$x_a \leq d_{as_1s_2} \leq u_a \quad , \tag{3.53}$$

which is again always feasible, since $x_a \leq u_a$ and $d_{as_1s_2} = x_a$ since by Inequation (3.48) it holds $d_{as_1s_2} \leq x_a$. □

Chapter 4

Planning Steps Lower Bounds

In Section 1.1 we gave an overview about the traditional planning steps in public transportation. In this chapter, we deal with four of them:

1. Network Design Where to put the stations and infrastructure?

2. Line Planning How to layout the lines, i.e. the vehicle paths?

3. Passenger Routing Which paths will passengers take?

4. Timetabling At which times will lines arrive/depart at the stations?

For the timetabling step our objective is to minimize the average traveling time w.r.t. a feasible timetable π resp. feasible durations x and a passenger distribution w as in Definition 2.12. However, to obtain x and w, we started with a public transportation network PTN, an origin destination matrix OD as well as line pool $\mathfrak{L}$, from which we derived a line concept LC and constructed an event activity network EAN on top of this LC. All these preliminary steps have an effect on the average traveling time and in this chapter we estimate it quantitatively by introducing easy-to-compute lower bounds w.r.t. a given origin destination matrix $\text{OD} = (w_{s_1 s_2})_{s_1,s_2 \in S}$.

Throughout the chapter we assume that for all $s_1, s_2 \in S$ with $w_{s_1 s_2} > 0$ in $\text{OD} = (w_{s_1 s_2})_{s_1,s_2 \in S}$ there are paths in the $\text{PTN} = (S, \vec{E})$ that lead from s_1 to s_2 as well as line pools resp. line concepts always ensure connectivity between those s_1 and s_2. If this is not the case, the model is broken anyway and something needs to be fixed first.

4.1 Public Transportation Network

Given a public transportation network $\text{PTN} = (S, \vec{E})$, what is the best line concept $\text{LC} = (\vec{L}, F)$ one could think of from the traveling time point of view?

4.1.1 General Lower Bound

Vehicles would never need to pay attention to each other, i.e. no headways. Furthermore, every passenger has a line that goes directly from origin to destination, so that changes are not needed at all and there are no stopovers. The only restriction: vehicles need is to pass the minimal driving time on the edges they use[1]. To obtain the average traveling time we then could simply take minimal driving times as weights for the PTN graph, compute pairwise shortest paths lengths $d^{\mathrm{PTN}}_{s_1 s_2}$, for all $s_1 s_2 \in S$ with $w_{s_1 s_2} > 0$ in $\mathrm{OD} = (w_{s_1 s_2})_{s_1, s_2 \in S}$ and look at

$$l_{\mathrm{PTN}} = \sum_{\substack{s_1, s_2 \in S \\ w_{s_1 s_2} > 0}} d^{\mathrm{PTN}}_{s_1 s_2} w_{s_1 s_2} \quad . \tag{4.1}$$

We call l_{PTN} the *General Public Transportation Network Average Traveling Time Lower Bound* or *General* PTN ATT *Lower Bound* for short.

Theorem 4.1 (General PTN ATT Lower Bound)**.** *Let* PTN *be a public transportation network and* OD *an origin destination matrix. For every feasible duration set x and* OD *derived passenger distribution w that can be obtained for* PTN *and* OD *by constructing an* EAN *with methods from Sections 2.3 or 3.1 on top of any line concept* LC *holds*

$$l_{\mathrm{PTN}} \leq \mathrm{ATT}^x_w \quad , \tag{4.2}$$

where ATT^x_w is the average traveling time from Definition 2.12.

Proof. Theorem 4.2 with $l^{\mathrm{wait}} = l^{\mathrm{change}} = 0$. □

4.1.2 Wait Aware Lower Bound

If additionally to the Section 4.1.1 before we take minimal stopover times into account, i.e. global minimal waiting times l^{wait} in our case, we can further increase this bound. Therefore, we replace every station $s \in S$ by an *incoming station* s_{In}, which gets the incoming edges of s, an *outgoing station* s_{Out} analogously and add $\mathrm{edge}^{\mathrm{wait}}_s$, an *virtual wait edge* in between, as in Figure 4.1. We introduce $\mathrm{In}_s = s_{\mathrm{In}}$ and $\mathrm{Out}_s = s_{\mathrm{Out}}$ as the *incoming station map* resp. *outgoing station map* as well as $\mathrm{Im}(\mathrm{In}(S)) = S_{\mathrm{In}}$ and $\mathrm{Im}(\mathrm{Out}(S)) = S_{\mathrm{Out}}$ as the *set of incoming stations* resp. *set of outgoing stations*. Let further $\vec{E}_{\mathrm{wait}} := \{\mathrm{edge}^{\mathrm{wait}}_s : s \in S\}$ the *set of virtual wait edges*. We call this expanded graph $\overline{\mathrm{PTN}} = (S_{\mathrm{In}} \cup S_{\mathrm{Out}}, \vec{E} \cup \vec{E}_{\mathrm{wait}})$ *wait expanded public transportation network*. We take minimal driving times as weights for all $e \in \vec{E}$ and l^{wait} as weight for all $e \in \vec{E}_{\mathrm{wait}}$, compute pairwise shortest paths lengths $\overline{d}^{\mathrm{PTN}}_{s_1 s_2}$, for all $\mathrm{Out}_{s_1} \in S_{\mathrm{Out}}$, $\mathrm{In}_{s_2} \in S_{\mathrm{In}}$ and

$$l^{\mathrm{wait}}_{\mathrm{PTN}} = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} \overline{d}^{\mathrm{PTN}}_{s_1 s_2} w_{s_1 s_2} \tag{4.3}$$

[1] We could also think of passengers with walk speed as fast as vehicles and never interfere.

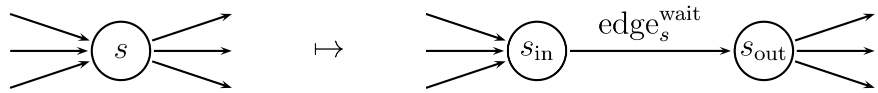


Figure 4.1: Station wait expansion.

where $w_{s_1 s_2}$ is an entry from the origin destination matrix OD for all $s_1, s_2 \in S$. We call $l_{\text{PTN}}^{\text{wait}}$ the *Wait Aware Public Transportation Network Average Traveling Time Lower Bound* resp. *Wait aware* PTN ATT *Lower Bound* for short.

Theorem 4.2 (Wait aware PTN ATT Lower Bound)**.** *Let* PTN *be a public transportation network,* l^{wait} *the minimal wait time,* l^{change} *the minimal change time with* $l^{\text{wait}} \leq l^{\text{change}}$*,* OD *an origin destination matrix. For every feasible duration set* x *and* OD *derived passenger distribution* w *that can be obtained for* PTN *and* OD *by constructing an* EAN *with methods from Sections 2.3 or 3.1 on top of any line concept* LC *holds*

$$l_{\text{PTN}}^{\text{wait}} \leq \text{ATT}_w^x \quad , \tag{4.4}$$

with ATT_w^x *being the average traveling time from Definition 2.12.*

Proof. By Definition 2.3, every OD derived passenger distribution w has the property that w_a, $a \in \mathcal{A}$ is the sum of coefficients from OD of a linear combination of EAN passenger routes $p_{s_1 s_2}^{\text{EAN}}$, $s_1, s_2 \in S$, $w_{s_1 s_2} > 0$. Every such passenger route posesses a PTN route trace $P_{s_1 s_2}^{\text{PTN}}$ which, after Lemma 1.45 is connected and thus contains a path $p_{s_1 s_2}^{\text{PTN}}$ from s_1 to s_2 in the PTN. By the Lines Roll Out methods from Sections 2.3 and 3.1 drive activities in the EAN inherit their lower bounds from edges in the PTN. Therefore, if $\overline{d}_{s_1 s_2}$ and $\overline{d}_{s_1 s_2}^{\text{PTN}}$ are the lengths of the shortest path between s_1 and s_2 in the PTN resp. of $p_{s_1 s_2}^{\text{PTN}}$, both w.r.t. the minimal driving times and wait times $l^{\text{wait}} \leq l^{\text{change}}$ at every station as well as $d_{s_1 s_2}^{l,\text{EAN}}$ and $d_{s_1 s_2}^{x,\text{EAN}}$ the lengths of $p_{s_1 s_2}^{\text{EAN}}$ w.r.t. the activity lower bounds resp. x, then it holds

$$\overline{d}_{s_1 s_2} \leq \overline{d}_{s_1 s_2}^{\text{PTN}} \leq d_{s_1 s_2}^{l,\text{EAN}} \leq d_{s_1 s_2}^{x,\text{EAN}} \tag{4.5}$$

for all $s_1, s_2 \in S$ with $w_{s_1 s_2} > 0$ and thus

$$l_{\text{PTN}}^{\text{wait}} = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} \overline{d}_{s_1 s_2} w_{s_1 s_2} \leq \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} d_{s_1 s_2}^{x,\text{EAN}} w_{s_1 s_2} = \text{ATT}_w^x \quad , \tag{4.6}$$

which yields the theorem. □

Theorem 4.3. *It holds* $l_{\text{PTN}} \leq l_{\text{PTN}}^{\text{wait}}$.

Proof. For $l^{\text{wait}} = 0$ it holds $l_{\text{PTN}} = l_{\text{PTN}}^{\text{wait}}$ and since $l^{\text{wait}} > 0$ compared to $l^{\text{wait}} = 0$ does not decrease $\overline{d}_{s_1 s_2}$ for any $s_1, s_2 \in S$ it does not decrease $l_{\text{PTN}}^{\text{wait}}$ and thus the theorem follows. □

4.2 Line Planning

4.2.1 Line Concept Lower Bound

For an actually given LC $= (\vec{L}, F)$ the best scenario for the average traveling time would be if all lines were taken and never interfered, i.e. no headways influence timetables later. Therefore we run the classical Lines Roll Out with `frequency_as_attribute` model, Generate Changes, which yields an event activity network $\text{EAN}_{\text{LC}} = (\mathcal{E}_{\text{LC}}, \mathcal{A}_{\text{LC}})$, take the activities lower bounds as initial duration assumption, i.e. $x_a^{\text{init}} = l_a$, for all $a \in \mathcal{A}_{\text{LC}}$ in the Passenger Distribution Algorithm 4, obtain pairwise distances[2] $d_{s_1 s_2}^{\text{LC}}$ and sum up

$$l_{\text{LC}} = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} d_{s_1 s_2}^{\text{LC}} w_{s_1 s_2} \tag{4.7}$$

where $w_{s_1 s_2}$ is from OD $= (w_{s_1 s_2})_{s_1 s_2 \in S}$. We call l_{LC} the *Line Concept Average Traveling Time Lower Bound* or LC ATT *Lower Bound* for short.

Theorem 4.4. *For event activity networks as constructed by a combination of methods from Sections 2.3 or 3.1, the line concept average traveling time lower bound l_{LC} is a lower bound for the* ODPESP *objective function.*

Proof. Let EAN $= (\mathcal{E}, \mathcal{A})$ be an event activity network derived from PTN $= (S, \vec{E})$, LC $= (\vec{L}, F)$, OD $= (w_{s_1 s_2})_{s_1, s_2 \in S}$ and l_a being lower bounds, for all $a \in \mathcal{A}$ as well as $\mathcal{A}_p \subset \mathcal{A}$ the passenger usable activities. In the Quadratic Program 3.11 of the ODPESP the objective function $\text{obj}_{\text{ODPESP}}$ may be estimated by

$$\sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0, \\ a \in \mathcal{A}_p}} w_{s_1 s_2} p_{a s_1 s_2} x_a \geq \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0, \\ a \in \mathcal{A}_p}} w_{s_1 s_2} p_{a s_1 s_2} l_a = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} w_{s_1 s_2} \sum_{a \in \mathcal{A}_p} p_{a s_1 s_2} l_a \tag{4.8}$$

and since $p_{a s_1 s_2}$ must satisfy path constraints

$$\sum_{a \in \mathcal{A}_p} p_{a s_1 s_2} l_a \geq \tilde{d}_{s_1 s_2} \quad , \qquad \forall s_1, s_2 \in S, \; w_{s_1 s_2} > 0, \tag{4.9}$$

where $\tilde{d}_{s_1 s_2}$ is the length of a shortest path from s_1 to s_2 w.r.t.[3] the weights l_a. Further it holds

$$\tilde{d}_{s_1 s_2} = d_{s_1 s_2}^{\text{LC}} \quad , \qquad \forall s_1, s_2 \in S, \; w_{s_1 s_2} > 0, \tag{4.10}$$

[2]Note that $d_{s_1 s_2}^{\text{LC}}$ is not a distance in a line concept, but in EAN_{LC}, therefore wait and change activities are considered.

[3]Note that in general, $\sum_{a \in \mathcal{A}_p} p_{a s_1 s_2} l_a \neq \tilde{d}_{s_1 s_2}$, since the $p_{a s_1 s_2}$, $a \in \mathcal{A}_p$ in a feasible ODPESP solution are arbitrary paths resp. shortest paths w.r.t. *feasible durations* in an optimal solution as stated in Theorem 3.12 and not necessarily shortest paths w.r.t. lower bounds as can be seen in Figure 5.12, `frequency_as_attribute` model to the left.

since headway activities are not passenger usable and thus negliable and the `frequency_as_multiplicity` Lines Roll Out model, if used for EAN, does only introduce parallel paths with equal lower bounds compared to the `frequency_as_attribute` model. Therefore

$$\mathrm{obj}_{\mathrm{ODPESP}} = \sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0,\\ a\in\mathcal{A}_p}} w_{s_1s_2} p_{as_1s_2} x_a \geq \sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0}} d^{\mathrm{LC}}_{s_1s_2} w_{s_1s_2} = l_{\mathrm{LC}} \quad . \tag{4.11}$$

□

Theorem 4.5 (LC ATT Lower Bound)**.** *Let* PTN *be a public transportation network,* OD *an origin destination matrix and* LC *a line concept for* PTN*. For every feasible duration set x and* OD *derived passenger distribution w that can be obtained for* PTN*,* LC *and* OD *by constructing an* EAN *with methods from Sections 2.3 or 3.1 on top of* LC *holds*

$$l_{\mathrm{LC}} \leq \mathrm{ATT}^x_w \quad , \tag{4.12}$$

where ATT^x_w *is the average traveling time from Definition 2.12.*

Proof. By Corollary 3.17 the ODPESP minimizes the average traveling time over the domain of feasible timetables and OD derived passenger distributions. Latter implies that there is a fixed, underlying EAN. Theorem 4.4 shows that l_{LC} is a lower bound for ODPESP, independent of how that EAN was obtained from LC and therefore the theorem follows. □

Theorem 4.6. *Let* $\mathrm{LC}' = (\vec{L}', F')$, $\mathrm{LC} = (\vec{L}, F)$ *be two line concepts with the property that $\vec{L} \subset \vec{L}'$ and that if $f_\ell > 0$ in F, then $f_\ell > 0$ in F'. Then it holds*

$$l_{\mathrm{LC}'} \leq l_{\mathrm{LC}} \quad . \tag{4.13}$$

Proof. By construction through the `frequency_as_attribute` Lines Roll Out model, the event activity network EAN′ derived from LC′ contains EAN derived from LC. Therefore, all passenger routes from EAN are contained in EAN′ as well and thus $l_{\mathrm{LC}'} \leq l_{\mathrm{LC}}$. □

Theorem 4.7. *Let* PTN *be a public transportation network,* OD *an origin destination matrix and* LC *a line concept for* PTN *and $l^{\mathrm{wait}} \leq l^{\mathrm{change}}$, where l^{wait} and l^{change} are the minimal wait resp. change times. Then it holds*

$$l^{\mathrm{wait}}_{\mathrm{PTN}} \leq l_{\mathrm{LC}} \quad . \tag{4.14}$$

Proof. By Theorem 4.2 $l^{\mathrm{wait}}_{\mathrm{PTN}}$ is a lower bound for the average traveling time for arbitrary line concepts on PTN, if $l^{\mathrm{wait}} \leq l^{\mathrm{change}}$ and therefore for LC as well. □

4.2.2 Line Pool Lower Bound

Let $\mathfrak{L}$ be a line pool. Since the set of lines $\vec{L}$ in a line concept $\mathrm{LC} = (\vec{L}, F)$ used later is taken from $\mathfrak{L}$, the best scenario for the average traveling time would be if all lines were taken, which can be seen as a maximal extension of the line concept from the preceding Section 4.2.1, in the sense of Theorem 4.6. We thus define $\overline{\mathrm{LC}} = (\vec{\mathfrak{L}}, F)$, where $F = \{\ell \mapsto 1 : \forall\ \ell \in \vec{\mathfrak{L}}\}$. Further, we construct a dummy event activity network $\mathrm{EAN}_{\mathfrak{L}} = (\mathcal{E}_{\mathfrak{L}}, \mathcal{A}_{\mathfrak{L}})$ on top of $\overline{\mathrm{LC}}$, by using classical methods Lines Roll Out and Generate Changes as described in Section 2.3. Since we are only interested in lower bounds the `frequency_as_attribute` Lines Roll Out model is sufficient. Again, we compute shortest paths lengths $d^{\mathfrak{L}}_{s_1 s_2}$ for all $s_1, s_2 \in S$, $w_{s_1 s_2} > 0$ this time with lower bounds from $\mathcal{A}_{\mathfrak{L}}$ as weights and therefore

$$l_{\mathfrak{L}} = \sum_{\substack{s_1, s_2 \in S, \\ w_{s_1 s_2} > 0}} d^{\mathfrak{L}}_{s_1 s_2} w_{s_1 s_2} \tag{4.15}$$

with $w_{s_1 s_2}$ being the origin destination matrix OD entry for all $s_1, s_2 \in S$ and delivers the *Line Pool Average Traveling Time Lower Bound* or for short *Line Pool* ATT *Lower Bound* for a given line pool $\mathfrak{L}$.

Corollary 4.8 (Line Pool ATT Lower Bound)**.** *Let* PTN *be a public transportation network,* OD *an origin destination matrix and $\mathfrak{L}$ a line pool for* PTN*. For every feasible duration set x and* OD *derived passenger distribution w that can be obtained for* PTN*, $\mathfrak{L}$ and* OD *by constructing an* EAN *with methods from Sections 2.3 or 3.1 on top any* LC *derived from $\mathfrak{L}$ holds*

$$l_{\mathfrak{L}} \leq \mathrm{ATT}^x_w \quad , \tag{4.16}$$

with ATT^x_w being the average traveling time from Definition 2.12.

Proof. The line pool $\mathfrak{L}$ may be considered as an extension of a line concept LC derived from that pool, therefore Theorem 4.6 is applicable. □

4.3 Summary

Let ODPESP be a problem instance with optimal objective function $\mathrm{obj}^*_{\mathrm{ODPESP}}$ on an origin destination matrix OD and an event activity network EAN, derived from a line concept LC, derived from a line pool $\mathfrak{L}$ on a public transportation network PTN as well as $l^{\mathrm{wait}} \leq l^{\mathrm{change}}$. This chapter yields lower bounds l_{PTN}, $l^{\mathrm{wait}}_{\mathrm{PTN}}$, $l_{\mathfrak{L}}$ as well as l_{LC} and shows that they satisfy

$$l_{\mathrm{PTN}} \leq l^{\mathrm{wait}}_{\mathrm{PTN}} \leq l_{\mathfrak{L}} \leq l_{\mathrm{LC}} \leq \mathrm{obj}^*_{\mathrm{ODPESP}} \quad . \tag{4.17}$$

An evaluation for our large scale networks yields that $l^{\mathrm{wait}}_{\mathrm{PTN}}$, $l_{\mathfrak{L}}$ and l_{LC} are close together, while l_{PTN} is rather low. With Retimetabling we get as close as 6 to 7% to l_{LC}. For more details see Chapter 6.

Chapter 5

Worst Case Error

Event Activity Networks can be modeled in different ways, if we compare Chapters 2 and 3. All classical models are simplifications to the actual ODPESP problem with `frequency_as_multiplicity` LINES ROLL OUT model mentioned in Sections 3.2 resp. 3.1.1 and we want to know how large the relative error can get if we simplify. An initial analytic estimation from Section 5.1 is very rough. Can that gigantic error really occur in some real network? Therefore, we construct parametrized example networks from public transportation network level on and consider different scenarios to at least get an estimate for the lower bound of the worst case.

Although we consider the general ODPESP, some results are independend of the passenger distribution and thus directly applicable to PESP as well, about which the reader will be informed. If passengers can take different paths, we have a look at what happens after an iteration of PESP timetabling. Since we compare objective functions in the rolled out and rerouted network, rerouting is included before the additional PESP step. However, for the Section *Overestimation* 5.5 we make an exception and compare two different ODPESP objective functions: One of the EAN with `frequency_as_attribute` LINES ROLL OUT model, the other with `frequency_as_multiplicity`.

To distinguish between actual example and general quantities, we use the superscript X, e.g. $\mathrm{PTN}^{\mathrm{X}}$ is a public transportation network depicted in some figure in this chapter, while PTN is some arbitrarily given, general PTN.

5.1 Analytic Point

In our event activity network, the only passenger usable drive, wait and change activities limit the average traveling time by their lower and upper bounds l_a resp. u_a. However, are these u_a arbitrary large? Generally yes, but, e.g. $u_a - l_a \geq T$ does not make sense from an average traveling time point of view, as seen in Section 2.5, which makes us take a closer look at l_a. For wait, change and drive activities, l_a is determined by the passengers/vehicle enter/vehicle leave/walk speed and distances within stations resp. the vehicles speed and edge length. Every public transportation network has some longest edge, longest walk distance, slowest vehicle and slowest passenger, which are all network intrinsic and are, a priori, independend of the choice of the period length or can at least be bound by a fixed multiple of T and thus u_a may also be limited.

Definition 5.1 (Bounded By Period Property). *Let T be a period length. An* EAN $= (\mathcal{E}, \mathcal{A})$ *with upper bounds u_a satisfies the* bounded by period property, *if with $\mathcal{A}_p \subset \mathcal{A}$ being the set of passenger usable activities, it holds*

$$\exists\, k^{\max} \in \mathbb{N} : u_a \leq k^{\max} T \quad , \qquad \forall\, a \in \mathcal{A}_p \quad . \tag{5.1}$$

An activity that does not satisfy the bounded by period property has bounds of e.g. $[l_a, u_a] = [T^2 - 1, T^2]$. Since $u_a - l_a = 1$, this actually influences feasibility. However, at least to the author no szenario in public transportation is know in which such a bound parametrization would be useful.

For an actual EAN $= (\mathcal{E}, \mathcal{A})$ we can calculate

$$k^{\max} = \left\lceil \frac{\max_{a \in \mathcal{A}_p} u_a}{T} \right\rceil \quad . \tag{5.2}$$

Let us find an estimation for the maximal error magnitude.

Theorem 5.2 (Objective Values Quotient Estimation). *Let T be the period length,* PTN $= (S, E)$, LC $= (\vec{L}, F)$, OD $= (w_{s_1 s_2})_{s_1, s_2 \in S}$ *and* EAN $= (\mathcal{E}, \mathcal{A})$ *an associated event activity network with $\mathcal{A}_p \subset \mathcal{A}$ being the passenger usable activities as well as lower and upper bounds l_a resp. u_a, for all $a \in \mathcal{A}$. Further* EAN *satisfies the bounded by period property from Definition 5.1 with $k^{\max}$. Let*

$$\begin{aligned} w^{\min} &= \min_{\substack{s_1, s_2 \in S,\\ w_{s_1 s_2} > 0}} w_{s_1 s_2} \quad , & l^{\min} &= \min_{a \in \mathcal{A}_p, l_a > 0} l_a \quad , & & \\ w^{\max} &= \max_{\substack{s_1, s_2 \in S,\\ w_{s_1 s_2} > 0}} w_{s_1 s_2} \quad , & u^{\max} &= \max_{a \in \mathcal{A}_p, u_a > 0} u_a \quad , & \|\,\mathrm{OD}_{>0}\,\| &= \sum_{\substack{s_1, s_2 \in S,\\ w_{s_1 s_2} > 0}} 1 \quad , \end{aligned} \tag{5.3}$$

as well as obj_1 *and* obj_2 *two nonvanishing objective function values for two solutions of* ODPESP. *It then holds*

$$\frac{\mathrm{obj}_1}{\mathrm{obj}_2} \leq \frac{w^{\max} k^{\max}}{w^{\min} l^{\min}} \|\,\mathrm{OD}_{>0}\,\| |\mathcal{A}_p| T \quad \textit{resp.} \quad \frac{\mathrm{obj}_1}{\mathrm{obj}_2} \in \mathcal{O}(\|\,\mathrm{OD}_{>0}\,\| |\mathcal{A}_p| T) \quad . \tag{5.4}$$

Proof. Let $p_{a s_1 s_2}, x_a$ with $s_1, s_2 \in S$, $w_{s_1 s_2} > 0$, $a \in \mathcal{A}$ be solution for the ODPESP problem with nonvanishing objective function

$$\sum_{\substack{s_1, s_2 \in S,\\ w_{s_1 s_2} > 0,\\ a \in \mathcal{A}_p}} w_{s_1 s_2} p_{a s_1 s_2} x_a \quad . \tag{5.5}$$

Since the sum does not vanish and all summands are nonnegative, at least one summand therefore must be positive. It holds

$$w_{s_1s_2}p_{as_1s_2}x_a = w_{s_1s_2}x_a \geq w^{\min}l^{\min} \tag{5.6}$$

for all $s_1, s_2 \in S$, $w_{s_1s_2} > 0$, $a \in \mathcal{A}_p$ for which $w_{s_1s_2}p_{as_1s_2}x_a \overset{!}{>} 0$ and therefore

$$\sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0,\\ a\in\mathcal{A}_p}} w_{s_1s_2}p_{as_1s_2}x_a \geq w^{\min}l^{\min} \quad . \tag{5.7}$$

On the other hand

$$\begin{aligned}
\sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0,\\ a\in\mathcal{A}_p}} w_{s_1s_2}p_{as_1s_2}x_a &\leq \sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0,\\ a\in\mathcal{A}_p}} w_{s_1s_2}p_{as_1s_2}u_a \leq \sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0,\\ a\in\mathcal{A}_p}} w_{s_1s_2}u_a && (5.8)\\
&= \sum_{\substack{s_1,s_2\in S,\\ w_{s_1s_2}>0}} w_{s_1s_2} \sum_{a\in\mathcal{A}_p} u_a \leq w^{\max}u^{\max} \| \mathrm{OD}_{>0} \| |\mathcal{A}_p| && (5.9)\\
&\leq w^{\max}k^{\max} \| \mathrm{OD}_{>0} \| |\mathcal{A}_p| T \quad , && (5.10)
\end{aligned}$$

so for two different solutions with different objective function values obj_1 and obj_2 holds

$$\frac{\mathrm{obj}_1}{\mathrm{obj}_2} \leq \frac{w^{\max}k^{\max}}{\mathrm{obj}_2} \| \mathrm{OD}_{>0} \| |\mathcal{A}_p| T \leq \frac{w^{\max}k^{\max}}{w^{\min}l^{\min}} \| \mathrm{OD}_{>0} \| |\mathcal{A}_p| T \tag{5.11}$$

and thus the theorem follows. □

We basically need Theorem 5.2 for the fact that errors cannot grow more than linearly in T. Under different conditions, this error magnitude is reached and gives reason to talk about an *unbounded error for practical application*, at least when it comes to public transportation. Therefore, we vary only T and line frequencies but not the number of nonzero OD entries or passenger usable activities which, by Theorem 5.2 could basically also affect the worst case error.

For the networks we shall consider, for the variable $k^{\max}$ from Defintion 5.1 it holds $k^{\max} \leq 2$, since the greatest upper bound used is $T + 2$ for a drive activity derived from an edge in Figure 5.13.

5.2 Fixed Passengers

The PESP as in Section 2.5 uses a fixed passenger distribution w_a, $a \in \mathcal{A}$ in its objective function. But since the traveling time is a major criterion to the passengers, their distribution may change as soon as the network operator uses a different timetable. This effect can and empirically *will*, as we shall see in

Chapter 6, lead to our timetable to not be in a ODPESP optimum if we use PESP timetabling.

In this section, we use a very simple $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} and OD^{X} from Figure 5.1, and consider an associated event activity network $\mathrm{EAN}^{\mathrm{X}}$ as in Figures 5.2a and 5.3a with `frequency_as_attribute` as LINES ROLL OUT model, since for the actual estimation frequencies are negligible. The period length is a parameter and figures see below text.

Let us consider the public transportation network $\mathrm{PTN}^{\mathrm{X}} = (V, E)$ as depicted in Figure 5.1 and assume that there is only one passenger in total, traveling from s_2 to s_1. The line ℓ_1 needs a whole period length T to get from s_3 to s_1, while ℓ_2 needs only one time unit. Our passenger must definitely take ℓ_1 to get from s_2 to s_3 and then either stay in ℓ_1 or change to ℓ_2 to get from s_3 to s_1.

Lemma 5.3. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$*,* LC^{X} *and* OD^{X} *as in Figure 5.1,* $\mathrm{EAN}^{\mathrm{X}}$ *an associated event activity network as in Figure 5.2 and* $\mathrm{obj}_{p_1}^{\mathrm{X}*}$ *the optimal* ODPESP *objective value for a fixed* OD^{X} *derived passenger distribution p_1 (effectively* PESP*). It holds*

$$\mathrm{obj}_{p_1}^{\mathrm{X}*} = T + 2 \tag{5.12}$$

and there exist timetables which cannot attain a better objective by an iteration of shortest duration rerouting and PESP *timetabling, if the solver used operates deterministially.*

Proof. Since all timespans are fixed, $\mathrm{obj}_{p_1}^{\mathrm{X}*} = T + 2$ is the only objective function value possible for p_1. Let the timetable π_{p_1} as in Figure 5.2b be obtained by a linear solver. There are only two OD^{X} derived passenger distributions possible: p_1 and p_2 as depicted in 5.2 resp. 5.3, where p_1 takes passengers $T+2$ and p_2 uses $T+3$ time units. Therefore duration shortest paths rerouting does not change the passenger distribution. Thus, an additional PESP timetabling step in the iteration yields π_{p_1} again if the solver operates deterministically. □

Lemma 5.4. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$*,* LC^{X} *and* OD^{X} *as in Figure 5.1,* $\mathrm{EAN}^{\mathrm{X}}$ *an associated event activity network as in Figure 5.3 and* $\mathrm{obj}_{p_2}^{\mathrm{X}*}$ *the optimal* ODPESP *objective value for a fixed* OD^{X} *derived passenger distribution p_2 (effectively* PESP*) and* $\mathrm{obj}^{\mathrm{X}*}$ *the global* ODPESP *optimum. It holds*

$$\mathrm{obj}^{\mathrm{X}*} = \mathrm{obj}_{p_2}^{\mathrm{X}*} = 3 \quad . \tag{5.13}$$

Proof. See caption of Figure 5.3. □

Theorem 5.5 (Fixed Passengers Worst Case Relative Error Lower Bound)**.** *Let* EAN *be an event activity network that satisfies the bounded by period property from definition 5.1 and derived from* PTN*,* LC*,* OD *and period length T. Let further* obj^* *be the* ODPESP *global optimum objective function value resp.* obj_p^* *the* ODPESP *optimum for a fixed* OD *derived passenger distribution p (effectively* PESP*). It holds*

1. $\max\limits_{\substack{\text{PTN,}\\ \text{LC,OD}}} \dfrac{\text{obj}_p^*}{\text{obj}^*} \geq \dfrac{T+2}{3}$.

2. The error magnitude is maximal in T.

Proof. 1. EAN^{X} satisfies the bounded by period property with $k^{\max} = 2$, since the greatest upper bound is $T + 1$. Therefore, Lemmata 5.3 and 5.4 may be combined to

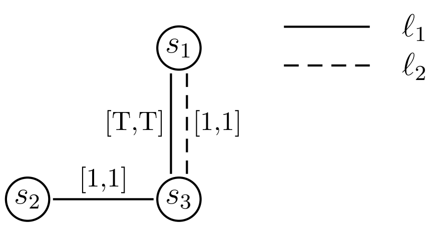
$$\max_{\substack{\text{PTN,}\\ \text{LC,OD}}} \frac{\text{obj}_p^*}{\text{obj}^*} \geq \frac{\text{obj}_{p_1}^{\text{X}*}}{\text{obj}_{p_2}^{\text{X}*}} = \frac{\text{obj}_{p_1}^{\text{X}*}}{\text{obj}^{\text{X}*}} = \frac{T+2}{3} \quad . \tag{5.14}$$

2. Apply Theorem 5.2.

□

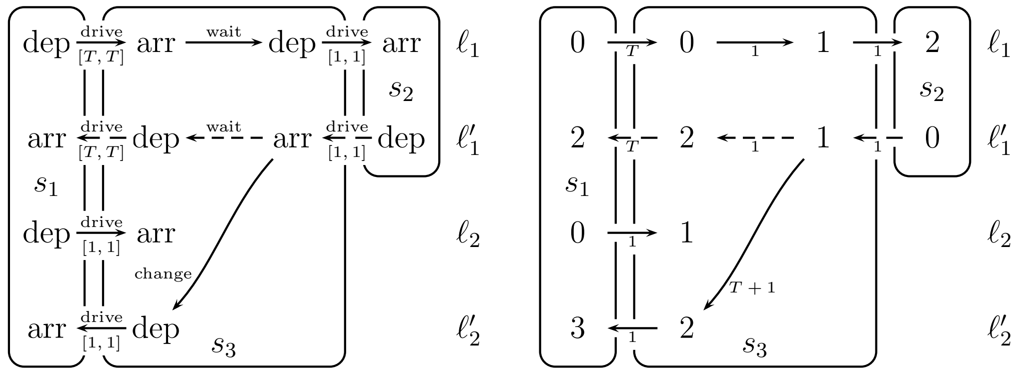
Table 5.1: Fixed passengers worst case relative error lower bounds for different period lengths T, see Theorem 5.5.

T	2	5	15	30	60	120	600	1200	2400
$\frac{T+2}{3}$	1.33	2.33	5.67	10.67	20.67	40.67	200.67	400.67	800.67



$$\text{change} \in [2, T+1], \quad \text{wait} \in [1,1], \quad \text{OD}_{ss'} = \begin{cases} 1 & (s,s') = (s_2, s_1) \\ 0 & \text{otherwise} \end{cases}, \quad \forall s, s' \in S.$$

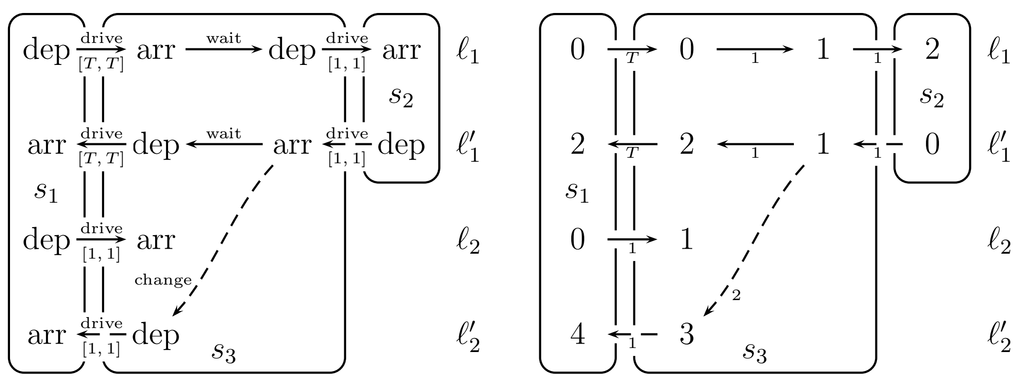
Figure 5.1: A public transportation network $\text{PTN}^{\text{X}} = (V, E)$ that has three stations $V = \{s_1, s_2, s_3\}$ and three edges $E = \{\overline{e_{13}}, \overline{e_{23}}, \bar{\overline{e_{13}}}\}$, each with equal lower and upper bounds $\overline{l_{23}} = \overline{u_{23}} = 1$, $\overline{l_{13}} = \overline{u_{13}} = T$ and $\bar{\overline{l_{13}}} = \bar{\overline{u_{13}}} = 1$ together with a line concept $\text{LC}^{\text{X}} = (L, F)$ with two lines $L = \{\ell_1, \ell_2\}$ as well as an asymmetric origin destination OD^{X} matrix that states that there is only one passenger in the network that wants to go from s_2 to s_1.



(a) Passenger Distribution p_1

(b) An Optimal Timetable π_{p_1} for Passenger Distribution p_1

Figure 5.2: (a) depicts the $\text{EAN}^{\text{X}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN^{X} from Figure 5.1, together with an OD^{X} derived passenger distribution p_1 (dashed line). Since change times span a period length, we consider only those change activities relevant to OD^{X}. (b) shows a timetable π_{p_1} (with durations) that minimizes the average traveling time $\text{obj}_{p_2}^{\text{X}*} = T + 3$ for p_1. Note that rerouting passengers by duration shortest paths and solving the PESP for the resulting distribution only allows to find a better timetable as in Figure 5.3 if the solver operates (pseudo-)nondeterministically.



(a) Passenger Distribution p_2

(b) An Optimal Timetable π_{p_2} for Passenger Distribution p_2

Figure 5.3: (a) depicts the $\text{EAN}^{\text{X}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN^{X} from Figure 5.1, together with an OD^{X} derived passenger distribution p_2 (dashed line). Since change times span a period length, we consider only those change activities relevant to OD^{X}. (b) shows a timetable π_2 (with durations) that minimizes the average traveling time for p_2 and yields the global optimum $\text{obj}^{\text{X}*} = \text{obj}_{p_2}^{\text{X}*} = 3$, since p_1 from Figure 5.2 and p_2 are the only OD^{X} derived distributions possible and $\text{obj}_{p_2}^{\text{X}*} < \text{obj}_{p_1}^{\text{X}*}$ for $T > 1$.

5.3 Fixed Moduli

When we have a feasible periodic timetable, we can fix the modulo parameters z_c from Section 2.5. This effectively turns the periodic event scheduling into an aperiodic event scheduling problem, which can be solved in polynomial time. However, this affects the average traveling time objective.

As in the section before, we use a very simple $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} and OD^{X} from Figure 5.4, and consider an associated event activity network $\mathrm{EAN}^{\mathrm{X}}$ as in Figure 5.5 with `frequency_as_attribute` as LINES ROLL OUT model, since for the actual estimation frequencies are negligible. Figures see below text.

Lemma 5.6. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$*,* LC^{X} *and* OD^{X} *as in Figure 5.4 and* $\mathrm{EAN}^{\mathrm{X}}$ *an associated event activity network as in Figure 5.5 with* $\mathrm{obj}^{\mathrm{X}*}$ *as the* ODPESP *global optimum objective value as well as* $\mathrm{obj}_0^{\mathrm{X}*}$ *and* $\mathrm{obj}_1^{\mathrm{X}*}$ *being optimal for $z = 0$ resp. $z = 1$ in Figure 5.5b. Independent of a* OD^{X} *derived passenger distribution, i.e. also for* PESP*, it holds*

$$\mathrm{obj}^{\mathrm{X}*} = \mathrm{obj}_0^{\mathrm{X}*} = 6 \quad , \qquad \mathrm{obj}_1^{\mathrm{X}*} = T + 6 \tag{5.15}$$

and $z \in \{0, 1\}$ are the only feasible modulo parameters.

Proof. In $\mathrm{EAN}^{\mathrm{X}}$ there is only one nontrivial cycle, which gives the constraint

$$x_1 - 1 + x_2 - 1 = zT \quad , \qquad x_1, x_2 \in \{2, \dots, T + 1\},\ z \in \{0, 1\}, \tag{5.16}$$

with x_1 and x_2 being the change durations and the modulo parameter $z \in \{0, 1\}$ due to bounds of x_1 and x_2, Lemma 2.24 (Odijk). Further

$$\mathrm{obj}^{\mathrm{X}}(x_1, x_2) = x_1 + x_2 + 4 \tag{5.17}$$

is the ODPESP objective function from Section 3.2 since there is only one OD^{X} derived passenger distribution possible and therefore coincides with the PESP objective. Setting $z = 1$ leads to $\mathrm{obj}_1^{\mathrm{X}} = T + 6$, so the global optimum is $z = 0$ with $\mathrm{obj}^{\mathrm{X}*} = \mathrm{obj}_0^{\mathrm{X}*} = 6$. □

Theorem 5.7 (Fixed Moduli Worst Case Relative Error Lower Bound)**.** *Let* EAN *be an event activity network that satisfies the bounded by period property from Definition 5.1 derived from* PTN*,* LC*,* OD *and period length T. Let further* obj^* *be the* ODPESP *global optimum objective function value resp.* obj_z^* *the* ODPESP *optimum for fixed moduli. Independend of a* OD *derived passenger distribution, i.e. also for* PESP*, it holds*

1. $\displaystyle\max_{\substack{\mathrm{PTN},\\ \mathrm{LC,OD}}} \frac{\mathrm{obj}_z^*}{\mathrm{obj}^*} \geq \frac{T}{6} + 1 \quad .$

2. *The error magnitude is maximal in T.*

3. *After an iteration of* PESP *timetabling non-fixed moduli, the error vanishes.*

Proof. 1. The greatest upper bound is $T+1$, therefore $\mathrm{EAN}^{\mathrm{X}}$ satisfies the bounded by period property with $k^{\max}=2$. Lemma 5.6 holds

$$\max_{\substack{\mathrm{PTN,}\\ \mathrm{LC,OD}}} \frac{\mathrm{obj}_p^*}{\mathrm{obj}^*} \geq \frac{\mathrm{obj}_1^{\mathrm{X}*}}{\mathrm{obj}_0^{\mathrm{X}*}} = \frac{\mathrm{obj}_1^{\mathrm{X}*}}{\mathrm{obj}^{\mathrm{X}*}} = \frac{T+6}{6} = \frac{T}{6}+1 \quad . \tag{5.18}$$

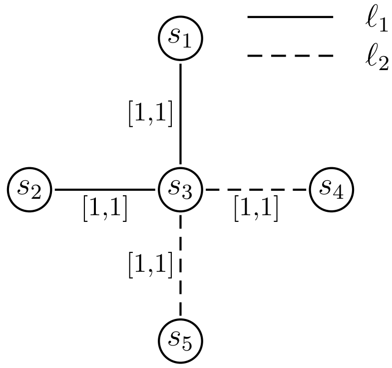
2. Theorem 5.2 is applicable.

3. The error vanishes by definition, since in the PESP model, modulo parameters are variables.

□

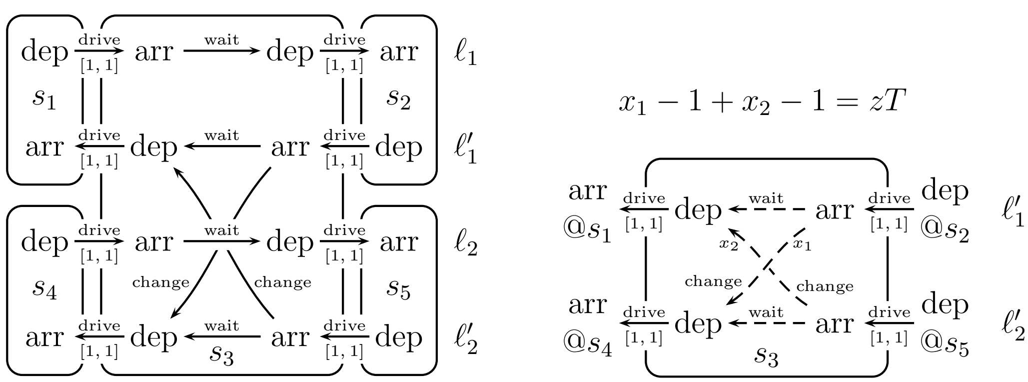
Table 5.2: Fixed moduli worst case relative error lower bounds for different period lengths T, see Theorem 5.7.

T	2	5	15	30	60	120	600	1200	2400
$\frac{T}{6}+1$	1.33	1.83	3.5	6	11	21	101	201	401



$$\begin{aligned} \text{change} &\in [1, T+1], \\ \text{wait} &\in [1, 1], \\ \mathrm{OD}^{\mathrm{X}}_{ss'} &= \begin{cases} 1 & (s,s') = (s_2, s_4) \\ & (s,s') = (s_5, s_1) \\ 0 & \text{otherwise} \end{cases}, \\ & \qquad\qquad \forall s, s' \in S. \end{aligned}$$

Figure 5.4: A public transportation network $\mathrm{PTN}^{\mathrm{X}} = (S, E)$ that has five stations $S = \{s_1, s_2, s_3, s_4, s_5\}$ and four edges $E = \{e_{13}, e_{23}, e_{34}, e_{35}\}$, all with equal lower and upper bounds $l_{13} = u_{13} = l_{23} = u_{23} = l_{34} = u_{34} = l_{35} = u_{35} = 1$ together with a line concept $\mathrm{LC}^{\mathrm{X}} = (L, F)$ with two lines $L = \{\ell_1, \ell_2\}$ as well as an asymmetric origin destination OD^{X} matrix that states that there is only two passengers in the network: one that wants to go from s_2 to s_4 and one from s_5 to s_1.



(a) Event Activity Network, Frequency as Attribute

(b) Nontrivial Cycles

Figure 5.5: (a) depicts the frequency as attribute $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}} = (\mathcal{A}, \mathcal{E})$ that belongs to the $\mathrm{PTN}^{\mathrm{X}}$ from Figure 5.4. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. The only cycle possible is shown in (b).

5.4 Line Concept

One may assume: the less and longer lines, the lower the ODPESP objective, since passengers have to change between lines less frequently and save valuable time by taking the, usually lower, waiting time. However, this is not true in general. The reason for this effect on the traveling time are bounds being too stiff. Especially, the line concept is cruical for timetabling.

As before, we use a very simple $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} and OD^{X} from Figure 5.6, and consider an associated event activity network $\mathrm{EAN}^{\mathrm{X}}$ as in Figure 5.7 with `frequency_as_attribute` as Lines Roll Out model, since for the actual estimation frequencies are negligible. Unless we refer to the Section 5.3 before, figures see below text.

Lemma 5.8. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} *and* OD^{X} *as in Figure 5.6 and* $\mathrm{EAN}^{\mathrm{X}}$ *an associated event activity network as in Figure 5.7 with* $\mathrm{obj}^{\mathrm{X}*}$ *as the* ODPESP *global optimum objective value. Independend of an* OD^{X} *derived passenger distribution, i.e. also for* PESP*, it holds*

$$\mathrm{obj}^{\mathrm{X}*} = 8 \quad . \tag{5.19}$$

Proof. In $\mathrm{EAN}^{\mathrm{X}}$ there is not a single nontrivial cycle, i.e. no constraints. Therefore, all activities attain their lower bounds and therefore the objective value is

$$\mathrm{obj}^{\mathrm{X}}(x_1, x_2) = x_1 + x_2 + 4 = 2 + 2 + 4 = 8 \quad , \tag{5.20}$$

since there is only one OD^{X} derived passenger distribution possible. □

Theorem 5.9 (Line Concept Worst Case Relative Error Lower Bound)**.** *Let* EAN, EAN′ *be two event activity networks for which holds the bounded by period property from Definition 5.1 and both derived from* PTN, LC *resp.* LC′, OD *and period length T. Let further* obj^* *be the* ODPESP *optimal objective function value for* EAN *resp.* obj'^* *for* EAN′*. Independend of the passenger distribution, i.e. also for* PESP*, it holds*

1. $$\max_{\substack{\mathrm{PTN,LC,}\\ \mathrm{LC',OD}}} \frac{\mathrm{obj}^*}{\mathrm{obj}'^*} \geq \frac{T+6}{8} \quad .$$

2. *The error magnitude is maximal in T.*

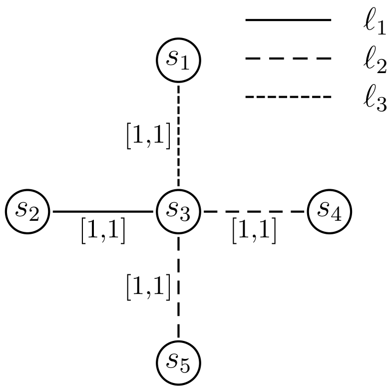
Proof.
1. Lemmata 5.6 and 5.8 together hold the theorem, since $k^{\max} = 2$ for the bounded by period property.

2. Apply Theorem 5.2.

□

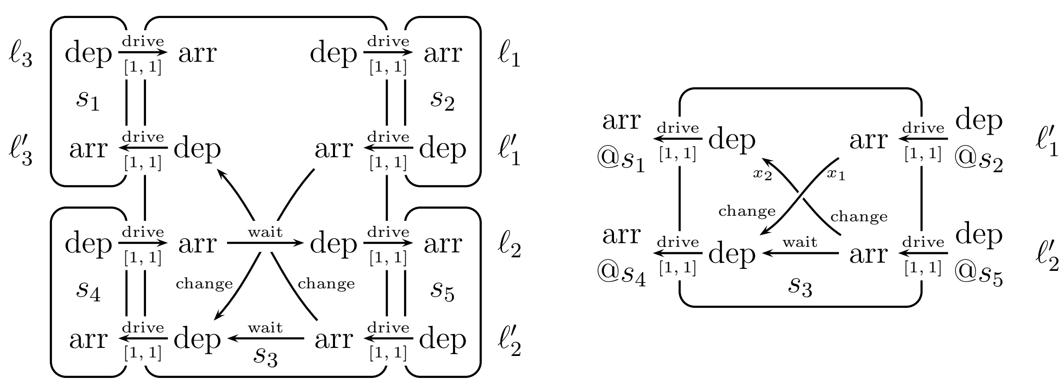
Table 5.3: Line concept worst case relative error lower bounds for different period lengths T, see Theorem 5.9.

T	2	5	15	30	60	120	600	1200	2400
$\frac{T+6}{8}$	1	1.375	2.625	4.500	8.25	15.75	75.75	150.75	300.75



$$\text{change} \in [2, T+1],$$
$$\text{wait} \in [1, 1],$$
$$\mathrm{OD}^{\mathrm{X}}_{ss'} = \begin{cases} 1 & (s, s') = (s_2, s_4) \\ & (s, s') = (s_5, s_1) \\ 0 & \text{otherwise} \end{cases}, \quad \forall s, s' \in S.$$

Figure 5.6: A public transportation network $\mathrm{PTN}^{\mathrm{X}} = (S, E)$ that has five stations $S = \{s_1, s_2, s_3, s_4, s_5\}$ and four edges $E = \{e_{13}, e_{23}, e_{34}, e_{35}\}$, all with equal lower and upper bounds $l_{13} = u_{13} = l_{23} = u_{23} = l_{34} = u_{34} = l_{35} = u_{35} = 1$ together with a line concept $\mathrm{LC}^{\mathrm{X}} = (L, F)$ with this time three lines $L = \{\ell_1, \ell_2, \ell_3\}$ as well as an asymmetric origin destination OD^{X} matrix that states that there is only two passengers in the network: one that wants to go from s_2 to s_4 and one from s_5 to s_1.



(a) Event Activity Network, Frequency as Attribute

(b) No Nontrivial Cycles

Figure 5.7: (a) depicts the frequency as attribute $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}} = (\mathcal{A}, \mathcal{E})$ that belongs to the $\mathrm{PTN}^{\mathrm{X}}$ from Figure 5.6. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. There are no nontrivial cycles, take (b) as comparison to 5.5b.

5.5 Overestimation

Unlike in the other sections, this time we compare two different ODPESP objective functions: the one of the EAN with `frequency_as_attribute` LINES ROLL OUT model, the other with `frequency_as_multiplicity` and show that the ODPESP timetabling objective in the former case overestimates the actual average traveling time obtained from the latter case. The example in this section has two nice properties to evaluate overestimation: In the `frequency_as_attribute` case, there is only one passenger distribution possible and the requirements are very low, since we need just two lines with arbitrary frequencies that cross at some station.

Therefore, throughout the section, we use a very simple $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} and OD^{X} from Figure 5.8, and consider an associated event activity networks $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ and $\mathrm{EAN}^{\mathrm{X}}_{f_1 f_2}$ as in Figures 5.9 resp. 5.11 with `frequency_as_attribute` resp. `frequency_as_multiplicity` as LINES ROLL OUT models, frequencies f_1, f_2 and period length being parameters. $\mathrm{PTN}^{\mathrm{X}}$ is identical to that of Section 5.3, but for self-containment purposes we mention it again. Figures see below text.

Lemma 5.10. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} *and* OD^{X} *as in Figure 5.8 and* $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ *an associated event activity network with the* LINES ROLL OUT *model* ***`frequency_as_attribute`*** *as in Figure 5.9 with* $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}}$ *as the optimal* ODPESP *objective value. Independent of an* OD^{X} *derived passenger distribution, i.e. also for* PESP*, it holds*

$$\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}} = T + 6 \quad . \tag{5.21}$$

Proof. In $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ there is only one OD^{X} derived passenger distribution possible and only one nontrivial cycle, which gives the constraint

$$x_1 - 1 + x_2 - 1 = zT \quad , \qquad x_1, x_2 \in \{2, \dots, T+1\},\ z \in \{1, 2\}, \tag{5.22}$$

where x_1 and x_2 are the change durations. Further, $z \in \{1, 2\}$ due to bounds of x_1 and x_2 or more generally the Lemma 2.24 (Odijk). Let

$$\mathrm{obj}^{\mathrm{X}}_{\mathrm{FA}}(x_1, x_2) = x_1 + x_2 + 4 \tag{5.23}$$

be the ODPESP objective function from Section 3.2. Since $z = 2$ leads to $\mathrm{obj}^{\mathrm{X}}_{\mathrm{FA}} = 2T + 6$, the choice is $z = 1$ with $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}} = T + 6$. □

Let us have a look at fixed line frequencies for introductory purposes.

Lemma 5.11. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} *and* OD^{X} *as in Figure 5.8 and* $\mathrm{EAN}^{\mathrm{X}}_{1,2}$ *an associated event activity network with the* LINES ROLL OUT *model* ***`frequency_as_multiplicity`*** *as in Figure 5.10 with* $\mathrm{obj}^{\mathrm{X}*}_{1,2}$ *as the optimal* ODPESP *objective value. It holds*

$$\mathrm{obj}^{\mathrm{X}*}_{1,2} = \tfrac{1}{2}T + 6 \quad . \tag{5.24}$$

Proof. There are only four OD^{X} derived passenger distributions possible, of which two have the same cycle as the former Figure 5.9b, and two are equivalent to Figure 5.10b. In the latter case, there is again is only one (nontrivial) cycle that yields the constraint

$$x_1 + \frac{T}{2} - 1 + x_2 - 1 = zT \quad , \qquad x_1, x_2 \in \{2, \dots, T+1\},\ z \in \{1, 2\}, \tag{5.25}$$

where x_1 and x_2 are the change durations, and again $z \in \{1, 2\}$ due to the bounds of x_1 and x_2. The objective function still looks same

$$\mathrm{obj}^{\mathrm{X}}_{1,2}(x_1, x_2) = x_1 + x_2 + 4 \tag{5.26}$$

and since $z = 2$ gives $\mathrm{obj}^{\mathrm{X}}_{1,2} = \frac{3}{2}T + 6$ the choice is $z = 1$ and at the end $\mathrm{obj}^{\mathrm{X}*}_{1,2} = \frac{1}{2}T + 6$. $\square$

We confront the general case.

Lemma 5.12. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$*,* LC^{X} *and* OD^{X} *as in Figure 5.8 and* $\mathrm{EAN}^{\mathrm{X}}_{f_1 f_2}$ *an associated event activity network with the* Lines Roll Out *model* `frequency_as_multiplicity` *as in Figure 5.11 with* $\mathrm{obj}^{\mathrm{X}*}_{f_1 f_2}$ *as the optimal* ODPESP *objective value and $l = \mathrm{lcm}(f_1, f_2)$. It holds*

$$\mathrm{obj}^{\mathrm{X}*}_{f_1 f_2} = \frac{(1 + \lfloor \frac{l}{T} \rfloor)}{l} T + 6 \quad . \tag{5.27}$$

Proof. The frequencies of lines ℓ_1 and ℓ_2 are f_1 resp. f_2, so there are up to $(f_1 f_2)^2$ possible OD^{X} derived passenger distributions, depending on which of the $f_1 f_2$ changes at s_3 the passenger from s_2 to s_4 resp. from s_5 to s_1 takes. Without loss of generality the change from ℓ_1 to ℓ_2 may be fixed as depicted in Figure 5.11. The remaining distributions can be parametrized by the second change, i.e. i and j as in Figure 5.11a which yields the only nontrivial cycle $\mathcal{C}$

$$x_1 - j\frac{T}{f_2} - 1 + x_2 + i\frac{T}{f_1} - 1 = zT \quad , \qquad x_1, x_2 \in \{2, \dots, T+1\},\ z \in \mathbb{Z}, \tag{5.28}$$

where x_1 and x_2 are again the change durations and bounds for z are yet to be determined. This is equivalent to the modulo equation

$$x_1 + x_2 - 2 + i\frac{T}{f_1} - j\frac{T}{f_2} = 0 \mod T \quad , \qquad x_1, x_2 \in \{2, \dots, T+1\}, \tag{5.29}$$

which the compact representation by lcm (Theorem 1.11) with $l := \mathrm{lcm}(f_1, f_2)$, $k \in \{0, \dots, l-1\}$ rewrites as

$$i\frac{T}{f_1} - j\frac{T}{f_2} = \frac{kT}{l} \mod T \tag{5.30}$$

and thus

$$x_1 + x_2 + \frac{kT}{l} - 2 = zT \quad , \qquad x_1, x_2 \in \{2, \ldots, T+1\},\ z \in \mathbb{Z}, \tag{5.31}$$

A lower bound estimation $a_{\mathcal{C}}$ for z yields

$$a_{\mathcal{C}} = \left\lceil \frac{2 + 2 + \frac{kT}{l} - 2}{T} \right\rceil = \left\lceil \frac{2}{T} + \frac{k}{l} \right\rceil \geq 1 \quad . \tag{5.32}$$

The upper bound $b_{\mathcal{C}}$ for z in a similar fashion

$$b_{\mathcal{C}} = \left\lfloor \frac{T + 1 + T + 1 + \frac{kT}{l} - 2}{T} \right\rfloor = \left\lfloor 2 + \frac{k}{l} \right\rfloor = 2 \quad . \tag{5.33}$$

The objective function again looks same

$$\mathrm{obj}^{\mathrm{X}}_{f_1 f_2}(x_1, x_2) = x_1 + x_2 + 4 \tag{5.34}$$

and replacing $x_1 + x_2$ with constraint $\mathcal{C}$ gives

$$\mathrm{obj}^{\mathrm{X}}_{f_1 f_2}(x_1, x_2) = \left(z - \frac{k}{l}\right) T + 6 \quad . \tag{5.35}$$

On the first sight, $k = l - 1$ and $z = 1$ are the most desirable. However, it is not always possible, because certain k can make $a_{\mathcal{C}} = 2$ and force $z = 2$. A closer look at the final term in the ceil function Equation (5.32) unveils that

$$\frac{2}{T} + \frac{k}{l} \leq 1 \tag{5.36}$$

must be satisfied for $a_{\mathcal{C}} = 1$. Substituting $k = l - \overline{k}$, $\overline{k} \in \{1, \ldots, l\}$ yields

$$\frac{2}{T} + \frac{l - \overline{k}}{l} \leq 1 \quad , \tag{5.37}$$

which is the case iff

$$l \leq \frac{\overline{k} T}{2} \quad . \tag{5.38}$$

Therefore, $\overline{k} = 2$ i.e. $k = l - 2$ always works and $k = l - 1$ functions iff

$$l \leq \frac{T}{2} \quad , \tag{5.39}$$

which is only violated for very high frequencies, i.e. since two is the smallest prime divisor, iff $l = T$. Thus for latter case

$$\mathrm{obj}^{\mathrm{X}}_{f_1 f_2} = \frac{T}{l} + 6 \tag{5.40}$$

for the former

$$\mathrm{obj}^{\mathrm{X}}_{f_1 f_2} = \frac{2T}{l} + 6 \quad , \tag{5.41}$$

generally

$$\mathrm{obj}^{\mathrm{X}*}_{f_1 f_2} = \frac{(1 + \lfloor \frac{l}{T} \rfloor)T}{l} + 6 \tag{5.42}$$

□

Theorem 5.13 (Overestimation Worst Case Error Lower Bound). *Let* $\mathrm{EAN}_{\mathrm{FA}}$ *and* $\mathrm{EAN}_{\mathrm{FM}}$ *be two event activity networks that satisfy the bounded by period property from Definition 5.1 with optimal* ODPESP *objective function values* $\mathrm{obj}^*_{\mathrm{FA}}$ *resp.* $\mathrm{obj}^*_{\mathrm{FM}}$ *derived from a common* $\mathrm{PTN} = (S, E)$, $\mathrm{LC}(l) = (\vec{L}, F(l))$, OD *and common period length T with only different construction models, i.e.* ***frequency_as_attribute*** *resp.* ***frequency_as_multiplicity****, where $f_1, f_2 \in F$, $f_1, f_2 | T$ are the frequencies of two lines ℓ_1 and ℓ_2 that cross at least at one station $s \in S$ and $l = \mathrm{lcm}(f_1, f_2)$ is maximal among all crossing lines.*

$$\max_{\substack{\mathrm{PTN},\\ \mathrm{LC}(l),\mathrm{OD}}} \frac{\mathrm{obj}_{\mathrm{FM}^*}}{\mathrm{obj}_{\mathrm{FA}^*}} \geq \frac{\frac{1+\lfloor l/T \rfloor}{l} T + 6}{T+6} \quad , \qquad \forall\, l = \mathrm{lcm}(f_1, f_2) \in \{1, \ldots, T\}. \tag{5.43}$$

Proof. The greatest upper bound is $T+1$, thus $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ and $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FM}}$ both satisfies the bounded by period property with $k^{\max} = 2$. Therefore combine Lemmata 5.10 and 5.12

$$\max_{\substack{\mathrm{PTN},\\ \mathrm{LC}(l),\mathrm{OD}}} \frac{\mathrm{obj}^*_{\mathrm{FM}}}{\mathrm{obj}^*_{\mathrm{FA}}} \geq \frac{\mathrm{obj}^{\mathrm{X}*}_{f_1 f_2}}{\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FM}}} = \frac{\frac{1+\lfloor l/T \rfloor}{l} T + 6}{T+6} \quad , \qquad \begin{array}{c} \forall\, l = \mathrm{lcm}(f_1, f_2) \\ \in \{1, \ldots, T\} \end{array}. \tag{5.44}$$

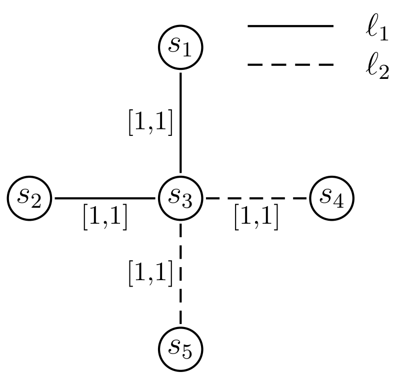
□

For Theorem 5.13 in numbers see Table 5.4.

Please note that we just compare objective function values. After a Periodic Rollout and rerouting passengers the objective most likely improves drastically, but not necessarily in general. Therefore, have a look at Section 5.6.

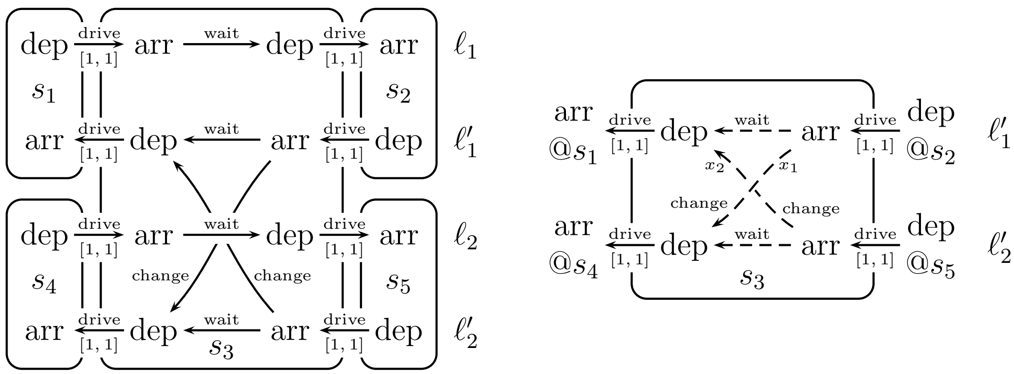
Table 5.4: Frequency representation worst case relative error lower bounds for different period lengths T and different maximal $\mathrm{lcm}(f_1, f_2)$, see Theorem 5.13.

	Period Length T								
$\mathrm{lcm}(f_1, f_2)$	2	5	15	30	60	120	600	1200	2400
2	1.00			1.71	1.83	1.91	1.98	1.99	2.00
3			1.91	2.25	2.54	2.74	2.94	2.97	2.99
4				2.67	3.14	3.50	3.88	3.94	3.97
5		1.38	2.33	3.00	3.67	4.20	4.81	4.90	4.95
6			2.47	3.27	4.13	4.85	5.72	5.85	5.93
12			2.90	4.24	6.00	7.88	10.82	11.38	11.68
15			2.63	4.50	6.60	9.00	13.17	14.02	14.49
20				4.80	7.33	10.50	16.83	18.27	19.10
30				4.50	8.25	12.60	23.31	26.22	27.98
60					8.25	15.75	37.88	46.38	52.30
120						15.75	55.09	75.38	92.54
600							75.75	150.75	240.60
1200								150.75	300.75
2400									300.75



$$\begin{aligned}\text{change} &\in [2, T+1],\\ \text{wait} &\in [1,1],\\ \mathrm{OD}^{\mathrm{X}}_{ss'} &= \begin{cases} 1 & (s,s') = (s_2, s_4) \\ & (s,s') = (s_5, s_1) \\ 0 & \text{otherwise} \end{cases}, \\ & \qquad\qquad \forall s, s' \in S.\end{aligned}$$

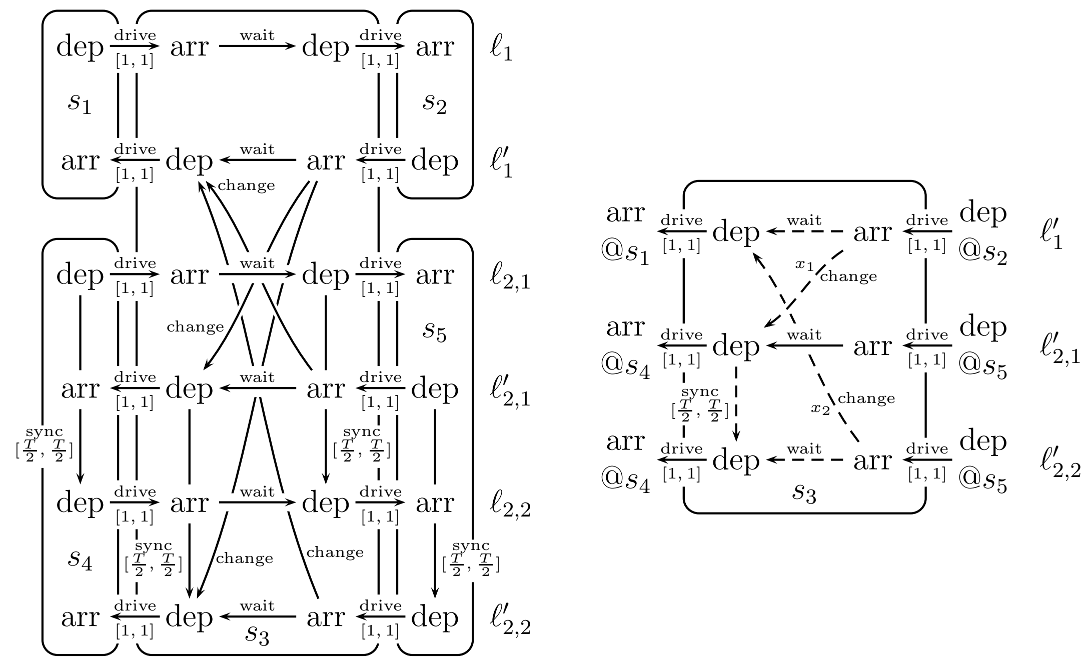
Figure 5.8: A public transportation network $\mathrm{PTN}^{\mathrm{X}} = (S, E)$ that has five stations $S = \{s_1, s_2, s_3, s_4, s_5\}$ and four edges $E = \{e_{13}, e_{23}, e_{34}, e_{35}\}$, all with equal lower and upper bounds $l_{13} = u_{13} = l_{23} = u_{23} = l_{34} = u_{34} = l_{35} = u_{35} = 1$ together with a line concept $\mathrm{LC}^{\mathrm{X}} = (L, F)$ with two lines $L = \{\ell_1, \ell_2\}$ as well as an asymmetric origin destination OD^{X} matrix that states that there is only two passengers in the network: one that wants to go from s_2 to s_4 and one from s_5 to s_1.



(a) Event Activity Network,
Frequency as Attribute

(b) Nontrivial Cycles

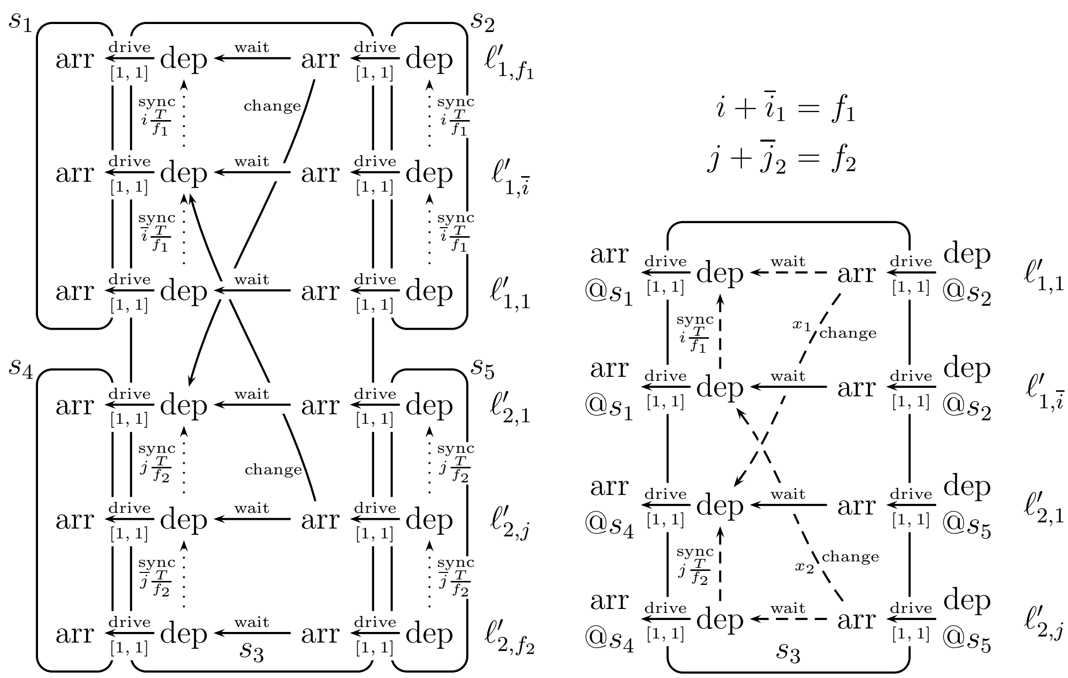
Figure 5.9: (a) depicts the frequency as attribute $\text{EAN}^{\text{X}}_{\text{FA}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN$^{\text{X}}$ from Figure 5.8. Since change times span a period length, we consider only those change activities relevant to the origin destination OD$^{\text{X}}$ matrix. The only cycle possible is shown in (b).



(a) Event Activity Network,
Frequency as Multiplicity

(b) Nontrivial Cycles

Figure 5.10: (a) depicts the frequency as multiplicity $\text{EAN}^{\text{X}}_{1,2} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN$^{\text{X}}$ from Figure 5.8 with line frequencies $F = (f_1, f_2) = (1, 2)$. Since change times span a period length, we consider only those change activities relevant to the origin destination OD$^{\text{X}}$ matrix. The besides permutation only nontrivial cycle different from the one in Figure 5.9b is shown in (b).



(a) Event Activity Network, Frequency as Multiplicity

(b) Nontrivial Cycles

Figure 5.11: (a) depicts the frequency as multiplicity $\mathrm{EAN}^{\mathrm{X}}_{f_1 f_2} = (\mathcal{A}, \mathcal{E})$ that belongs to the $\mathrm{PTN}^{\mathrm{X}}$ from Figure 5.8 with line frequencies $F = (f_1, f_2)$. For space reasons, only one direction per line is visible. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. (b) shows the only cycle possible, parametrized by i and j.

5.6 Timetablers Nightmare

In this section we show that an optimum for the ODPESP from Section 2.5 using the `frequency_as_attribute` LINES ROLL OUT model does not necessarily have to yield a global optimum for the ODPESP with `frequency_as_multiplicity` model. Only depending on the frequency of one line involved, performing a PERIODIC ROLLOUT and rerouting passengers according to shortest paths does not fix this issue, independend of solver characteristics and the shortest paths method used and the error generally has a magnitude of T. Even worse, the slack of the suboptimal solution may be zero and the actual optimum can only be reached by increasing the duration of a drive activity. This totally breaks intuition, and in case it breaks common practices in timetabling, it may be considered as the *timetablers nightmare*. And there seems to be no awakening.

Theorem 5.14. *In general, the* ODPESP *optimum is* conceptionally invisible *to the* ***`frequency_as_attribute`*** *model, i.e. no matter how nonnegative coefficients are chosen for a* PESP *objective function, no choice guarantees that* ODPESP *optimality is reached in the rolled out network.*

Proof. This holds for the example network in this section, see Lemmata 5.16 and 5.17. □

Since we cannot prevent the optimum from being chosen by chance for a zero objective, we cannot strengthen the statement that way. This states that even if coeffcients are not from an OD derived passenger distribution, the `frequency_as_multiplicity` ODPESP optimum generally cannot be found with the `frequency_as_attribute` model and the way we model change activities.

The difference of this approach to the one from Section 5.13 before is that the OD derived passenger distribution is not unique and the ODPESP objective function has to decide which to pick to get into a global optimum. One could assume that if one somehow knew a distribution in the `frequency_as_attribute` construction model that leads to the ODPESP global optimum, then the problem is somehow reducible to PESP for the `frequency_as_multiplicity` model. However, this is not possible (or at least not in an obvious way).

To have some picture of the situation in mind when reading the relatively technical lemmata and proofs, have a look at Figure 5.12.

As in the section before, we use a very simple $\mathrm{PTN}^{\mathrm{X}}$, LC^{X} and OD^{X} from Figure 5.13, and consider an associated event activity networks with our two different construction models. The period length T, the frequency f_1 and a lower bound for a drive activity l^{d} are parameters.

Lemma 5.15. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X}, OD^{X}, *be as in Figure 5.13 and* $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ *an associated event activity network with the* LINES ROLL OUT *model* ***`frequency_as_attribute`*** *as in Figure 5.14 with* $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}}$ *as the optimal*

ODPESP *objective value. Let further* $f_1 > 1$, $f_2 = 1$, $\tau = T/f_1$, $l^{\mathrm{d}} = \tau$. *It holds*

$$\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X}*} = \tau + 6 \geq 7 \quad . \tag{5.45}$$

Proof. There is only one (nontrivial) cycle $\mathcal{C}$ in Figure 5.14 that yields the constraint

$$x_{\mathrm{c}} + 1 + x'_{\mathrm{c}} - 1 - x_{\mathrm{d}} - 1 = zT \quad , \qquad x_{\mathrm{c}}, x'_{\mathrm{c}} \in \{2, \ldots, T+1\},\ x_{\mathrm{d}} \in \{\tau+2, \ldots, T+2\},\ z \in \mathbb{Z}, \tag{5.46}$$

where x_{c}, x'_{c}, x_{d} as depicted in the Figure. There are only two OD^{X} derived passenger distributions possible: either changing from line ℓ_1 to ℓ_2 and back or staying in line ℓ_1. This yields the following objective function

$$\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X}}(x_{\mathrm{c}}, x'_{\mathrm{c}}, x_{\mathrm{d}}) = \min(x_{\mathrm{d}} + 4, x_{\mathrm{c}} + x'_{\mathrm{c}} + 3) \quad . \tag{5.47}$$

Since the sum $x_{\mathrm{c}} + x'_{\mathrm{c}}$ occurs both in the objective as in the only cycle, it makes sense to substitute it by $\tilde{x}_{\mathrm{c}} = x_{\mathrm{c}} + x'_{\mathrm{c}}$. Therefore, $\mathcal{C}$ looks like

$$\tilde{x}_{\mathrm{c}} - x_{\mathrm{d}} - 1 = zT \quad , \qquad \tilde{x}_{\mathrm{c}} \in \{4, \ldots, 2T+2\},\ x_{\mathrm{d}} \in \{\tau+2, \ldots, T+2\},\ z \in \mathbb{Z}, \tag{5.48}$$

and the objective like

$$\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X}}(\tilde{x}_{\mathrm{c}}, x_{\mathrm{d}}) = \min(x_{\mathrm{d}} + 4, \tilde{x}_{\mathrm{c}} + 3) \quad , \tag{5.49}$$

from which arises the limitation $\tilde{x}_{\mathrm{c}} \in \{4, \ldots, T+3\}$, since this still spans T and on the other hand values greater than $T+3$ do not make sense for the objective. It holds

$$\left\lceil \frac{1}{T} - 1 \right\rceil = 0 \leq z \leq \left\lfloor \frac{T + 3 - 1 - \tau - 2}{T} \right\rfloor = \left\lfloor 1 - \frac{\tau}{T} \right\rfloor = 0 \quad , \tag{5.50}$$

thus $z = 0$ and

$$\tilde{x}_{\mathrm{c}} = 1 + x_{\mathrm{d}} \tag{5.51}$$

which simplifies the objective to

$$\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X}}(x_{\mathrm{d}}) = \min(x_{\mathrm{d}} + 4, x_{\mathrm{d}} + 4) = x_{\mathrm{d}} + 4 = \tau + 6 \quad . \tag{5.52}$$

A `frequency_as_attribute` ODPESP optimum is therefore reached iff

$$x_{\mathrm{d}} = \tau + 2 \tag{5.53}$$

and the lemma follows. □

Lemma 5.16. *For Lemma 5.15, no choice of the coefficients in an* PESP *objective function prevents* $x_d = \tau + 2$ *from being an optimum.*

Proof. The most general objective function is

$$\min \tilde{\alpha} x_{\mathrm{d}} + \tilde{\beta} x_{\mathrm{c}} + \tilde{\gamma} x'_{\mathrm{c}} + \tilde{C} \tag{5.54}$$

with $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{C} \geq 0$. Without the substitution, (5.51) writes as

$$x_{\mathrm{c}} + x'_{\mathrm{c}} - 1 = x_{\mathrm{d}} \tag{5.55}$$

and thus the objective transforms into

$$\min(\tilde{\beta} + \tilde{\alpha}) x_{\mathrm{c}} + (\tilde{\gamma} + \tilde{\alpha}) x'_{\mathrm{c}} + (\tilde{C} - \tilde{\alpha}) \quad . \tag{5.56}$$

With $\beta = \tilde{\beta} + \tilde{\alpha}$ and $\gamma = \tilde{\gamma} + \tilde{\alpha}$ which can still be chosen freely and without constant there is

$$\min \beta x_{\mathrm{c}} + \gamma x'_{\mathrm{c}} \quad . \tag{5.57}$$

If β and γ are both zero, then $x_{\mathrm{d}} = \tau + 2$ is still optimal as in Lemma 5.15 before. Let w.l.o.g. $\gamma > 0$. It remains

$$\min \alpha x_{\mathrm{c}} + x'_{\mathrm{c}} \quad , \tag{5.58}$$

with $\alpha = \frac{\beta}{\gamma}$. There are four possibilities: $\alpha = 0$, $\alpha = 1$, $\alpha \in (0, 1)$ and $\alpha > 1$. Lemma 5.15 already considered the case $\alpha = 1$, for $\alpha = 0$ it holds $x'_{\mathrm{c}} = 2$, so still $x_{\mathrm{d}} = \tau + 2$ would be optimal, the latter two are equivalent trough permutation of x_{c} and $x_{\mathrm{c}'}$. Therefore, only $\alpha > 1$ remains, which yields $x_{\mathrm{c}} = 2$ and since then $x'_{\mathrm{c}} = x_{\mathrm{d}} + 3$ again $x_{\mathrm{d}} = \tau + 2$ may take on its lower bound. □

Lemma 5.17. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X}, OD^{X}, *be as in Figure 5.13 and* $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FM}}$ *an associated event activity network with the* LINES ROLL OUT *model* ***`frequency_as_multiplicity`*** *as in Figure 5.15 with* $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FM}}$ *as the optimal* ODPESP *objective value. Let further $f_1 > 1$, $f_2 = 1$, $\tau = T/f_1$, $l^{\mathrm{d}} = \tau + 2$. It holds*

$$\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FM}} = 7 \quad . \tag{5.59}$$

Proof. In Figure 5.15 there is only one nontrivial cycle $\mathcal{C}$ that yields the constraint

$$\begin{aligned} &x_{\mathrm{c}} + 1 + x'_{\mathrm{c}} - k\tau - 1 - x_{\mathrm{d}} - 1 = 0 \mod T \quad , \\ &\quad x_{\mathrm{c}}, x'_{\mathrm{c}} \in \{2, \dots, T+1\},\ x_{\mathrm{d}} \in \{\tau + 2, \dots, T+2\},\ k \in \{0, \dots, f_1 - 1\}, \end{aligned} \tag{5.60}$$

with x_{c}, x'_{c}, x_{d} as displayed in the Figure and after Lemma 1.6 may be reduced to

$$\begin{aligned} &x_{\mathrm{c}} + 1 + x'_{\mathrm{c}} - 1 - x_{\mathrm{d}} - 1 = 0 \mod \tau \quad , \\ &\qquad x_{\mathrm{c}}, x'_{\mathrm{c}} \in \{2, \dots, T+1\},\ x_{\mathrm{d}} \in \{\tau + 2, \dots, T+2\}, \end{aligned} \tag{5.61}$$

Again, there are only two passenger OD$^{\mathrm{X}}$ derived distributions possible: either staying in line ℓ_1 or changing between lines, which yields the objective function

$$\mathrm{obj}_{\mathrm{FM}}^{\mathrm{X}}(x_{\mathrm{c}}, x'_{\mathrm{c}}, x_{\mathrm{d}}) = \min(x_{\mathrm{d}} + 4, x_{\mathrm{c}} + x'_{\mathrm{c}} + 3) \quad . \tag{5.62}$$

This time substitute $\overline{x}_{\mathrm{c}} + 4 = x_{\mathrm{c}} + x'_{\mathrm{c}}$, $\overline{x}_{\mathrm{c}} \in \{0, \dots, 2T-2\}$, which can in a similar way as in the proof before be limited to $\overline{x}_{\mathrm{c}} \in \{0, \dots, \tau - 1\}$. The cycle constraint thus looks like

$$\overline{x}_{\mathrm{c}} = x_{\mathrm{d}} - 3 \mod \tau \quad , \qquad \overline{x}_{\mathrm{c}} \in \{0, \dots, \tau - 1\},\ x_{\mathrm{d}} \in \{\tau + 2, \dots, T + 2\}. \tag{5.63}$$

For the lemma to be true is now needed that $\tau + 3 \in \{\tau + 2, \dots, T + 2\}$, which is equivalent to $3 \in \{2, \dots, (f_1 - 1)\frac{T}{f_1} + 2\} \supset \{2, 3\}$, since $f_1 \geq 2$ and $\frac{T}{f_1} \geq 1$ since $f_1 | T$. Therefore $x_{\mathrm{d}} = \tau + 3$ may be chosen and leads to

$$\overline{x}_{\mathrm{c}} = 0 \mod \tau \quad , \tag{5.64}$$

which is satisfied for $\overline{x}_{\mathrm{c}} = 0$, i.e. $x_{\mathrm{c}} + x'_{\mathrm{c}} = 4$ and thus

$$\mathrm{obj}_{\mathrm{FM}}^{\mathrm{X}*} = 7 \quad , \tag{5.65}$$

which is always attained if passengers change from ℓ_1 to ℓ_2 and back. □

Lemma 5.18. *The* ODPESP *objective function value from Lemma 5.15 can not be improved by a sequence of* PERIODIC ROLLOUT*, rerouting passengers by duration shortest paths and* PESP *timetabling, independend of solver characteristics and shortest paths methods.*

Proof. If $\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X}*} = 7$, then as per Lemma 5.17 the ODPESP optimum for the rolled out network is attained and nothing can be improved anyway. Otherwise, shortest duration rerouting considers $1 + f_1$ possible paths after PERIODIC ROLLOUT: one that uses the x_{d} and f_1 that use the change provided by line ℓ_2, parametrized by k. This is equivalent to calculate

$$\min(x_{\mathrm{d}} + 4, [x_{\mathrm{c}} + x'_{\mathrm{c}}](k) + 3) \quad . \tag{5.66}$$

By Equation (5.53) $x_{\mathrm{d}} = \tau + 2$ with $\tau = T/f_1$ is given and the f_1 cycles constrain the possible values of $[x_{\mathrm{c}} + x'_{\mathrm{c}}](k)$ to

$$x_{\mathrm{c}} + x'_{\mathrm{c}} - k\tau - x_{\mathrm{d}} - 1 = 0 \mod T \quad , \tag{5.67}$$

$$x_{\mathrm{c}} + x'_{\mathrm{c}} = x_{\mathrm{d}} + 1 = \tau - 1 \mod \tau \quad , \tag{5.68}$$

after Lemma 1.6, so any shortest path that uses ℓ_2 is still as long as staying in ℓ_1. If rerouting picks the ℓ_1 path, PESP timetabling cannot improve the objective, since x_{d} is already on its lower bound. Otherwise, the PESP is

$$\min x_{\mathrm{c}} + x'_{\mathrm{c}} + 3 \tag{5.69}$$

$$\text{s.t. } x_{\mathrm{c}} + x'_{\mathrm{c}} - x_{\mathrm{d}} - 1 = zT \quad , \tag{5.70}$$

$$x_{\mathrm{c}}, x'_{\mathrm{c}} \in \{2, \dots, T + 1\},\ z \in \mathbb{Z} \quad , \tag{5.71}$$

which is just a subproblem from Lemma 5.15, Lines Roll Out model `frequency_as_attribute`, i.e. the old schedule stays optimal and therefore again no improvement. The lemma follows. □

Theorem 5.19 (Timetablers Nightmare Worst Case Error Lower Bound)**.** *Let* $\mathrm{EAN}_{\mathrm{FA}}$ *and* $\mathrm{EAN}_{\mathrm{FM}}$ *be two event activity networks that satisfy the bounded by period property from Definition 5.1, latter with optimal* ODPESP *objective function value* $\mathrm{obj}^*_{\mathrm{FM}}$ *derived from a common* $\mathrm{PTN} = (S, E)$, $\mathrm{LC}(f_1) = (\vec{L}, F(f_1))$, OD *and common period length* T *with only different* Lines Roll Out *models, i.e.* ***`frequency_as_attribute`*** *resp.* ***`frequency_as_multiplicity`****, where* $f_1 > 1, f_2 = 1 \in F$, $f_1|T$ *are the frequencies of two lines* ℓ_1 *and* ℓ_2 *that cross at least at one station* $s \in S$ *and* T/f_1 *is maximal among all crossing lines. Let further* $\mathrm{EAN}'_{\mathrm{FM}}$ *be the* Periodic Rollout *of* $\mathrm{EAN}_{\mathrm{FA}}$ *with objective function* $\mathrm{obj}^*_{\mathrm{FA}}$ *after shortest durations rerouting and* PESP *timetabling. Independent of the solver characteristics and shortest paths methods, it holds*

$$\max_{\substack{\mathrm{PTN},\\ \mathrm{LC}(f_1),\mathrm{OD}}} \frac{\mathrm{obj}^*_{\mathrm{FA}}}{\mathrm{obj}^*_{\mathrm{FM}}} \geq \frac{\frac{T}{f_1} + 6}{7} \quad , \qquad \forall\ f_1 \in \{2, \ldots, T\}. \tag{5.72}$$

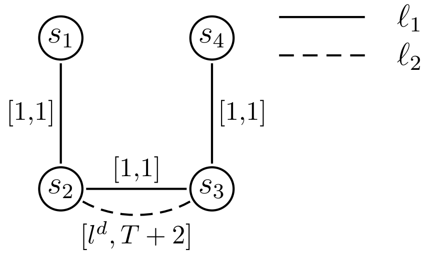
Numbers for Theorem 5.19 see Table 5.5.

Table 5.5: ODPESP after rollout worst case relative error lower bounds for different period lengths T and different f_1, see Theorem 5.19.

	Period Length T								
f_1	2	5	15	30	60	120	600	1200	2400
2	1.00			3.00	5.14	9.43	43.71	86.57	172.29
3			1.57	2.29	3.71	6.57	29.43	58.00	115.14
4				1.93	3.00	5.14	22.29	43.71	86.57
5		1.00	1.29	1.71	2.57	4.29	18.00	35.14	69.43
6			1.21	1.57	2.29	3.71	15.14	29.43	58.00
12			1.04	1.21	1.57	2.29	8.00	15.14	29.43
15			1.00	1.14	1.43	2.00	6.57	12.29	23.71
20				1.07	1.29	1.71	5.14	9.43	18.00
30				1.00	1.14	1.43	3.71	6.57	12.29
60					1.00	1.14	2.29	3.71	6.57
120						1.00	1.57	2.29	3.71
600							1.00	1.14	1.43
1200								1.00	1.14
2400									1.00

	ODPESP Method							
	Frequency as Attribute		Frequency as Attribute, Rollout, Reroute, PESP Timetabling			Frequency as Multiplicity		
t	ℓ_1	ℓ_2	$\ell_{1,1}$	$\ell_{1,2}$	ℓ_2	$\ell_{1,1}$	$\ell_{1,2}$	ℓ_2
0	s_1, dep		s_1, dep			s_1, dep		
1	s_2, arr		s_2, arr			s_2, arr		
2	s_2, dep		s_2, dep			s_2, dep		
3		s_2, dep			s_2, dep			s_2, dep
4		s_3, arr		s_3, arr	s_3, arr			s_3, arr
5				s_3, dep			s_3, arr	
6				s_4, arr			s_3, dep	
7							s_4, arr	
8								
9								
10				s_1, dep			s_1, dep	
11				s_2, arr			s_2, arr	
12				s_2, dep			s_2, dep	
13								
14	s_3, arr		s_3, arr					
15	s_3, dep		s_3, dep			s_3, arr		
16	s_4, arr		s_4, arr			s_3, dep		
17						s_4, arr		
18								
19								

Figure 5.12: An actual timetable for Lemma 5.18 that shows the different solution methods, $T = 20$, $f_1 = 2$. Solid lines denote the optimal solution if passengers stick to ℓ_1, while dashed lines if they use ℓ_2.



$$\text{change} \in [2, T+1],$$
$$\text{wait} \in [1,1],$$
$$\text{OD}^{\text{X}}_{ss'} = \begin{cases} 1 & (s,s') = (s_1, s_4) \\ 0 & \text{otherwise} \end{cases}, \quad \forall s, s' \in S.$$

Figure 5.13: A public transportation network $\text{PTN}^{\text{X}} = (S, E)$ that has four stations $S = \{s_1, s_2, s_3, s_4\}$ and five edges $E = \{e_{13}, e_{23}, \overline{e_{23}}, e_{34}, e_{35}\}$, most with equal lower and upper bounds $l_{13} = u_{13} = l_{34} = u_{34} = l_{35} = u_{35} = 1$, $\overline{l_{23}} = l^{\text{d}}$, $\overline{u_{23}} = T+2$ together with a line concept $\text{LC}^{\text{X}} = (L, F)$ with two lines $L = \{\ell_1, \ell_2\}$, $F = \{f_1, f_2\}$, $f_1, f_2 | T$ where T is the period length as well as an asymmetric origin destination OD^{X} matrix that states that there is only one passenger in the network: one that wants to go from s_1 to s_4.

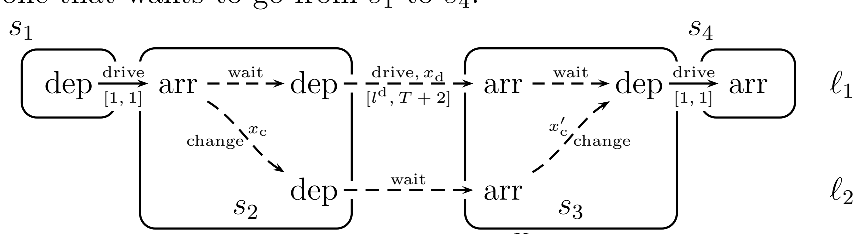


Figure 5.14: The frequency as attribute $\text{EAN}^{\text{X}}_{\text{FA}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN^{X} from Figure 5.13, `frequency_as_attribute` construction model. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. The only cycle possible is dashed; only one direction per line for simplicity purposes.

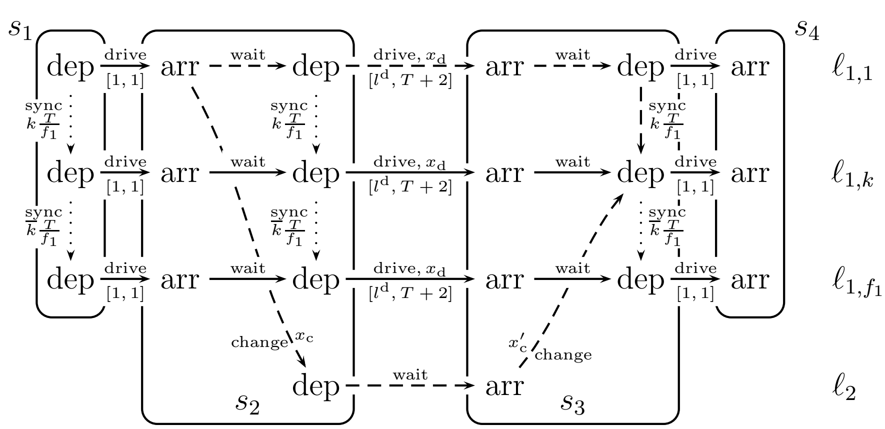


Figure 5.15: The frequency as multiplicity $\text{EAN}^{\text{X}}_{\text{FM}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN^{X} from Figure 5.13, `frequency_as_multiplicity` construction model. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. The only cycle possible is dashed; only one direction per line for simplicity purposes.

5.7 Always Best Changes

One could assume that in an event activity network with LINES ROLL OUT model `frequency_as_attribute` passengers may always use the change with the shortest duration among all changes between a pair of lines at each station, as in Lemma 3.4. As usually however, this is not correct in general and again may lead to suboptimal timetables. The reason for this effect is the passenger distribution being inconsistent, i.e. passengers needed to *beam* between two frequency instances of a line and thus are generally not able to take every shortest change between every pair of lines on their way from one station to another, which can be seen in Figure 5.17 in the second column, where the dashed line is not connected.

Using best changes is like introducing an extended ODPESP based on the EPESP. However, since in the example network we consider there are only two distributions possible, we simply have a look at both instead of formulating a new model.

Attention: this time we are using the $\mathrm{PTN}^{\mathrm{X}}$ from the Section 5.6 before. The author highly recommends Figure 5.17 for a visualization of the actual issue.

Lemma 5.20. *Let T be the period length,* $\mathrm{PTN}^{\mathrm{X}}$, LC^{X}, OD^{X}, *be as in Figure 5.13 and* $\mathrm{EAN}^{\mathrm{X}}_{\mathrm{FA}}$ *an associated event activity network with the* LINES ROLL OUT *model* ***`frequency_as_attribute`*** *as in Figure 5.14 with* $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}}$ *as the optimal* ODPESP *objective value with the assumption that change durations take their best change duration in* $\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}}$ *as in Lemma 3.4. Let further $f_1 = 1$, $f_2 > 1$, $\tau = T/f_2$ and $l^{\mathrm{d}} = \tau + 3$. It holds*

$$\mathrm{obj}^{\mathrm{X}*}_{\mathrm{FA}} = 7 \quad . \tag{5.73}$$

Proof. In Figure 5.14 there is only one nontrivial cycle that yields the constraint

$$x_{\mathrm{c}} + 1 + x'_{\mathrm{c}} - 1 - x_{\mathrm{d}} - 1 = zT \quad , \\ x_{\mathrm{c}}, x'_{\mathrm{c}} \in \{2, \dots, T+1\},\ x_{\mathrm{d}} \in \{\tau+2, \dots, T+2\},\ z \in \mathbb{Z}, \tag{5.74}$$

with x_{c}, x'_{c}, x_{d} as in the Figure 5.17. With only two OD^{X} derived passenger distributions possible and durations of best changes $\overline{x}_{\mathrm{c}}$, $\overline{x}'_{\mathrm{c}}$, the objective function looks like

$$\mathrm{obj}^{\mathrm{X}}_{\mathrm{FA}}(x_{\mathrm{c}}, x'_{\mathrm{c}}, x_{\mathrm{d}}) = \min(x_{\mathrm{d}} + 4, \overline{x}_{\mathrm{c}} + \overline{x}'_{\mathrm{c}} + 3) \quad . \tag{5.75}$$

Since there always are representations $x_{\mathrm{c}} = \overline{x}_{\mathrm{c}} + i\tau$, $x'_{\mathrm{c}} = \overline{x}'_{\mathrm{c}} + j\tau$, $i, j \in \{0, \dots, \tau - 1\}$ the cycle constraint writes as

$$\overline{x}_{\mathrm{c}} + i\tau + \overline{x}'_{\mathrm{c}} + j\tau - x_{\mathrm{d}} - 1 = 0 \mod T \quad , \tag{5.76}$$

is after Lemma 1.6 equivalent to

$$\overline{x}_{\mathrm{c}} + \overline{x}'_{\mathrm{c}} = x_{\mathrm{d}} + 1 \mod \tau \quad , \tag{5.77}$$

and after substitution $\tilde{x}_\mathrm{c} + 4 = \overline{x}_\mathrm{c} + \overline{x}'_\mathrm{c}$ the variable $\tilde{x}_\mathrm{c}$ is limitable to $\tilde{x}_\mathrm{c} \in \{0, \ldots, \tau - 1\}$ since it still spans τ and bigger values do not make sense for the objective. Thus

$$\tilde{x}_\mathrm{c} = x_\mathrm{d} - 3 \mod \tau \quad . \tag{5.78}$$

The equation

$$x_\mathrm{d} = 3 + (f_2 - 1)\tau \tag{5.79}$$

is needed for Lemma 5.22 later and therefore must $3+(f_2-1)\tau \in \{\tau+3, \ldots, T+2\}$, which with $\tau = T/f_2$ given is equivalent to

$$\begin{aligned} 3 &\overset{!}{\in} \{\tfrac{T}{f_2} + 3 - (f_2 - 1)\tfrac{T}{f_2}, \ldots, T - (f_2 - 1)\tfrac{T}{f_2} + 2\} && (5.80)\\ &= \{3 - T + 2\tfrac{T}{f_2}, \ldots, \tfrac{T}{f_2} + 2\} \supset \{3 - T + 2\lceil\tfrac{T}{2}\rceil, \ldots, 3\} && (5.81)\\ &= \{3 - \lceil\tfrac{T}{2}\rceil + \lfloor\tfrac{T}{2}\rfloor, \ldots, 3\} \supset \{3\} && (5.82) \end{aligned}$$

and thus always ensured and with Equation (5.79) holds $\tilde{x}_\mathrm{c} = 0$ from which the lemma follows. □

Lemma 5.21. *Let T be the period length,* PTN^X, LC^X, OD^X*, be as in Figure 5.13 and* $\mathrm{EAN}^\mathrm{X}_\mathrm{FM}$ *an associated event activity network with the* Lines Roll Out *model* ***frequency_as_multiplicity*** *as in Figure 5.17 with* $\mathrm{obj}^{\mathrm{X}*}_\mathrm{FM}$ *as the optimal* ODPESP *objective value. Let further $f_1 = 1$, $f_2 > 1$, $\tau = T/f_2$ and $l^\mathrm{d} = \tau + 3$. It holds*

$$\mathrm{obj}^{\mathrm{X}*}_\mathrm{FM} = \tau + 7 \tag{5.83}$$

and all timetables yield the same objective for the two OD^X *derived passenger distributions possible.*

Proof. The cycle equation this time is

$$x_\mathrm{c} + 1 + x'_\mathrm{c} - 1 - x_\mathrm{d} - 1 = zT \quad , \qquad x_\mathrm{c}, x'_\mathrm{c} \in \{2, \ldots, T+1\},\ x_\mathrm{d} \in \{\tau+2, \ldots, T+2\},\ z \in \mathbb{Z}, \tag{5.84}$$

just as the proof for Lemma 5.15, so this time with slightly different bounds for x_d and z it holds

$$\left\lceil \frac{1}{T} - 1 \right\rceil = 0 \le z \le \left\lfloor \frac{T + 3 - 1 - \tau - 3}{T} \right\rfloor = \left\lfloor 1 - \frac{\tau + 1}{T} \right\rfloor = 0 \quad , \tag{5.85}$$

and thus

$$x_\mathrm{c} + x'_\mathrm{c} = x_\mathrm{d} + 1 \quad , \tag{5.86}$$

so both paths possible take equal duration for any timetable and the objective

$$\mathrm{obj}^{\mathrm{X}*}_\mathrm{FM}(x_\mathrm{c}, x'_\mathrm{c}, x_\mathrm{d}) = \min(x_\mathrm{c}+x'_\mathrm{c}+3, x_\mathrm{d}+4) = x_\mathrm{c}+x'_\mathrm{c}+3 = x_\mathrm{d}+4 = \tau+7 \quad , \tag{5.87}$$

from which the lemma follows. □

Lemma 5.22. *Let* $\mathrm{obj}_{\mathrm{FA}}^{\mathrm{X+}}$ *be the objective from Lemma 5.20 after a* Periodic Rollout *and shortest duration passenger rerouting. Depending on the solver characteristics*

$$\mathrm{obj}_{\mathrm{FM}}^{\mathrm{X+}} = 7 + (f_2 - 1)\tau \tag{5.88}$$

is possible. However, an additional PESP *timetabling step reduces this to the* ODPESP *optimum from Lemma 5.21.*

Proof. As in Equation (5.79) from Lemma 5.20,

$$x_{\mathrm{d}} = 3 + (f_2 - 1)\tau \tag{5.89}$$

yields an optimal solution. However Lemma 5.21 states that in the rolled out network both paths possible have the same duration, therefore a change does not improve this value, but PESP timetabling does, because

$$x_{\mathrm{d}} = 3 + \tau \tag{5.90}$$

is better for both OD^{X} derived passenger distributions. The lemma follows. □

Theorem 5.23 (Always Best Changes Worst Case Error Lower Bound)**.** *Let* $\mathrm{EAN}_{\mathrm{FA}}$ *and* $\mathrm{EAN}_{\mathrm{FM}}$ *be two event activity networks, latter with optimal* ODPESP *objective function value* $\mathrm{obj}_{\mathrm{FM}}^{*}$ *derived from a common* $\mathrm{PTN} = (S, E)$, $\mathrm{LC}(f_2) = (\vec{L}, F(f_2))$, OD *and common period length* T *with only different* Lines Roll Out *models, i.e.* ***`frequency_as_attribute`*** *resp.* ***`frequency_as_multiplicity`****, where* $f_1 = 1, f_2 > 1 \in F$, $f_2|T$ *are the frequencies of two lines* ℓ_1 *and* ℓ_2 *that cross at least at one station* $s \in S$ *and* T/f_2 *is maximal among all crossing lines. Let further* $\mathrm{EAN}'_{\mathrm{FM}}$ *be the* Periodic Rollout *of* $\mathrm{EAN}_{\mathrm{FA}}$ *with objective function* $\mathrm{obj}_{\mathrm{FA}}^{*}$ *after shortest durations rerouting. Depending on the solver characteristics, it holds*

$$\max_{\substack{\mathrm{PTN},\\ \mathrm{LC}(f_2),\mathrm{OD}}} \frac{\mathrm{obj}_{\mathrm{FA}}^{*}}{\mathrm{obj}_{\mathrm{FM}}^{*}} \geq \frac{7 + (f_2 - 1)\frac{T}{f_2}}{7 + \frac{T}{f_2}} \quad , \qquad \forall\ f_2 \in \{2, \dots, T\}. \tag{5.91}$$

For Theorem 5.23 in numbers see Table 5.6.

	ODPESP Method							
	Frequency as Attribute, "Best Changes"		Frequency as Attribute, Rollout, Reroute			Frequency as Attribute, Rollout, Reroute, PESP Timetabling		
t	ℓ_1	ℓ_2	ℓ_1	$\ell_{2,1}$	$\ell_{2,2-4}$	ℓ_1	$\ell_{2,1}$	$\ell_{2,2-4}$
0	s_1, dep		s_1, dep			s_1, dep		
1	s_2, arr		s_2, arr			s_2, arr		
2	s_2, dep		s_2, dep			s_2, dep		
3		s_2, dep		s_2, dep			s_2, dep	
4		s_3, arr		s_3, arr			s_3, arr	
5								
6								
7								
8		s_2, dep			s_2, dep			s_2, dep
9		s_3, arr			s_3, arr			s_3, arr
10						s_3, arr		
11						s_3, dep		
12						s_4, arr		
13		s_2, dep			s_2, dep			s_2, dep
14		s_3, arr			s_3, arr			s_3, arr
15								
16								
17								
18		s_2, dep			s_2, dep			s_2, dep
19		s_3, arr			s_3, arr			s_3, arr
20	s_3, arr		s_3, arr					
21	s_3, dep		s_3, dep					
22	s_4, arr		s_4, arr					

Figure 5.16: An actual timetable for Lemma 5.22 that shows the different solution methods, $T = 20$, $f_2 = 4$. Solid lines denote the optimal solution if passengers stick to ℓ_1, while dashed lines if they use ℓ_2.

Table 5.6: ODPESP after rollout worst case relative error lower bounds for different period lengths T and different f_1, see Theorem 5.19.

	Period Length T								
f_1	2	5	15	30	60	120	600	1200	2400
2	1.00			1.00	1.00	1.00	1.00	1.00	1.00
3			1.42	1.59	1.74	1.85	1.97	1.98	1.99
4				2.03	2.36	2.62	2.91	2.95	2.98
5		1.38	1.90	2.38	2.89	3.32	3.83	3.91	3.96
6			2.05	2.67	3.35	3.96	4.74	4.86	4.93
12			2.52	3.63	5.17	6.88	9.77	10.35	10.66
15			2.63	3.89	5.73	7.93	12.06	12.95	13.46
20				4.18	6.40	9.31	15.59	17.12	18.01
30				4.50	7.22	11.18	21.74	24.83	26.75
60					8.25	13.89	35.12	43.96	50.36
120						15.75	50.17	70.41	88.41
600							75.75	133.89	218.45
1200								150.75	267.22
2400									300.75

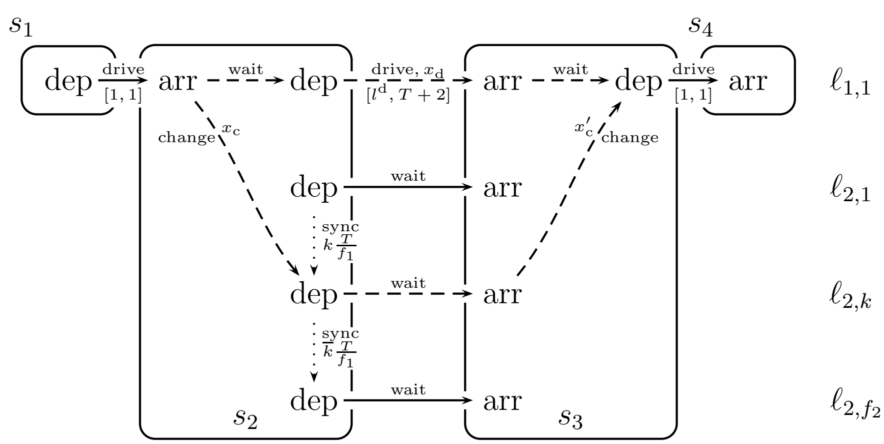


Figure 5.17: The frequency as attribute $\text{EAN}^{\text{X}}_{\text{FM}} = (\mathcal{A}, \mathcal{E})$ that belongs to the PTN^{X} from Figure 5.13, `frequency_as_multiplicity` construction model. Since change times span a period length, we consider only those change activities relevant to the origin destination OD^{X} matrix. The only cycle possible is dashed; only one direction per line for simplicity purposes.

5.8 Summary

In this section, we summarize the results of Chapter 5 mainly in table 5.7.

From an analytical point, Theorem 5.2 yields an upper bound for the worst case error

$$\frac{\mathrm{obj}_1}{\mathrm{obj}_2} \leq \frac{w^{\max} k^{\max}}{w^{\min} l^{\min}} \|\, \mathrm{OD}_{>0} \,\| |\mathcal{A}_p| T \quad \text{resp.} \quad \frac{\mathrm{obj}_1}{\mathrm{obj}_2} \in \mathcal{O}(\|\, \mathrm{OD}_{>0} \,\| |\mathcal{A}_p| T) \quad .$$

A worst case overestimation lower bound for the `frequency_as_attribute` vs. the `frequency_as_multiplicity` Lines Roll Out model with only one passenger distribution possible in former network is

$$\frac{\frac{1+\lfloor \mathrm{lcm}(f_1,f_2)/T \rfloor}{\mathrm{lcm}(f_1,f_2)} T + 6}{T+6} \quad .$$

Table 5.7: A summary for Chapter 5.

	Property						
Section	Only one passenger distribution possible/ applicable to PESP	Worst case relative error lower bound after rollout and rerouting	Independend of line frequencies	Independend of initial passenger distribution	Independend of ODPESP solution	Additional PESP timetabling step	Independend of timetabling solution
Fixed Passengers	✘	$\frac{T+2}{3}$	✔	✘	✔	$\frac{T+2}{3}$	✘
Fixed Moduli	✔	$\frac{T}{6}+1$	✔	✔	✔	1	✔
Line Concept	✔	$\frac{T+6}{8}$	✔	✔	✔	$\frac{T+6}{8}$	✔
Timetablers Nightmare	✘	$\frac{T/f_1+6}{7}$	✘	✔	✔	$\frac{T/f_1+6}{7}$	✔
Alw. Best Changes	✘	$\frac{7+\frac{(f_2-1)T}{f_2}}{7+\frac{T}{f_2}}$	✘	✔	✘	1	✔

Chapter 6

Computational Results

6.1 Test Instances

We work with four LinTim[1] test instances:

Spiel A tiny test network.

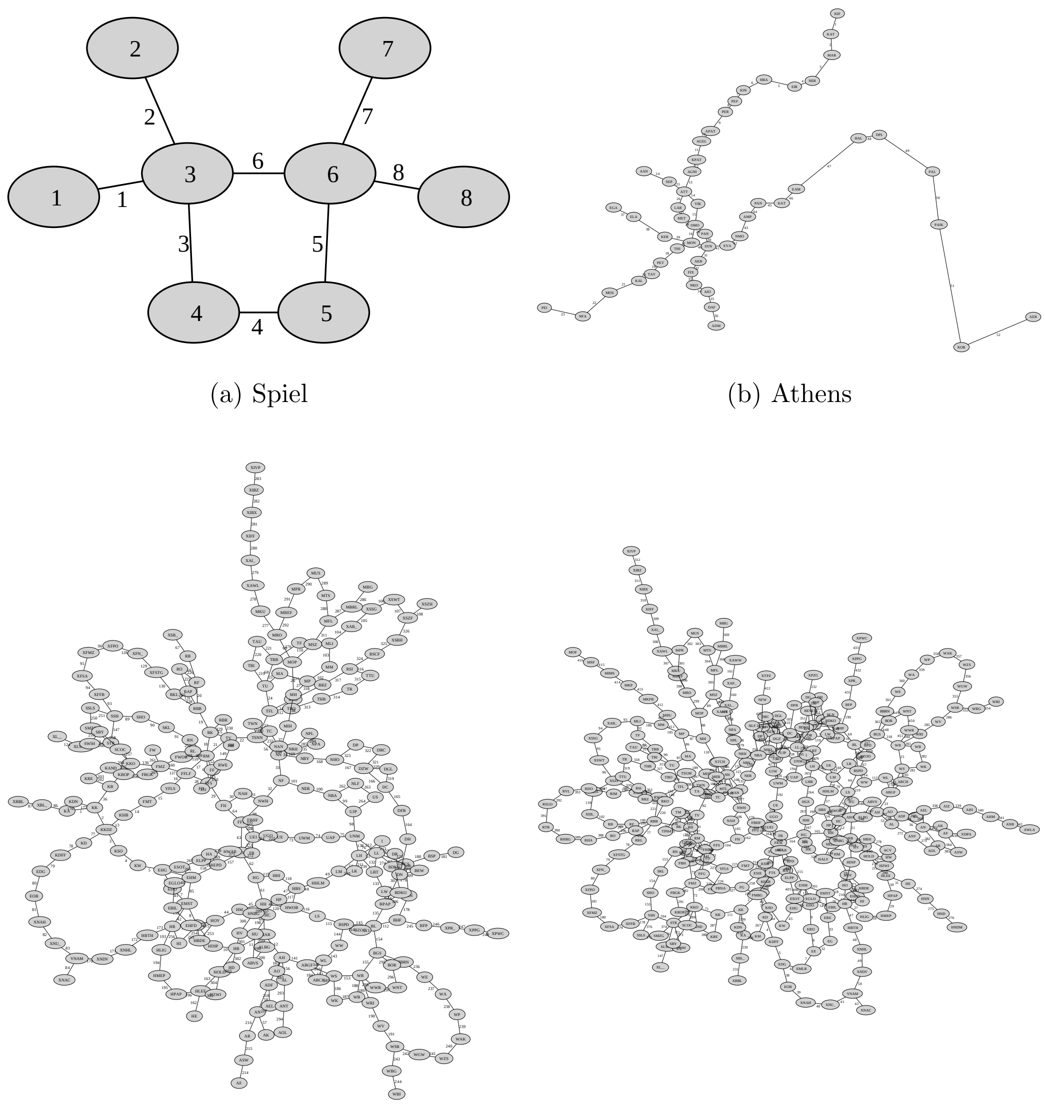
Athens Metro Network of Athens, capital of Greece. Integrated with K. Gkoumas in February 2010. The line pool is derived from splitting lines in a default line concept, therefore $l_{\mathfrak{L}} = l_{\mathrm{LC}}$. There are no headways since in our line concept no two lines share an edge in the PTN. Moderate size.

Bahn-klein/gross Based on Germany's intercity railway network.

Every instance consists of a public transportation network PTN, an origin destination matrix OD as well a line pool $\mathfrak{L}$ and a line concept LC $= (\vec{L}, F)$ with $\vec{L} \subset \mathfrak{L}$ as well as bounds required to construct event activity networks with methods from Sections 2.3 and 3.1, like l^{wait}, u^{wait}, l^{change} and u^{change}. See Table 6.1 for a detailed listing.

We do not use the option to evaluate different line concepts and focus on the average traveling time for a fixed PTN, LC and OD setup. Our evaluation starts at EAN construction level. The line concepts of Bahn-klein and Bahn-gross differ in that in the former there are less lines, but with higher frequencies, which as a large scale network makes it more interesting for `frequency_as_attribute` vs. `frequency_as_multiplicity` comparisons.

[1] http://lintim.math.uni-goettingen.de/



(a) Spiel (b) Athens

(c) Bahn-klein (d) Bahn-gross

Figure 6.1: Illustration of used LinTim instances.

Draw by Neato: http://www.graphviz.org/

Table 6.1: Properties of the LinTim datasets. Cyclebases widths with fundamental improvement from [Lie03]. *Feasibility* refers to a network where all activities that span a period have been removed while *Objective* keeps those with more than zero passengers, numbers obtained by taking a random feasible timetable, rerouting passengers and formulating the PESP for the resulting OD derived passenger distribution.

Property			Dataset			
			Spiel	Athens	Bahn-klein	Bahn-gross
PTN		Stations	8	51	250	319
		Edges	8	52	326	452
		Line Pool	8	481	132	2770
		Line Concept	5	4	53	86
		OD passengers	2620	63323	3147382	4183088
		OD pairs > 0	44	2385	48842	77878
Lower Bounds		l_{PTN}	12560	7797809	4.8271E8	6.6780E8
		$l_{\text{PTN}}^{\text{wait}}$	14540	9061697	5.0234E8	6.9445E8
		$l_{\mathfrak{L}}$	15460	9371511	5.0800E8	7.0024E8
		l_{LC}	15940	9371511	5.0887E8	7.0345E8
EAN		Events	52	208	3664	4932
		Actvities	182	234	24670	33446
		Drive	26	104	1832	2466
		Wait	16	96	1722	2294
		Change	64	34	14636	22418
		Headways	76	0	6480	6268
		$\mathfrak{t}$/Minute	1	10	1	1
		Period T	60	600	60	60
		$[l^{\text{wait}}, u^{\text{wait}}]$	$[1, 3]$	$[3, 6]$	$[1, 5]$	$[1, 3]$
		$[l^{\text{change}}, u^{\text{change}}]$	$[3, 62]$	$[10, 609]$	$[3, 62]$	$[3, 62]$
PESP	Feasibility	Activities	118	200	10034	10928
		Cyclebase	20.82	0.0	2012.26	1387.66
	Objective	Activities	126	226	11340	13157
		Cyclebase	24.05	7.67	2551.71	2118.14

6.2 Test Environment

Our computational evaluations took place on two machines.

laptop A 13.3" MacBook Pro from mid 2010, 2.4 GHz Intel Core 2 Duo Processor, 4 GB main memory and running the 10.6.7 version of Mac OS X.

c3 A 4 core Intel Xeon E5520, 2.27GHz, 24 GB main memory running the server edition of Ubuntu 10.04.2 LTS.

6.3 Test Set-Up

In table 6.1, the cyclebase width is of particular importance, since it is an indicator for the runtime for solving the PESP. For Spiel and Athens it is low, but for Bahn-klein and Bahn-gross it is astronomical compared to the numbers in [Lie03], where the greatest width was 88.4. This leads us to have the primal-dual gap stuck at 75% for maybe the next billion years. Ignoring headways would solve that problem but is not an option, since it makes the timetable effectively infeasible. Only for Spiel we succeeded in evaluating the ODPESP optimum while in general we rely on using a Retimetabling (ReTim, Definition 3.21) based heuristic approach and compare our results to l_{LC}, which at least gives us an idea on how far our ODPESP global optimum may be away.

Definition 6.1 (Average Traveling Time Gap to l_{LC}). *For a timetable π and w obtained by* Passenger Distribution *Algorithm 4, shortest paths w.r.t. π derived durations, the* average traveling time gap to l_{LC} in percent *is*

$$\mathrm{gap}^{\pi}_{\mathrm{ATT}} = 100\left[\frac{\mathrm{ATT}^{\pi}_{w}}{l_{\mathrm{LC}}} - 1\right] \quad , \tag{6.1}$$

where ATT^{π}_{w} is the average traveling time from Definition 2.12.

Definition 6.2 (Periodic Rollout Gap Quotient). *Let $\pi = \pi_{\mathrm{FA}}$ be a timetable and π_{FM} its* Periodic Rollout *as well as w_{FA} and w_{FM} obtained by* Passenger Distribution*, shortest paths w.r.t. π_{FA} resp. π_{FM} derived durations. Then the* periodic rollout gap quotient in percent *is defined as*

$$Q^{\mathrm{FM}}_{\pi} = 100\left[\frac{\mathrm{ATT}^{\pi_{\mathrm{FA}}}_{w_{\mathrm{FA}}}}{\mathrm{ATT}^{\pi_{\mathrm{FM}}}_{w_{\mathrm{FM}}}} - 1\right] \quad . \tag{6.2}$$

Thanks to abscon[2] we can obtain feasible timetables within seconds, even for our large networks. However, although it is able to provide multiple solutions for one problem, they are too similar and often differ just in one variable. To

[2] May be found on `http://www.cril.univ-artois.fr/~lecoutre/software.html`. Great thanks to Marc Goerigk for this discovery!

obtain a higher diversity, we apply a random permutation to the events index set, find a feasible PESP solution and permute it back. With this procedure, ten initial timetables were generated per dataset, some properties in Table 6.2. The randomness seems to be reasonably high, since the ReTim results vary heavily. Of course, a sample size of ten is too small to make strong statements about the underlying sample space, but still better than just relying on a single timetable.

Table 6.2: Properties of the ten initial timetables used. Values obtained after a Periodic Rollout and Passenger Distribution.

Property		Spiel	Athens	Bahn-klein	Bahn-gross
$\mathrm{gap}_{\mathrm{ATT}}^{\pi^0}$	∅	26.08	27.97	11.86	11.45
	σ	4.28	6.56	0.45	0.25
	best	20.58	17.92	11.06	10.93
	worst	34.76	35.47	12.35	11.84
$Q_{\pi^0}^{\mathrm{FM}}$	∅	52.22	87.99	2.81	0.82
	σ	14.51	17.65	0.20	0.11
	best	26.86	62.79	2.57	0.70
	worst	81.17	117.38	3.20	1.01

(Dataset spans Spiel, Athens, Bahn-klein, Bahn-gross.)

6.3.1 Modulo Simplex

In our ReTim approach we use and evaluate the modulo simplex [GS11] as well. Roughly said, it uses properties of the space

$$\mathcal{Q} = \text{conv.hull}\left(\left\{\begin{pmatrix}\pi \\ z\end{pmatrix} : l_a \leq \pi_{\varepsilon'} - \pi_\varepsilon + T z_a \leq u_a, (\varepsilon', \varepsilon) = a,\ z \in \mathbb{Z}^{|\mathcal{A}|},\ \pi \in \mathbb{R}^{|\mathcal{E}|}\right\}\right)$$

to heuristically improve a given timetable. It is set up to use *multi node cuts* since they perform best in [GS11]. The only thing we need to know for our purposes is that the modulo simplex for an initially given timetable π never yields worse results than solving the PESP for modulo parameters fixed to those of π.

6.3.2 Retimetabling

ReTim consists of two steps: timetabling with a fixed passenger distribution and rerouting with a fixed timetable. In this section we go into detail on which configurations we evaluate. A summary may be found in Table 6.3.

We use randomized shortest paths, i.e. when constructing the shortest path tree in Algorithm 4 in each step we take a (pseudo)random node with minimal distance and not a fixed node predetermined by a heap structure. Therefore, we obtain different passenger routes even for the same timetable, which prevents the

ReTim iteration from getting stuck too soon. Also, for the timetabling step we reuse only modulo parameters and not the whole previous timetable as initial solution to avoid situations as in the example network in Section 5.2. We want to evaluate the influence of randomness as well and thus perform three runs for every initial timetable and configuration. Tables 6.4 to 6.7 contain the average $\varnothing$ and maximum of the *gap span* $\Delta_{\min}^{\max}$ *(of* ATT *over an) initial timetable*, where $\Delta_{\min}^{\max} = \max(S) - \min(S)$ with $S = \{\mathrm{ATT}^1, \dots, \mathrm{ATT}^n\}$ being the set of average traveling times of the $n = 3$ runs for an initial timetable.

If shortest paths are randomized and timetables not guaranteed to repeat, how then to ultimately know that the average traveling time ATT converged? It is impossible, we can never know. Therefore, we talk of *convergence* if ATT does not improve in three consecutive ReTim steps. However, we do not give up and, depending on the configuration, try different heuristics to break through and further improve ATT.

The event activity network on which ReTim takes place can either be modeled with the `frequency_as_attribute` or the `frequency_as_multiplicity` Lines Roll Out model. We evaluate both, but since the average traveling time in the latter is less-equal to that in the former by Theorem 3.2 resp. the other way round if we assume always best changes as in Section 5.7 and we want to compare both models, after each rerouting step, we back up the event activity network, passenger distribution and timetable, perform a Periodic Rollout, reroute passengers, measure ATT in the rolled out network w.r.t. actual durations, i.e. all frequency instances visible and without always-best-changes assumption, restore our backup and continue the iteration. We substract the time needed to perform this *rollout peek* from the runtime, since this step is for evaluation purposes only, does not affect the outcome and could thus be skipped in production systems, but only at the first glance, as we can be seen in the test results for Spiel and Athens in Sections 6.4.1 resp. 6.4.2.

In consecutive timetabling and passenger rerouting steps the Periodic Rollout would give the same result, since the rolled out timetable stays the same. To make our *iteration plots* like in Figure 6.2 richer in information, in timetabling steps, we use the passenger distribution from the previous rollout peek and do not reroute passengers, which yields a generally higher gap for the peek and gives us an idea of how the new rolled out timetable would have performed for the old rolled out distribution.

We evaluate our both Lines Roll Out models as initial ReTim configurations. Since our initial timetables are for `frequency_as_attribute`, we perform a Periodic Rollout to get a respective feasible `frequency_as_multiplicity` timetable. In latter case the rollout peek does not make sense since the network already is in a periodically rolled out state and we just measure the ATT directly.

As linear models we have a look at the PESP as in Linear Program 2.4 and the EPESP as in Linear Program 2.25. For latter, we chose $T_a = T/\operatorname{lcm}(f_1, f_2)$ and $[l_a, u_a] = [l^{\text{change}}, l^{\text{change}} + T_a - 1]$ for all $a \in \mathcal{A}_{\text{change}}$, which yields the best change between two lines at one station as by Lemma 3.4. To ensure ReTim convergence, we thus also have to set shortest path weights to the best change as in Lemma 3.4, which is why we call the EPESP a *change model* as well, like in Table 6.4. For headway activities, we set $[l_a, u_a] = [h_a, T_a - h_a]$ for all $a \in \mathcal{A}_{\text{headway}}$ where $h_a := h_{\text{edge}_a}$ is the headway of the corresponding edge in the PTN, which ensures feasibility in the periodically rolled-out network by Corollary 3.9. Waits and drives keep their bounds and remain as in the PESP.

If `frequency_as_attribute` is the LINES ROLL OUT model, we apply a PERIODIC ROLLOUT once ATT converges, the so called *convergence rollout*. This does not worsen ATT if the linear model is PESP, as by Theorem 3.2. In case we used the EPESP, this step may increase ATT, but since it occurs only once, it does not influence convergence. The ReTim iteration continues using PESP for timetabling and shortest paths w.r.t. to actual durations in the rolled out network until ATT converges again, which we denote by *final convergence* like in Table 6.4.

By Theorem 3.22 we may apply heuristics that do not increase the average traveling time in the PESP timetabling step and ReTim still converges. In our case it is the modulo simplex [GS11] and we evaluate three ways of using it.

No Modulo Simplex, NoMs Also denoted as *fixed modulo timetabling*, as in Tables 6.4 to 6.7. Chose the PESP or EPESP as linear model, obtain modulo parameters from the previous timetable, solve the problem for fixed parameters. Since we have no other heuristic for the modulo parameters of the EPESP, this is the only choice that case.

Modulo Simplex on Convergence, MsConv At iteration start, set a *modulo simplex usage state* `modulo_simplex_used=false`. If fixed modulo timetabling and randomized shortest paths rerouting cannot improve the objective in three consecutive ReTim steps, use the modulo simplex in the timetabling step with the modulo parameters of the previous timetable as initial solution and set `modulo_simplex_used=true`. Set it back to `false` again if any consecutive ReTim step further reduces ATT, to denote that the modulo simplex may be used again. A PERIODIC ROLLOUT or termination may only be performed if we assume that another modulo simplex iteration cannot further improve ATT, i.e. `modulo_simplex_used` is true.

Modulo Simplex Only, MsOnly Use the modulo simplex as heuristic in every timetabling step, initial solution from previous timetable.

Every configuration we evaluate at may thus be summarized by three attributes:

Initial Frequency Model Either Frequency as Attribute or Multiplicity.

Linear Model/Change Model Can be PESP or EPESP.

Modulo Simple Model Either NoMs, MsConv or MsOnly.

To save computation time but also to compare results, we may reduce the NoMs case to MsConv, for `frequency_as_multiplicity` being the initial frequency model, i.e. former would have stopped anyway before the first application of the latter or if the linear model is EPESP, since there is no modulo simplex available before PERIODIC ROLLOUT. This is the reason why column two to four in Tables 6.4 to 6.7 the rows one and two as well as seven and eight contain the same values. In other cases however, it is unclear to what iteration the rollout would have occured, since the Modulo Simplex delays convergence and therefore the reduction then is impossible. A comparison of ATT for MsConv vs. NoMs for PESP and `frequency_as_attribute` shows different values after the first ReTim step, although the timetabling method is the same, which happens because of the randomized passenger distribution.

Our goal is not only to minimize $\text{gap}^{\pi,w}_{\text{ATT}}$, and thus the average traveling time, but also to evaluate the error from Chapter 5.

Table 6.3 gives an overview over the configuration space. With reductions and removal of meaningless/not converging setups, there remain six configurations we run on ten timetables three times, which makes 180 ReTim tests per instance and 720 in total.

Table 6.3: An overview over the possible configurations.

Property			
Freq. as Multiplicity ✔ Freq. as Attribute ✘	PESP ✔ EPESP ✘	MsOnly ✔✔ MsConv ✔ NoMs ✘	Meaningful ✔ Not Meaningful ✘ M. S. after Rollout R Subtest of MsConv S
✘	✘	✘	S
✘	✘	✔	R
✘	✘	✔✔	✘
✘	✔	✘	✔
✘	✔	✔	✔
✘	✔	✔✔	✔
✔	✘	✘	✘
✔	✘	✔	✘
✔	✘	✔✔	✘
✔	✔	✘	S
✔	✔	✔	✔
✔	✔	✔✔	✔

6.4 Test Results

This section contains results of the test from Section 6.3 instancewise with short reviews that refer to subsections of next Section 6.5, where the overall analysis is done. When reading, instances should not be skipped, since we basically do not mention effects that occur in all networks twice.

6.4.1 Spiel

Since the Spiel network is small enough, we may evaluate the ODPESP optimum

$$\mathrm{obj}^*_{\mathrm{ODPESP}} = l_{\mathrm{LC}} = 15940 \quad , \tag{6.3}$$

already for the `frequency_as_attribute` Lines Roll Out model and can indeed be verified by solving the PESP and rerouting passengers, i.e. our implementation of Linear Program 3.23 actually works. However, this global optimum could not be reached by any ReTim configuration in our test run.

An interesting aspect shows up when studying Table 6.4 and Figure 6.2. For `frequency_as_attribute`, PESP, NoMs the best gap to l_{LC} on final convergence is worse than after the first run. The reason for this is that the mentioned configuration does not operate on a periodically rolled out network, thus is not aware of what happens there and can increase the average traveling time by accident as can be seen in Section 5.6 as well and we discuss in Section 6.5.4.

Also noteworthy about Table 6.4 is that for the Lines Roll Out model `frequency_as_multiplicity`, the ATT average on final convergence and after first ReTim step is worse for the MsOnly model than for MsConv resp. the subtest NoMs and $\Delta^{\max}_{\min}$ over the initial timetable rises higher than some average gap to l_{LC} sizes after the first ReTim step. Both effects occur because of the shortest paths randomization which we discuss in Section 6.5.3.

The EPESP and `frequency_as_multiplicity` models perform rather bad in final convergence and after the first ReTim step compared to PESP with `frequency_as_attribute`. This may be observed in all other networks as well and we discuss in Sections 6.5.6 resp. 6.5.4.

Figure 6.3 shows a ReTim run in which the EPESP change model underestimated ATT, which can happen as stated Section 5.7. Further, after a Periodic Rollout, ATT drops even below the already underestimated value. This happens to networks as small as Spiel and all other instances as well, which we discuss this effect in Section 6.5.6.

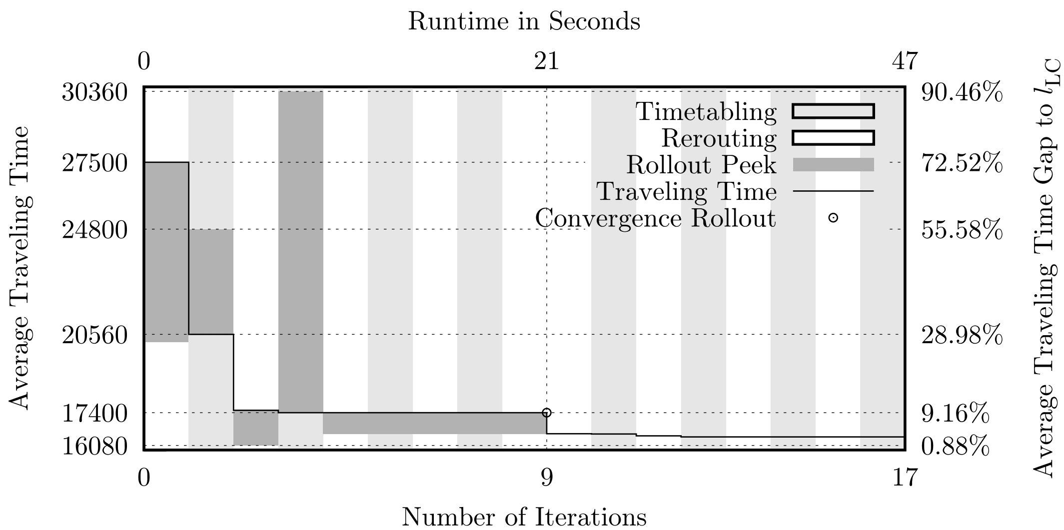


Figure 6.2: Spiel better ATT peek than final convergence, timetable 4, third run, `frequency_as_attribute`, PESP, NoMs. This is not the least peek from Table 6.4, but the easiest to spot in the plot. Be reminded that in the timetabling step rollout peek, we measure the ATT w.r.t. the new timetable, but the old rolled out old passenger distribution, which explains the gigantic peek in iteration three.

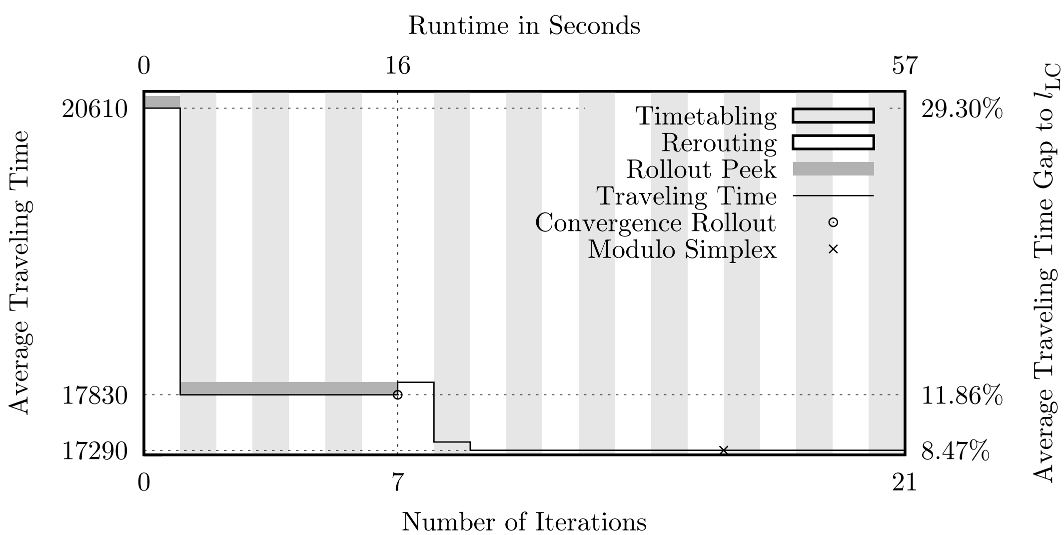


Figure 6.3: Spiel EPESP run, timetable 5, first run, PESP, MsConv. As stated in Section 5.7, ATT may be underestimated by the EPESP model, since passengers in general can not take all best changes along their route. Note that after a Periodic Rollout, ATT drops below the already underestimated value of the EPESP change model.

Table 6.4: Results for the Spiel instance, machine: laptop.

Configuration			Average Traveling Time Gap to l_{LC} measured in rolled-out network with rerouted passengers									
Frequency	Changes	Modulo Simplex	Size on Final Convergence		$\Delta_{\min}^{\max}$ Initial Timetable		After first ReTim Step		Size before Rollout		Configuration Runtime in sec	
			best	∅	max	∅	best	∅	best	∅	max	∅
Frequency as Attribute	EPESP	NoMs	1.38	8.66	0.25	0.03	7.53	13.43	7.53	13.21	58	44.90
		MsConv	1.38	8.60	0.25	0.03	7.53	13.43	7.53	13.21	87	62.80
	PESP	NoMs	1.00	4.24	6.27	1.22	0.50	5.26	1.25	4.58	74	47.57
		MsConv	0.50	2.99	1.13	0.23	0.50	5.28	0.50	3.32	134	96.83
		MsOnly	0.50	2.42	3.01	0.90	0.50	3.50	0.50	2.51	42	34.80
	∅ step in sec			3.42				1.69				2.26
Frequency as Multiplicity	PESP	NoMs	1.38	8.91	1.94	0.19	3.14	9.35	-	-	39	23.00
		MsConv	1.38	8.85	1.94	0.19	3.14	9.35	-	-	60	40.67
		MsOnly	1.51	9.50	3.64	0.56	3.76	9.90	-	-	30	16.63
	∅ step in sec			3.60				1.88				2.30
	Spiel			Reroute Passengers Runtime				Fixed Modulo				Modulo Simplex
							Timetabling Runtime					

6.4.2 Athens

After days of computation, Cplex 12.1.0 could not get the primal-dual gap of the ODPESP from Linear Program 3.23 below 18% for Athens even when using the smaller `frequency_as_attribute` LINES ROLL OUT model. Although in our test we obtained 0.84% as the smallest gap to l_{LC}, this may still not be the ODPESP optimum, as seen for the Spiel instance in the previous Section 6.4.1, where we came as close as 0.5% with our best ReTim run.

For Athens, the ATT deviation in Table 6.5 is the lowest among all networks, that means that randomized paths have the least influence on the ATT value on final convergence compared to other instances.

As it is the case for Spiel, there are test runs for which the rollout peek yields a better ATT than on final convergence. However, for Athens, that peek yields the best ReTim timetable in the whole test and can be seen in Figure 6.4.

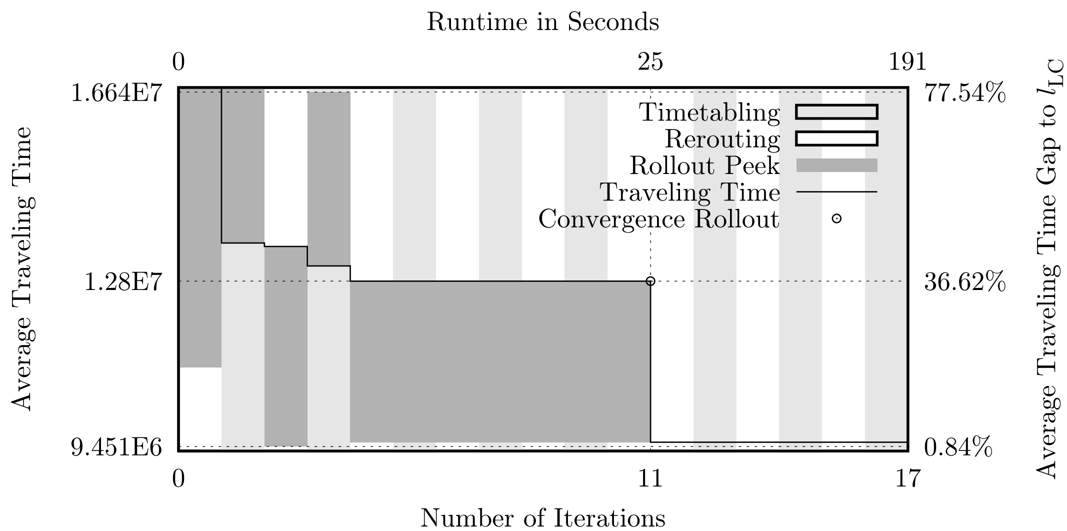


Figure 6.4: Athens, timetable 6, first run, `frequency_as_attribute`, PESP, Modulo Simplex only. In iteration number two the rollout peek drops to 0.84%, which is more than half below average traveling time gap to l_{LC} on final convergence. A similar scenario occurs to Spiel as can be seen in Figure 6.2, where the visibility of the difference between the peek and the final convergence value is better.

Table 6.5: Results for the Athens instance, machine: laptop.

Configuration			Average Traveling Time Gap to l_{LC} (measured in rolled-out network with rerouted passengers)								Configuration Runtime in sec	
Frequency	Changes	Modulo Simplex	Size on Final Convergence		$\Delta^{\max}_{\min}$ Initial Timetable		After first ReTim Step		Size before Rollout			
			best	∅	max	∅	best	∅	best	∅	max	∅
Frequency as Attribute	EPESP	NoMs	1.71	3.62	0.09	0.01	4.10	13.09	4.10	13.08	135	86.90
	EPESP	MsConv	1.71	3.62	0.09	0.01	4.10	13.09	4.10	13.08	218	169.60
	PESP	NoMs	1.32	2.05	0.10	0.01	1.53	2.16	1.39	2.14	111	72.07
	PESP	MsConv	1.30	1.81	0.04	0.00	1.53	2.16	1.30	1.81	219	192.33
	PESP	MsOnly	1.45	1.80	0.04	0.00	0.84	1.76	1.51	1.85	383	219.83
	∅ step in sec			3.71				2.29				2.70
Frequency as Multiplicity	PESP	NoMs	1.71	3.58	0.09	0.01	1.71	3.61	-	-	106	52.87
	PESP	MsConv	1.71	3.58	0.01	0.00	1.71	3.61	-	-	266	139.03
	PESP	MsOnly	1.71	3.64	0.00	0.00	1.71	3.64	-	-	242	217.47
	∅ step in sec			6.98				4.22				52.36
Athens				Reroute Passengers Runtime				Fixed Modulo (Timetabling Runtime)				Modulo Simplex (Timetabling Runtime)

6.4.3 Bahn Instances

Bahn-klein is the first instance which is already too large for the PESP to be solved to full optimality, as stated in Section 6.3, so solving the ODPESP for it is rather utopic, for Bahn-gross as well. However, with ReTim we could reduce the gap to l_{LC} to 6.52% for former resp. 6.90% for the latter network.

Unlike in Spiel and Athens in Sections 6.4.1 resp. 6.4.2 before, the rollout peek is never better than ATT on final convergence, but sometimes better than before convergence rollout, as summarized in Section 6.5.4.

The EPESP underestimation happens in Bahn-klein as well, but can be observed among all runs unlike in Spiel or Athens, where only certain runs are affected. After a Periodic Rollout, ATT drops below the underestimated value as well, which is visualized in Figure 6.7.

What can not be seen in this section but catches the eye when flying through the iteration plots is that for Bahn-klein timetable 5 performs much better then all other initial timetables throughout all configurations. We have a closer look at this in Section 6.5.1. The ReTim iteration loop of the best result is depicted in Figure 6.5.

Interestingly, although MsOnly performs best in average, for Bahn-klein we obtained the overall best timetable with MsConv, at which we again have a look in Section 6.5.5.

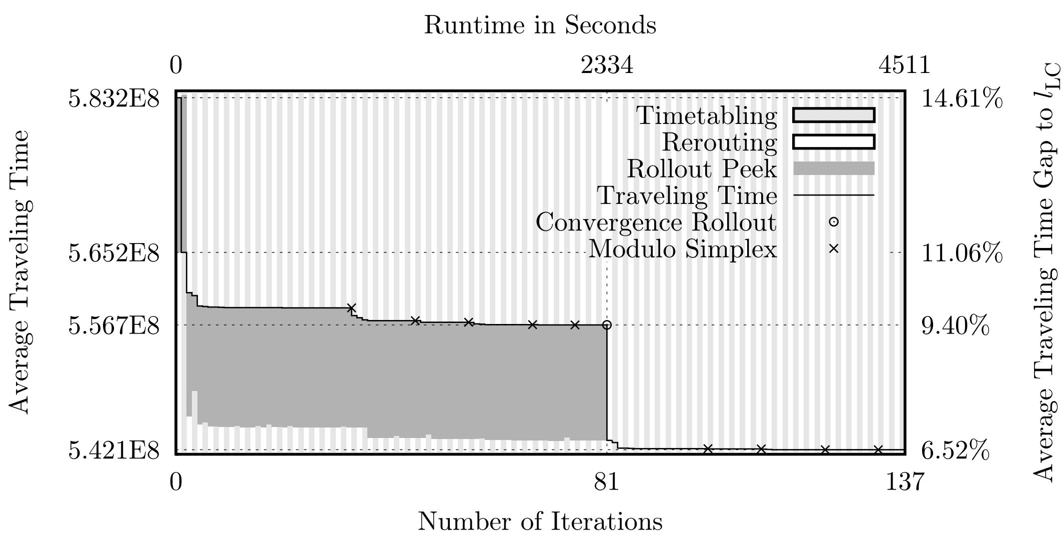


Figure 6.5: Bahn-klein overall best run, `frequency_as_attribute`, timetable 5, second run, PESP, MsConv. Actually, all methods perform very well for that initial timetable as can be seen in Figure 6.8 in Section 6.5.1.

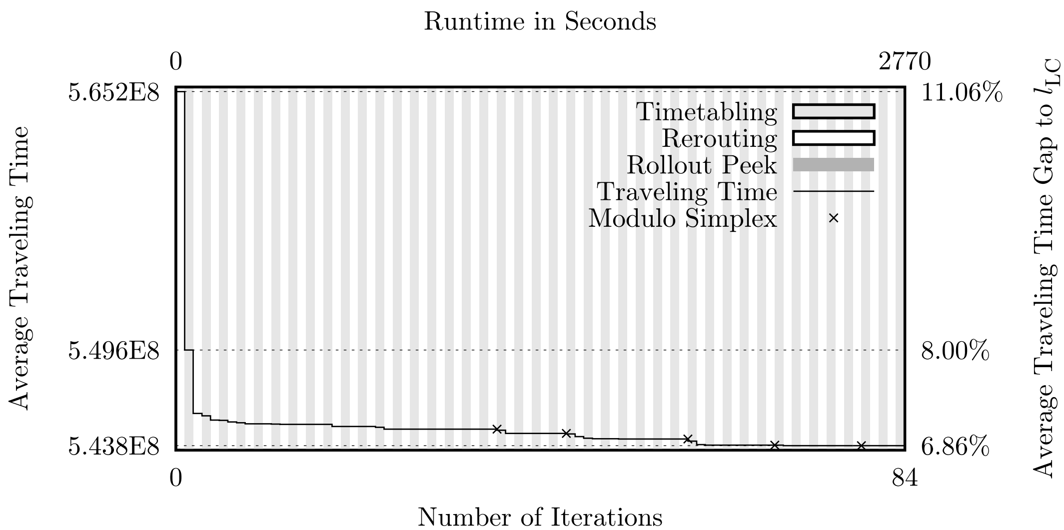


Figure 6.6: Bahn-klein best `frequency_as_multiplicity` run, timetable 5, first run, PESP, MsConv. This figure is an example plot for a typical `frequency_as_multiplicity` ReTim iteration loop.

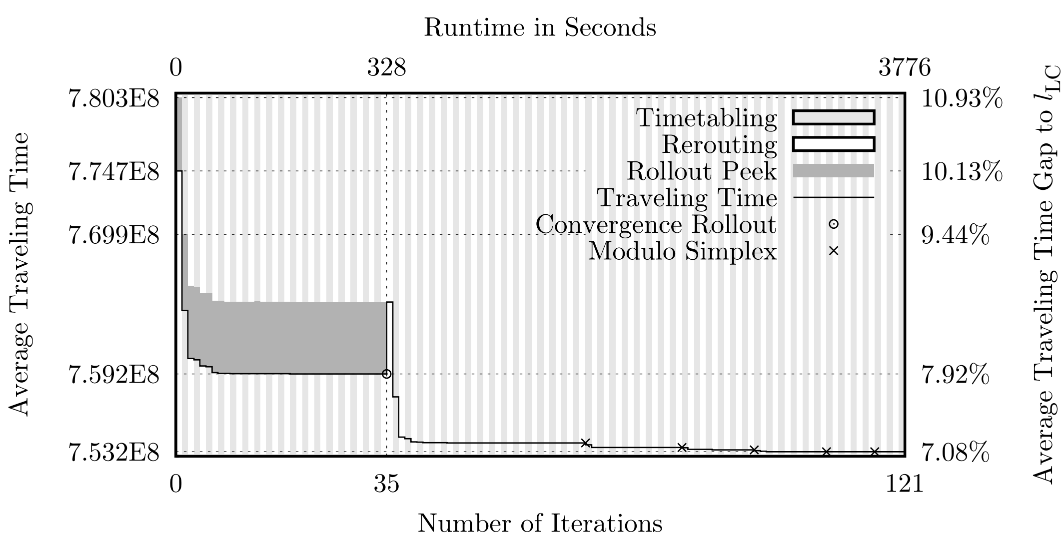


Figure 6.7: Bahn-gross best EPESP run, timetable 7, second run, Modulo Simplex on Convergence. As in Figure 6.3 the EPESP underestimates ATT, but a Periodic Rollout allows to get it even below the underestimation.

Table 6.6: Results for the Bahn-klein instance, machine: c3

Configuration: Frequency	Changes	Modulo Simplex	Average Traveling Time Gap to l_{LC} (measured in rolled-out network with rerouted passengers): Size on Final Convergence best	∅	$\Delta_{\min}^{\max}$ Initial Timetable max	∅	After first ReTim Step best	∅	Size before Rollout best	∅	Configuration Runtime in sec max	∅
Frequency as Attribute	EPESP	NoMs	7.10	7.80	0.05	0.02	9.31	9.89	9.12	9.75	731	356.10
		MsConv	6.93	7.55	0.14	0.06	9.31	9.89	9.12	9.75	5621	2906.50
	PESP	NoMs	6.72	7.39	0.04	0.02	7.29	7.86	7.05	7.67	673	403.37
		MsConv	6.52	7.18	0.12	0.06	7.29	7.86	6.73	7.36	5075	3198.73
		MsOnly	6.61	7.15	0.19	0.10	7.06	7.65	6.78	7.32	16438	9157.00
	∅ step in sec			7.92				5.45				257.82
Frequency as Multiplicity	PESP	NoMs	7.03	7.75	0.06	0.02	7.25	7.97	-	-	355	228.60
		MsConv	6.86	7.47	0.24	0.08	7.25	7.97	-	-	5779	3033.70
		MsOnly	6.74	7.42	0.17	0.08	7.06	7.79	-	-	11105	7026.50
	∅ step in sec			9.53				6.48				482.20
Bahn-klein			Reroute Passengers Runtime				Timetabling Runtime: Fixed Modulo				Timetabling Runtime: Modulo Simplex	

Table 6.7: Results for the Bahn-gross instance, machine: c3

Configuration			Average Traveling Time Gap to l_{LC} measured in rolled-out network with rerouted passengers								Configuration Runtime in sec	
Frequency	Changes	Modulo Simplex	Size on Final Convergence		$\Delta_{\min}^{\max}$ Initial Timetable		After first ReTim Step		Size before Rollout			
			best	∅	max	∅	best	∅	best	∅	max	∅
Frequency as Attribute	EPESP	NoMs	7.17	7.66	0.09	0.03	8.88	9.44	8.70	9.36	1149	592.80
		MsConv	7.08	7.51	0.18	0.06	8.88	9.44	8.70	9.36	6255	3778.83
	PESP	NoMs	7.11	7.59	0.05	0.02	7.39	7.86	7.14	7.67	1162	575.57
		MsConv	7.03	7.45	0.07	0.03	7.39	7.86	7.09	7.52	7802	4554.03
		MsOnly	6.90	7.41	0.09	0.03	7.27	7.76	6.97	7.48	23754	12463.17
	∅ step in sec			12.27				8.22				418.87
Frequency as Multiplicity	PESP	NoMs	7.13	7.68	0.05	0.02	7.41	7.87	-	-	737	314.40
		MsConv	7.01	7.56	0.08	0.05	7.41	7.87	-	-	5479	2702.43
		MsOnly	7.00	7.50	0.10	0.05	7.28	7.77	-	-	17437	8716.03
	∅ step in sec			12.97				8.62				490.96
Bahn-gross			Reroute Passengers Runtime				Fixed Modulo (Timetabling Runtime)					Modulo Simplex (Timetabling Runtime)

6.5 Review

In this section we discuss inter-instance observations. Be reminded that when we talk of the ATT gap to l_{LC}, we mean the one of rollout peek resp. in the rolled out network, since this makes outcomes of different models comparable.

6.5.1 Initial Timetable

What influence does the initial timetable have on ATT on final convergence? In Figure 6.8, although we cannot really distinguish between single configurations, in all except the small Spiel network, there seem to exist a pattern: if ATT is low on final convergence, it has already been low initially. However, a good initial timetable does not guarantee a good final outcome.

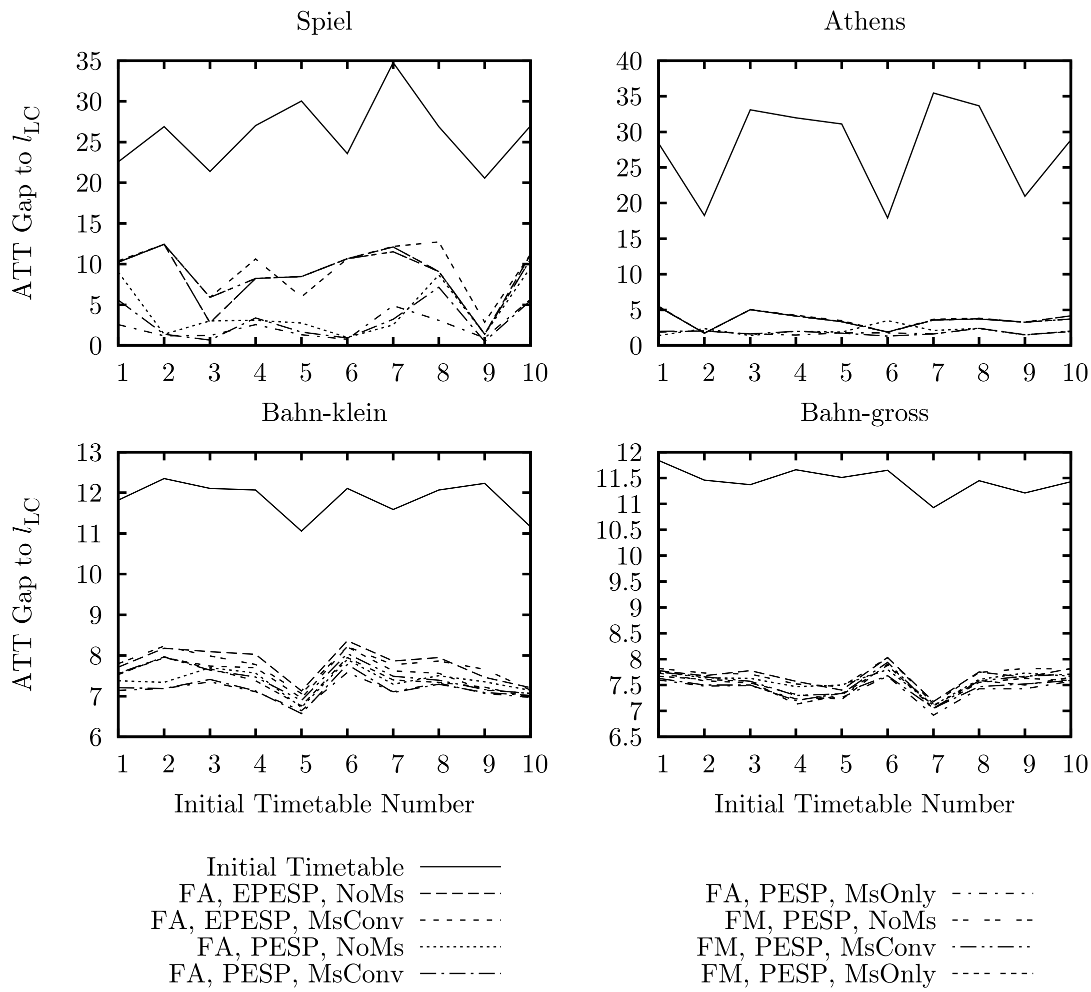


Figure 6.8: Per timetable results, initial timetable and average over three runs per configuration.

6.5.2 Eightfold Improvement

On page 10 the author wrote

> Further, we introduce a scaleable, extensible and iterative heuristic method that for practice-relevant large scale networks can improve results more than an additional threefold in average and more than an additional eightfold if combined with a statistic framework compared to what had been possible before with state-of-the-art methods for timetabling [GS11].

However, in Tables 6.6 and 6.7 there is no eightfold improvement in direct sight. So where is that magniude from? The modulo simplex, considered as state-of-the art, improves the ATT no less then fixed modulo parameters, which may be considered as the trivial method. Average values reduce the effect of the initial timetable choice and for our both large scale networks Bahn-klein as well as Bahn-gross, using MsOnly performs best in average, both on final convergence as well as after the first ReTim step for the `frequency_as_attribute` initial Lines Roll Out model. For latter network, i.e. Bahn-gross, the difference $\Delta^{\text{MsOnly}}_{\text{NoMs}}$ between l_{LC} gaps for using resp. not using the modulo simplex in the first ReTim step is

$$\Delta^{\text{MsOnly}}_{\text{NoMs}} = 7.86 - 7.76 = 0.10 \quad , \tag{6.4}$$

which we call the *average timetabling heuristic l_{LC} gap improvement*. On the other hand $\Delta^{\text{final}}_{\text{first}}$, the *average iteration l_{LC} gap improvement*, i.e. the difference between the average l_{LC} gap after the first ReTim step using modulo simplex and on final convergence with MsOnly is

$$\Delta^{\text{final}}_{\text{first}} = 7.76 - 7.41 = 0.35 \quad , \tag{6.5}$$

where the *more than an additional threefold* improvement comes from, because $\Delta^{\text{final}}_{\text{first}}/\Delta^{\text{MsOnly}}_{\text{NoMs}} > 3$ and $\Delta^{\text{final}}_{\text{first}}$ we obtained w.r.t. Modulo Simplex in a single timetabling step. On the other hand, working with several initial timetables and configurations and taking the best performance over all of them may be considered as a method by itsself. The *statistic framework l_{LC} gap improvement* $\Delta^{\text{best}}_{\text{mean}}$ is the difference between the l_{LC} gap of the overall best timetable on final convergence and the average l_{LC} gap after the first Modulo Simplex ReTim step and takes on

$$\Delta^{\text{best}}_{\text{mean}} = 7.76 - 6.90 = 0.86 \quad , \tag{6.6}$$

which, since $\Delta^{\text{best}}_{\text{mean}}/\Delta^{\text{MsOnly}}_{\text{NoMs}} > 8$, is *more than an additional eightfold*. A summary of those l_{LC} *gap improvements* for all instances may be found in Table 6.8.

From Table 6.8 also arises one question: How can it be that $\Delta^{\text{MsOnly}}_{\text{NoMs}}$ and $\Delta^{\text{final}}_{\text{first}}$ of Spiel and Athens are negative? The reason for this effect is the shortest paths randomization we discuss in the next Section 6.5.3.

Table 6.8: Absolute and relative l_{LC} gap improvements of different methods, obtained from Tables 6.4 to 6.7.

Timetabling		Spiel	Athens	Bahn-klein	Bahn-gross
		Dataset			
Frequency as Attribute	$\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$	1.763	0.401	0.215	0.101
	$\Delta_{\mathrm{first}}^{\mathrm{final}}$	1.083	-0.035	0.499	0.352
	$\Delta_{\mathrm{mean}}^{\mathrm{best}}$	2.999	0.460	1.125	0.862
	$\Delta_{\mathrm{first}}^{\mathrm{final}}/\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$	0.614	-0.086	2.316	3.478
	$\Delta_{\mathrm{first}}^{\mathrm{final}}/\Delta_{\mathrm{mean}}^{\mathrm{best}}$	1.701	1.146	5.221	8.526
Frequency as Multiplicity	$\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$	-0.554	-0.034	0.181	0.760
	$\Delta_{\mathrm{first}}^{\mathrm{final}}$	0.406	0.003	0.373	0.760
	$\Delta_{\mathrm{mean}}^{\mathrm{best}}$	8.524	1.934	0.926	0.760
	$\Delta_{\mathrm{first}}^{\mathrm{final}}/\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$	-0.732	-0.094	2.059	2.753
	$\Delta_{\mathrm{first}}^{\mathrm{final}}/\Delta_{\mathrm{mean}}^{\mathrm{best}}$	-15.381	-56.960	5.115	7.766

6.5.3 Randomized Shortest Paths

The maximum $\max \Delta_{\mathrm{min}}^{\mathrm{max}}$ over all configurations of the maximal deviation $\Delta_{\mathrm{min}}^{\mathrm{max}}$ over an initial timetable caused by randomized shortest paths in Tables 6.4 to 6.7 looks small, but in fact is gigantic when compared with $\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$, i.e. average timetabling heuristic l_{LC} gap improvement.

Table 6.9: Deviation over initial passenger distributions in comparison to $\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$, `frequency_as_attribute` Lines Roll Out model.

Quantity	Spiel	Athens	Bahn-klein	Bahn-gross
	Dataset			
$\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$	1.763	0.401	0.215	0.101
$\max \Delta_{\mathrm{min}}^{\mathrm{max}}$	6.274	0.097	0.238	0.178
$\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}/\max \Delta_{\mathrm{min}}^{\mathrm{max}}$	3.559	0.242	1.104	1.762

Why do we compare $\max \Delta_{\mathrm{min}}^{\mathrm{max}}$ with $\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$? It gives us an idea on how much influence randomized shortest paths have compared to solving the PESP in a single ReTim step, i.e. the classical way and Table 6.9 states that for large scale networks running the ReTim iteration on the same initial timetable another time with different randomized shortest paths may improve ATT by the same amount a state-of-the art PESP heuristic heuristic does in average, which shows how cruical the passenger distribution is in the process of timetable optimization.

On the other hand, Table 6.8 from Section 6.5.2 states that for the `frequency_as_multiplicity` model, Spiel and Athens have a negative $\Delta^{\text{MsOnly}}_{\text{NoMs}}$. Does this not contradict the property of the Modulo Simplex to always perform better than just using fixed modulo parameters? There are ten initial timetables, but since the test runs for NoMs and MsOnly are independent of each other, the shortest paths randomization yields different initial passenger distributions and therefore changes the PESP solution. The author double checked this and indeed, when using Modulo Simplex on the initial timetables plus passenger distributions of the NoMs runs, it does not yield a worse ATT, so this is not a bug, but states that our sample of ten timetables times three runs is too small and thus our average values differ strongly from the true average of the timetable/passenger distribution sample space. We could fix the numbers by resuing the passenger distributions of the NoMs run in the MsOnly run, but the significance does not get better if we do not increase the sample size drastically, which increases runtimes drastically as well, especially if we incorporate our large networks. Further, we cannot judge whether our shortest paths randomization resp. initial timetable randomization as described in Section 6.3 are unbiased, i.e. all passenger distributions/timetables have the same chance of being chosen, which they most likely are not. Therefore conclusions about their distribution seem generally difficult to make. What our results definitely show is that even if we take a feasible timetable π and derive w and w' by a randomized version of PASSENGER DISTRIBUTION Algorithm 4, i.e. by shortest paths w.r.t. π, PESP solutions may still vary heavily for w and w', even if the network is not one of the worst case scenarios from Chapter 5, which again points out the importance of the passenger distribution.

What happens to the eightfold improvement from Section 6.5.2? There is nothing wrong about that our methods *can improve* results an additional eightfold, since we observed it in out test, but of course they *do not have to*. Especially, this is not a statement about the distribution of feasible timetables and passenger distributions.

6.5.4 Rollout

The rollout peek can be better than ATT before convergence rollout. This happens to all networks and for Spiel and Athens, it occurs that for some runs it is even less than the ATT on final convergence as can be seen in Sections 6.4.1 and 6.4.2, while for Bahn-klein and Bahn-gross the improvement after the PERIODIC ROLLOUT is always greater than the losses through ignoring frequencies and in the ReTim steps before as can be seen in Table 6.10.

In all instances, `frequency_as_multiplicity` has a worse performance than `frequency_as_attribute` as initial LINES ROLL OUT model, especially in the average ATT gap to l_{LC}, which can be seen in Table 6.11.

Since this effect occurs for fixed modulo timetabling as well, an explanation could be that ReTim iteration generally gets stuck too soon or just because of our small sample size. However, by Theorem 3.2 there is an improvement of ATT after a Periodic Rollout, which is shown in Table 6.12.

Table 6.10: Minimal rollout peek gap to l_{LC} divided by ATT gap to l_{LC} on convergence rollout resp. on final convergence, minimum over all test runs with `frequency_as_attribute` as initial Lines Roll Out model.

	Dataset			
Step	Spiel	Athens	Bahn-klein	Bahn-gross
rollout	0.226	0.479	0.992	0.995
final	0.226	0.479	1.010	1.001

Table 6.11: PESP change model, ∅ `frequency_as_attribute` ATT gap to l_{LC} divided by ∅ `frequency_as_multiplicity` ATT gap to l_{LC}.

	Dataset			
Timetabling	Spiel	Athens	Bahn-klein	Bahn-gross
NoMs	2.102	1.750	1.049	1.011
MsConv	2.955	1.974	1.041	1.015
MsOnly	3.929	2.026	1.038	1.012

6.5.5 Timetabling Step

How well does the Modulo Simplex perform w.r.t. solving Linear Program 2.4 to full optimality? By the data we have we cannot judge that aspect. However, some findings indicate that additional improvement beyond solving the fixed modulo PESP in the timetabling step does not guarantee a better overall outcome, as is the case for Bahn-klein in Table 6.6, where the overall best l_{LC} gap of 6.52% we obtained with MsConv, although MsOnly performs better in average and in every timetabling step. The distance to the best MsOnly solution of 6.61% is 0.09% and thus almost half of $\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}} = 0.215$. For Bahn-gross however, MsOnly yields better results in both average and almost a whole $\Delta_{\mathrm{NoMs}}^{\mathrm{MsOnly}}$ improvement in the best case w.r.t. MsConv. For Spiel and Athens the situation is similar with results being unclear. Therefore, increasing the sample size strategically as proposed in Section 6.5.1 may be considered as an alternative to modulo parameter tuning in a single ReTim step.

6.5.6 Change Model

For all networks, ATT after a periodic rollout is generally even lower than the underestimation by EPESP resp. always best changes, which as effect is even worse than the theory developed in Section 5.7 from a qualitative point of view.

Table 6.12: ∅ ATT gap to l_{LC} on final convergence divided by ∅ ATT gap to l_{LC} before rollout.

Timetabling		Dataset			
		Spiel	Athens	Bahn-klein	Bahn-gross
EPESP	NoMs	0.655	0.277	0.800	0.819
	MsConv	0.651	0.277	0.774	0.803
	∅	0.653	0.277	0.787	0.811
PESP	NoMs	0.926	0.955	0.963	0.990
	MsConv	0.901	1.000	0.976	0.991
	MsOnly	0.962	0.971	0.977	0.990
	∅	0.929	0.975	0.972	0.990

This could have the reason that passengers take more changes, since their duration is low, making the timetable more difficult to optimize. Nevertheless in quantity, the ATT gap to l_{LC} is less than hundred, thus the quotient with l_{LC} is less than two, i.e. we do not have an error magnitude of $T \geq 2$.

However, results on final convergence obtained with the EPESP as change model are as bad as those with `frequency_as_multiplicity` as initial LINES ROLL OUT model, which indicates that before the PERIODIC ROLLOUT happened, the timetable could not be improved, as summarized in Table 6.13.

Table 6.13: ∅ ATT gap to l_{LC} of the EPESP divided by the ∅ ATT gap to l_{LC}, both on final convergence.

Method	Dataset			
	Spiel	Athens	Bahn-klein	Bahn-gross
NoMs	0.972	1.010	1.006	0.998
MsConv	0.972	1.011	1.011	0.994

When it comes to which ATT is taken on (in the rollout peek) before convergence rollout, the EPESP performance may be six times worse than that of PESP, see Table 6.14.

Due to these findings, the author discourages from using the EPESP resp. always best changes for modeling different frequencies.

Table 6.14: $\varnothing$ ATT gap to l_{LC} of the EPESP divided by the $\varnothing$ ATT gap to l_{LC} of the PESP, both before convergence rollout.

	Dataset			
Method	Spiel	Athens	Bahn-klein	Bahn-gross
NoMs	2.552	6.054	1.258	1.201
MsOnly	2.545	6.054	1.258	1.201

6.5.7 Recommendation

We derive a ReTim strategy to solve the ODPESP for heuristically from the observations in Section 6.5.

The PESP is preferable in favor of the EPESP as discussed in Section 6.5.6.

We use `frequency_as_attribute` as initial Lines Roll Out model, as results from Section 6.5.4 propose. We perform the convergence rollout on the timetable with the best rollout peek before we continue the iteration with `frequency_as_multiplicity`.

The findings in Section 6.5.1 suggest to generate a large pool $\mathfrak{T}$ of initial timetables the same way we obtained our initial ten as described in Section 6.3 and select those that have a low initial average traveling time into a *filtered timetable pool* $\mathfrak{T}_{\mathrm{f}}$.

To all timetables in $\mathfrak{T}_{\mathrm{f}}$ we then apply the fast running NoMs ReTim with randomized shortest paths and randomized PESP timetabling in every step with a single run only, since for Athens and the Bahn instances, the maximal NoMs deviation $\delta_{\min}^{\max}$ is rather low. The best e.g. ten timetables w.r.t. ATT on final NoMs convergence we keep in the *final pool* $\mathcal{T}_{\mathrm{f}}$ and start over with both MsConv and MsOnly, using a larger number of runs per initial timetable, e.g. hundred.

Over all timetables we obtain during the process we select the one with the best ATT as our final solution.

Chapter 7

Conclusion

> Why do we simply ignore frequencies in the event activity network? Why is the passenger distribution in the PESP fixed? Are these legitimate simplifications? If not: how can classical methods ever yield good average traveling times?

Indeed, the author had lots of doubts when he started his work in the LinTim team and even was about to switch to pure mathematics. Passion for applied problems returned when he was able to formulate a linear program that incorporates line planning and periodic timetabling, since he saw a way to finally make things *optimal*. However, with it having astronomic dimensions, he started to study its ODPESP subproblem, which was still too large for practice-relevant instances, no matter how hard he tried to simplify it, so he finally ended with retimetabling.

The planning steps lower bounds, as simple as they are, turned out to be a valuable tool in the ODPESP analysis and ReTim could reduce the gap to l_{LC} to 6-7% for large scale networks.

Initial doubts about classical methods turned out to be justified. Qualitatively, the errors predicted in synthetical worst case networks occur in our test instances as well, from tiny to large scale networks. For Athens Metro, the best solution could only be found by evaluating a quantity initially considered as a test-only waste product: the rollout peek. Indeed, a timetablers nightmare.

However, `frequency_as_multiplicity` remains a double-edged sword: on the one hand it is the only way to identify good timetables, on the other it has a bad ReTim performance as initial LINES ROLL OUT model so that the PERIODIC ROLLOUT turns out to be more than just an evaluation tool and, in a ReTim framework, can improve the average traveling time as much as state-of-the-art timetabling heuristics.

Surprisingly, the EPESP performs very poor, but is widely mentioned in literature. So there is either something wrong with our test or nobody actually checked it out.

All in all, the ODPESP seems not to be understood very well, since its overall performance seem to depend more on randomness than the actual methods used. It also remains unclear whether the average traveling time still may or may not be improved much further. However, statistical retimetabling can improve solutions an eightfold compared to classical approaches, which may be considered as a success.

Indeed, from a practical point of view and given that our observations extrapolate smoothly enough to real world networks, the results are stunning, make the author agree with [Lue09] and let him state

> Never, ever, even think about taking some arbitrary passenger distribution, optimize a feasible timetable by tuning modulo parameters, getting a *low slack*, stop and say *it's optimal*. Timetables obtained that way, no matter whether for small or large networks are generally *highly suboptimal* and may be significantly improved by means as simple as fixed modulo retimetabling.

Why using such drastic words? The answer may be found on Page 7:

> Public transportation affect the daily life of billions of people and besides ineffectiveness producing more costs and wasting more resources on the operators side, it wastes billions of hours of valueable time on the customers side and therefore is of global economical as well as ecological interest.

Therefore, we may talk of *responsibility in mathematics*, which in our case lies in analysing problems conscientiously and not making too many assumptions about their nature. To the authors point of view, this is the actual achievement of this work and he hopes that his contributions may help to find better timetables.

List of Figures

List of Tables

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