# Prony's Method: Parameter identification and sparse approximation

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**Emmy Noether Lecture**

# Emmy Noether in Göttingen

Supported by David Hilbert and Felix Klein, Emmy Noether submitted her Habilitation thesis at the University of Göttingen on July 20, 1915.
Her venia legendi was approved on June 4, 1919, allowing her to obtain the rank of a Privatdozent.

Official request on 26 of November 1915 to the Prussian Minister:

> "Eure Exzellenz bittet die mathematisch-naturwissenschaftliche Abteilung der philosophischen Fakultät der Göttinger Universität ehrerbietigst, ihr im Falle des Habilitationsgesuches von Fräulein Dr. Emmy Noether (für Mathematik) Dispens von dem Erlaß des 29. Mai 1908 gewähren zu wollen, nach welchem die Habilitation von Frauen unzulässig ist."

In the negative reply of the Minister of 5 November 1917, it was written:

> "Die Zulassung von Frauen zur Habilitation als Privatdozent begegnet in akademischen Kreisen nach wie vor erheblichen Bedenken. Da die Frage nur grundsätzlich entschieden werden kann, vermag ich auch die Zulassung von Ausnahmen nicht zu genehmigen, selbst wenn im Einzelfall dadurch gewisse Härten unvermeidbar sind. Sollte die grundsätzliche Stellungnahme der Fakultäten, mit der der Erlaß vom 29. Mai 1908 rechnet, eine andere werden, bin ich gern bereit, die Frage erneut zu prüfen."

# Emmy Noether in Göttingen

On 27 December 1918 Einstein wrote to Klein:

"Beim Empfang der neuen Arbeit von Frl. Noether empfand ich es wieder als grosse Ungerechtigkeit, dass man ihr die venia legendi vorenthält. Ich wäre sehr dafür, dass wir beim Ministerium einen energischen Schritt unternähmen. Halten Sie dies aber nicht für möglich, so werde ich mir allein Mühe geben."

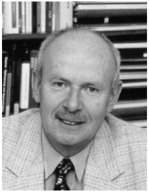
van der Waerden in his obituary:

"Völlig unegoistisch und frei von Eitelkeit, beanspruchte sie niemals etwas für sich selbst, sondern förderte in erster Linie die Arbeiten ihrer Schüler. Sie schrieb für uns alle immer die Einleitungen, in denen die Leitgedanken unserer Arbeiten erklärt wurden, die wir selbst anfangs niemals in solcher Klarheit bewußtmachen und aussprechen konnten. Sie war uns eine treue Freundin und gleichzeitig eine strenge, unbestechliche Richterin."

# Outline

- Original Prony method: Reconstruction of sparse exponential sums
- Ben-Or & Tiwari method: Reconstruction of sparse polynomials
- The generalized Prony method
- Application to shift and dilation operator
- Recovery of sparse vectors

**Collaborations**

Thomas Peter, Daniela Rosca, Manfred Tasche, Katrin Wannenwetsch, Marius Wischerhoff

## Prony method: Reconstruction of sparse exponential sums

**Function** $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

**We have** $M$, $f(\ell)$, $\ell = 0, \ldots, 2M-1$

**We want** $c_j$, $T_j \in \mathbb{C}$, where $-\pi \leq \mathrm{Im}\, T_j < \pi$, $j = 1, \ldots, M$.

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Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left( z - e^{T_j} \right) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters $T_j$ and $p_M = 1$.

$$\sum_{\ell=0}^{M} p_\ell f(\ell + m) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{T_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{T_j m} \sum_{\ell=0}^{M} p_\ell \, e^{T_j \ell}$$

$$= \sum_{j=1}^{M} c_j e^{T_j m} P(e^{T_j}) = 0, \qquad m = 0, \ldots, M-1.$$

# Reconstruction algorithm

**Input**: $f(\ell)$, $\ell = 0, \ldots, 2M-1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \ldots & f(M-1) \\ f(1) & f(2) & \ldots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \ldots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^{M} p_\ell z^\ell$ and extract the parameters $T_j$ from its zeros $z_j = e^{T_j}$, $j = 1, \ldots, M$.
- Compute $c_j$ solving the linear system

$$f(\ell) = \sum_{j=1}^{M} c_j e^{T_j \ell}, \qquad \ell = 0, \ldots, 2M-1.$$

**Output**: Parameters $T_j$ and $c_j$, $j = 1, \ldots, M$.

| | |
|---|---|
| [Prony] (1795): | Reconstruction of difference equations |
| [Schmidt] (1979): | **MUSIC** (Multiple Signal Classification) |
| [Roy, Kailath] (1989): | **ESPRIT** (Estimation of signal parameters via rotational invariance techniques) |
| [Hua, Sakar] (1990): | **Matrix-pencil method** |
| [Stoica, Moses] (2000): | **Annihilating filters** |
| [Potts, Tasche] (2010, 2011): | **Approximate Prony method** |

Sidi ('75,'82,'85); Golub, Milanfar, Varah ('99);
Vetterli, Marziliano, Blu ('02); Maravić, Vetterli ('04);
Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10);
Andersson, Carlsson, de Hoop ('10), Berent, Dragotti, Blu ('10), Batenkov, Sarg, Yomdin ('12,'13); Filbir, Mhaskar, Prestin ('12);
Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13); ...

# Generalized Prony method (Peter, Plonka (2013))

Let $V$ be a normed vector space and let $\mathcal{A} : V \to V$ be a linear operator. Let $\{e_n \,:\, n \in I\}$ be a set of eigenfunctions of $\mathcal{A}$ to **pairwise different** eigenvalues $\lambda_n \in \mathbb{C}$,

$$\mathcal{A}\, e_n = \lambda_n\, e_n.$$

Let

$$f = \sum_{j \in J} c_j\, e_j, \qquad J \subset I \,\text{with}\, |J| = M,\; c_j \in \mathbb{C}.$$

Let $G : V \to \mathbb{C}$ be a linear functional with $G(e_n) \neq 0$ for all $n \in I$.

**We have** $M,\; G(\mathcal{A}^\ell f)$ for $\ell = 0, \ldots, 2M-1$
**We want** $J \subset I,\; c_j \in \mathbb{C}$ for $j \in J$

# Prony polynomial

$$P(z) := \prod_{j\in J}(z-\lambda_j) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

with unknown $\lambda_j$, i.e., with unknown $J$. Hence, for $m = 0, 1, \ldots$

$$\sum_{\ell=0}^{M} p_\ell G(\mathcal{A}^{\ell+m} f) = \sum_{\ell=0}^{M} p_\ell G\left(\sum_{j\in J} c_j \mathcal{A}^{\ell+m} e_j\right) = \sum_{\ell=0}^{M} p_\ell G\left(\sum_{j\in J} c_j \lambda_j^{\ell+m} e_j\right)$$

$$= \sum_{j\in J} c_j \lambda_j^m \left(\sum_{\ell=0}^{M} p_\ell \lambda_j^\ell\right) G(e_j)$$

$$= \sum_{j\in J} c_j \lambda_j^m P(\lambda_j) G(e_j) = 0.$$

Thus, if $G(\mathcal{A}^\ell f)$, $\ell = 0, \ldots, 2M-1$ is known, then we can compute the index set $J \subset I$ of the active eigenfunctions and the coefficients $c_j$.

# Algorithm (Recovery of $f$)

**Input**: $M$, $G(\mathcal{A}^k f)$, $k = 0, \ldots, 2M-1$.

- Solve the linear system

$$\sum_{k=0}^{M-1} p_k G(\mathcal{A}^{k+m} f) = -G(\mathcal{A}^{M+m} f), \qquad m = 0, \ldots, M-1.$$

- Form the Prony polynomial $P(z) = \sum_{k=0}^{M} p_k z^k$. Compute the zeros $\lambda_j$, $j \in J$, of $P(z)$ and determine $e_j$, $j \in J$.
- Compute the coefficients $c_j$ by solving the overdetermined system

$$G(\mathcal{A}^k f) = \sum_{j \in J} c_j \lambda_j^k e_j \qquad k = 0, \ldots, 2M-1.$$

**Output**: $c_j$, $e_j$, $j \in J$, determining $f$.

# How to apply the generalized Prony method ?

# Application to linear operators: shift operator

Choose the **shift operator** $\mathcal{S}_h : C(\mathbb{R}) \to C(\mathbb{R})$, $h > 0$

$$\mathcal{S}_h f(x) := f(x+h)$$

Eigenfunctions of $\mathcal{S}_h$

$$\mathcal{S}_h \, \mathrm{e}^{T_j x} = \mathrm{e}^{T_j (x+h)} = \mathrm{e}^{T_j h} \, \mathrm{e}^{T_j x}, \qquad T_j \in \mathbb{C}, \, \mathrm{Im} \, T_j \in [-\frac{\pi}{h}, \frac{\pi}{h}).$$

**Prony method**: For the reconstruction of

$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{T_j x}$ we need $G(\mathcal{S}_h^{\ell} f) = G(f(\cdot + h\ell)), \; \ell = 0, \ldots, 2M-1.$

Put $G(f) := f(x_0)$ $\qquad G(\mathcal{S}_h^{\ell} f) = f(x_0 + h\ell).$

Put $G(f) := (h * f)(x_0)$ $\qquad G(\mathcal{S}_h^{\ell} f) = (h * f)(x_0 + h\ell).$

# Application to linear operators: dilation operator

Choose the **dilation operator** $\mathcal{D}_h : C(\mathbb{R}) \to C(\mathbb{R})$,

$$\mathcal{D}_h f(x) := f(hx)$$

Eigenfunctions of $\mathcal{D}_h$: $\mathcal{D}_h\, x^{p_j} = (hx)^{p_j} = h^{p_j} x^{p_j}, \quad p_j \in \mathbb{C},\, x \in \mathbb{R}$.

We need: $\quad h^{p_j}$ are pairwise different for all $j \in I$.

**Ben-Or & Tiwari method**: For reconstruction of

$$f(x) = \sum_{j=1}^{M} c_j\, x^{p_j},$$

we need $G(\mathcal{D}_h^{\ell} f) = G(f(h^{\ell} \cdot)),\ \ \ell = 0, \ldots, 2M-1$.

Put $G(f) := f(x_0)$. $\qquad G(\mathcal{D}_h^{\ell} f) = f(h^{\ell} x_0)$

Put $G(f) := \int_0^1 f(x) dx$. $\qquad G(\mathcal{D}_h^{\ell} f) = \frac{1}{h^{\ell}} \int\limits_0^{h^{\ell}} f(x) dx$.

# Example: dilation operator

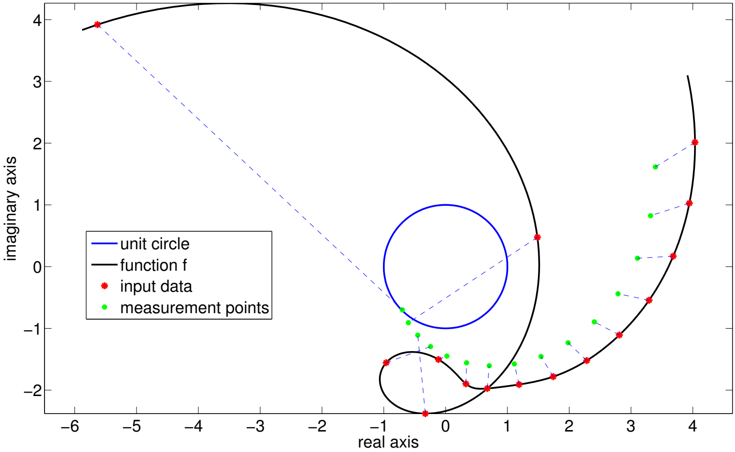
Consider

$$f(x) = \frac{6}{x^9} + \frac{\sqrt{x}}{5} + 1.3\,x.$$

Choose $G(f) := f(x_0)$ with $x_0 = -0.7 - 0.7\mathrm{i}$ and $h = 1.1\mathrm{e}^{\mathrm{i}/5}$

Input data: $f(h^k x_0)$, $k = 0, \ldots, 14$

# Example: dilation operator



# Application to linear operators: Recovery of sparse vectors

Choose the operator $\mathbf{D} : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathbf{D}\mathbf{x} := \operatorname{diag}(d_0, \ldots, d_{N-1})\,\mathbf{x}$$

with pairwise different $d_j$.
Eigenvectors of $\mathbf{D}$: $\quad \mathbf{D}\,\mathbf{e}_j = d_j \mathbf{e}_j \qquad j = 0, \ldots, N-1.$
We want to reconstruct

$$\mathbf{x} = \sum_{j=1}^{M} c_{n_j}\,\mathbf{e}_{n_j} \quad c_{n_j} \in \mathbb{C},\ 0 \le n_1 < \ldots < n_M \le N-1.$$

We need $G(\mathbf{D}^\ell \mathbf{x})$, $\ell = 0, \ldots 2M-1$.
Let $G(\mathbf{x}) := \mathbf{1}^{\mathrm{T}}\mathbf{x} := \sum_{j=0}^{N-1} x_j$.

Then $G(\mathbf{D}^\ell \mathbf{x}) = \mathbf{1}^{\mathrm{T}}\mathbf{D}^\ell \mathbf{x} = (d_0^\ell, \ldots, d_{N-1}^\ell)\mathbf{x}$, $\ell = 0, \ldots 2M-1$.

# Example: Sparse vectors

Choose

$$\mathbf{D} = \mathrm{diag}(\omega_N^0, \omega_N^1, \ldots, \omega_N^{N-1}),$$

where $\omega_N := \mathrm{e}^{-2\pi\mathrm{i}/N}$ denotes the $N$-th root of unity.
Then an $M$-sparse vector $\mathbf{x}$ can be recovered from

$$\mathbf{y} = \mathbf{F}_{2M,N}\,\mathbf{x},$$

where $\mathbf{F}_{2M,N} = (\omega_N^{k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$.

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where $\mathbf{F}_{2M,N} = (\omega_N^{k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$.
More generally, choose $\sigma \in \mathbb{N}$ with $(\sigma, N) = 1$ and

$$\mathbf{D} = \mathrm{diag}(\omega_N^0, \omega_N^{\sigma}, \ldots, \omega_N^{\sigma(N-1)}),$$

Then an $M$-sparse vector $\mathbf{x}$ can be recovered from

$$\mathbf{y} = \widetilde{\mathbf{F}}_{2M,N}\,\mathbf{x},$$

where $\widetilde{\mathbf{F}}_{2M,N} = (\omega_N^{\sigma k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$.

# Reconstruction of linear combinations of translates of $\Phi$

Consider

$$f(x) = \sum_{j=1}^{M} c_j \, \Phi(x - T_j)$$

with unknown parameters $c_j \in \mathbb{R}$, $T_j \in \mathbb{R}$, and known $\Phi$.

# Reconstruction of linear combinations of translates of $\Phi$

Consider

$$f(x) = \sum_{j=1}^{M} c_j \, \Phi(x - T_j)$$

with unknown parameters $c_j \in \mathbb{R}$, $T_j \in \mathbb{R}$, and known $\Phi$. Choose the operator $\mathcal{A}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$,

$$\mathcal{A}_h f(x) := F^{-1} \left( \frac{\widehat{\Phi}(\cdot)}{\widehat{\Phi}(\cdot + h)} \cdot \widehat{f}(\cdot + h) \right) (x)$$

Then

$$\begin{aligned} \mathcal{A}_h[\Phi(\cdot - T_j)](x) &= F^{-1} \left[ \frac{\widehat{\Phi}(\cdot)}{\widehat{\Phi}(\cdot + h)} \cdot e^{iT_j(\cdot + h)} \widehat{\Phi}(\cdot + h) \right] (x) \\ &= F^{-1} \left[ \widehat{\Phi}(\cdot) \, e^{iT_j(\cdot + h)} \right] (x) \\ &= e^{iT_j h} \, \Phi(x - T_j) \end{aligned}$$

# Application to linear operators: Chebyshev-shift operator

Choose the **Chebyshev-shift** $\mathcal{S}_h : C(\mathbb{R}) \to C(\mathbb{R})$ with $h \in [-1, 1]$,

$$\mathcal{S}_h f(x) := \frac{1}{2}\left( f(hx - \sqrt{(1-h^2)(1-x^2)}) + f(hx + \sqrt{(1-h^2)(1-x^2)}) \right)$$

Let $T_n(x) := \cos n(\arccos x)$. With $x = \cos t$, $h = \cos \alpha$,

$$\mathcal{S}_h T_n(x) = \frac{1}{2}\left( T_n(\cos(t+\alpha)) + T_n(\cos(t-\alpha)) \right) = \cos(n\alpha)\, T_n(x).$$

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The sparse Chebyshev expansion

$$f = \sum_{j=1}^{M} c_{n_j}\, T_{n_j}$$

can be recovered from $G(\mathcal{S}_h^\ell f)$, $\ell = 0, \ldots, 2M-1$.
Choose $G(f) := f(1)$ and $h := \cos(\frac{\pi}{2N-1})$. Then $G(\mathcal{S}_h^\ell f)$ are linear combinations of $f\left(\cos(\frac{\ell\pi}{2N-1})\right)$, $\ell = 0, \ldots, 2M-1$.

# Application to linear operators: Sturm-Liouville operator

Chose $\mathcal{L}_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

$$\mathcal{L}_{p,q} f(x) := p(x) f''(x) + q(x) f'(x),$$

where $p(x), q(x)$ are polynomials of degree 2 and 1 respectively. Eigenfunctions are orthogonal polynomials, where $\mathcal{L}_{p,q} Q_n = \lambda_n Q_n$.

| $p(x)$ | $q(x)$ | $\lambda_n$ | name |
|---|---|---|---|
| $(1-x^2)$ | $(\beta-\alpha-(\alpha+\beta+2)x)$ | $n(n+\alpha+\beta+1)$ | Jacobi |
| $(1-x^2)$ | $-(2\alpha+1)x$ | $n(n+2\alpha)$ | Gegenbauer |
| $(1-x^2)$ | $-2x$ | $n(n+1)$ | Legendre |
| $(1-x^2)$ | $-x$ | $n^2$ | Chebyshev 1. |
| $(1-x^2)$ | $-3x$ | $n(n+2)$ | Chebyshev 2. |
| $1$ | $-2x$ | $2n$ | Hermite |
| $x$ | $(\alpha+1-x)$ | $n$ | Laguerre |

# Sparse sums of orthogonal polynomials

**Function** $$f(x) = \sum_{j=1}^{M} c_{n_j}\, Q_{n_j}(x).$$

**We want:** $c_{n_j} \in \mathbb{C} \setminus \{0\}$, indices $n_j$ of "active" basis polynomials $Q_{n_j}$
Now, $f$ can be uniquely recovered from

$$G(\mathcal{L}_{p,q}^k f) = \mathcal{L}_{p,q}^k f(x_0) = \sum_{j=1}^{M} c_{n_j} \lambda_{n_j}^k Q_{n_j}(x_0), \quad k = 0, \ldots, 2M-1.$$

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**Theorem.**
For each polynomial $f$ and each $x \in \mathbb{R}$, the values $\mathcal{L}_{p,q}^k f(x)$, $k = 0, \ldots, 2M-1$, are uniquely determined by $f^{(m)}(x)$, $m = 0, \ldots, 4M-2$.
If $p(x_0) = 0$, then $\mathcal{L}_{p,q} f(x_0)$ reduces to $\mathcal{L}_{p,q} f(x_0) = q(x_0) f'(x_0)$, and the values $\mathcal{L}_{p,q}^k f(x_0)$, $k = 0, \ldots, 2M-1$, can be determined uniquely by $f^{(m)}(x_0)$, $m = 0, \ldots, 2M-1$.

# Example: Sparse Laguerre expansion

**Operator equation**

$$x(L_n^{(\alpha)})''(x) + (\alpha + 1 - x)(L_n^{(\alpha)})'(x) = -n\, L_n^{(\alpha)}(x)$$

**Sparse Laguerre expansion**

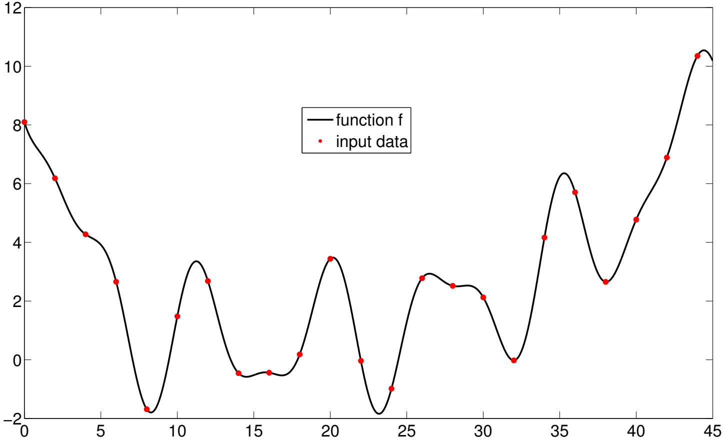
$$f(x) = \sum_{j=1}^{6} c_{n_j}\, L_{n_j}^{(0)}(x) \qquad \text{(with } \alpha = 0\text{)}$$

Given values: $f(0)$, $f'(0)$, $\ldots$, $f^{(11)}(0)$.

| $j$ | $n_j$ | $c_{n_j}$ | $\tilde{n}_j$ | $\tilde{c}_{n_j}$ |
|---|---|---|---|---|
| 1 | 142 | $-3$ | 142.0000000018223 | $-2.999999999999987$ |
| 2 | 125 | $-1$ | 125.0000000494359 | $-1.000000000000034$ |
| 3 | 91 | 2 | 90.9999998114290 | 2.000000000000063 |
| 4 | 69 | $-3$ | 69.0000003316075 | $-3.000000000000058$ |
| 5 | 53 | $-1$ | 53.0000003445395 | $-0.999999999999988$ |
| 6 | 11 | 2 | 10.9999999973030 | 2.000000000000004 |

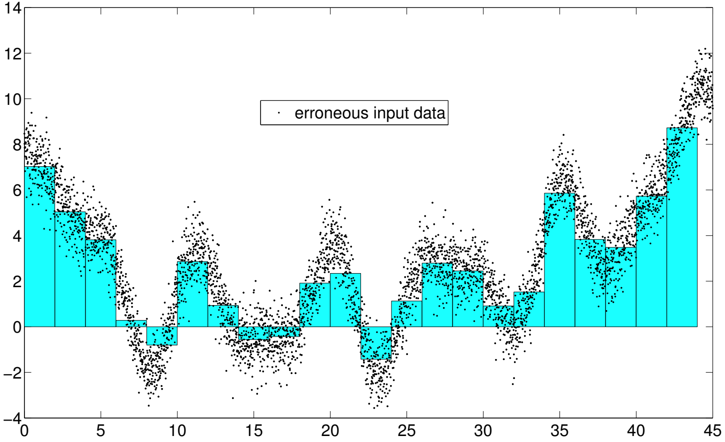
\thankyou

# Application to linear operators: Integrating Functional

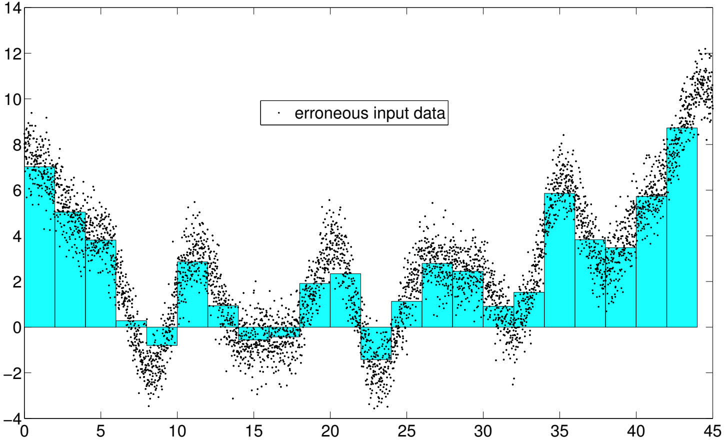



$$f(x) = \sin(1.3x) - 2\cos(0.8x) + 0.1\mathrm{e}^{x/10} + 10\mathrm{e}^{-0.3x}$$

# Application to linear operators: Integrating Functional

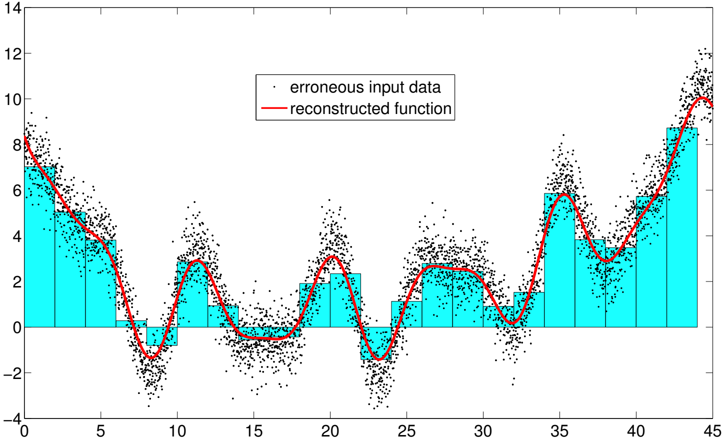


# Application to linear operators: Integrating Functional




| $\|\mathbf{c}-\tilde{\mathbf{c}}\|_\infty$ | $\|\mathbf{T}-\tilde{\mathbf{T}}\|_\infty$ | $\|f-\tilde{f}\|_\infty$ |
|---|---|---|
| $4.34 \cdot 10^{-2}$ | $4.79 \cdot 10^{-3}$ | |

# Application to linear operators: Integrating Functional




| $\|\mathbf{c}-\tilde{\mathbf{c}}\|_\infty$ | $\|\mathbf{T}-\tilde{\mathbf{T}}\|_\infty$ | $\|f-\tilde{f}\|_\infty$ |
|---|---|---|
| $4.34 \cdot 10^{-2}$ | $4.79 \cdot 10^{-3}$ | $5.42 \cdot 10^{-1}$ |