Reconstruction of non-stationary signals by the generalized Prony method

Gerlind Plonka

Institute for Numerical and Applied Mathematics, University of Göttingen

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Outline

- The Prony method: Reconstruction of sparse exponential sums
- Revisiting Prony's method using the shift operator
- Generalized shift operators
- Recovery of sparse trigonometric expansions
- Recovery of sparse expansions of shifted Gaussians
- Recovery of sparse Gabor expansions
- Recovery of sparse expansions of Chebyshev polynomials
- Recovery of non-stationary signals

Joint work with Kilian Stampfer and Ingeborg Keller

The Prony method for sparse exponential sums

Signal model $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

We have M, $f(\ell)$, $\ell = 0, \ldots, 2M-1$

We want $c_j, T_j \in \mathbb{C}$, where $-\pi \le \operatorname{Im} T_j < \pi$, $j = 1, \ldots, M$.

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Consider

$$P(z) := \prod_{j=1}^{M} \left(z - e^{T_j}\right) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters T_j and $p_M = 1$.

$$\sum_{\ell=0}^{M} p_\ell f(\ell + m) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{T_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{T_j m} \sum_{\ell=0}^{M} p_\ell \, e^{T_j \ell}$$

$$= \sum_{j=1}^{M} c_j e^{T_j m} P(e^{T_j}) = 0, \qquad m = 0, \ldots, M-1.$$

Reconstruction algorithm

Input: $f(\ell)$, $\ell = 0, \ldots, 2M-1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \ldots & f(M-1) \\ f(1) & f(2) & \ldots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \ldots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^{M} p_\ell z^\ell$ and extract the parameters T_j from its zeros $z_j = \mathrm{e}^{T_j}$, $j = 1, \ldots, M$.
- Compute c_j solving the linear system

$$f(\ell) = \sum_{j=1}^{M} c_j \mathrm{e}^{T_j \ell}, \qquad \ell = 0, \ldots, 2M-1.$$

Output: Parameters T_j and c_j, $j = 1, \ldots, M$.

(Almost) equivalent models

If we can reconstruct

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x},$$

then we can also reconstruct

$$g(t) = \sum_{j=1}^{M} c_j \, \delta(t - t_j) \qquad \Rightarrow \qquad \widehat{g}(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i} t_j x}$$

$$g(t) = \sum_{j=1}^{M} c_j \, \phi(t - t_j) \qquad \Rightarrow \qquad \widehat{g}(x) = \Big(\sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i} t_j x} \Big) \widehat{\phi}(x)$$

$$f(s) = \sum_{j=1}^{M} \frac{c_j}{s - \alpha_j} \qquad \Rightarrow \qquad \mathcal{L}^{-1}(g)(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x}$$

[Prony] (1795):	Reconstruction of difference equations
[Schmidt] (1979):	**MUSIC** (Multiple Signal Classification)
[Roy, Kailath] (1989):	**ESPRIT** (Estimation of signal parameters via rotational invariance techniques)
[Hua, Sakar] (1990):	**Matrix-pencil method**
[Stoica, Moses] (2000):	**Annihilating filters**
[Vetterli, Marziliano, Blu (2002):	**Finite rate of innovation signals**
[Potts, Tasche] (2010, 2011):	**Approximate Prony method**
[Peter, Plonka] (2013):	**Generalized Prony Method**

Sidi ('75,'82,'85); Golub, Milanfar, Varah ('99); Maravić, Vetterli ('04); Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10); Andersson, Carlsson, de Hoop ('10), Berent, Dragotti, Blu ('10), Batenkov, Sarg, Yomdin ('12,'13); Filbir, Mhaskar, Prestin ('12); Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13); Plonka, Tasche ('14); Kunis, Peter, Römer, von der Ohe ('16); Wei, Dragotti ('16); Sauer ('17); Cuyt, Lee ('17), Mourrain ('17)

Revisiting Prony's method using the shift operator

Let

$$S_h f := f(\cdot + h), \qquad h \in \mathbb{R} \setminus \{0\}.$$

Then

$$(S_h \mathrm{e}^{\alpha \cdot})(x) = \mathrm{e}^{\alpha(h+x)} = \mathrm{e}^{\alpha h} \, \mathrm{e}^{\alpha x}.$$

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For $f(x) = \sum\limits_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j x}$ and $\lambda_j = \mathrm{e}^{\alpha_j h}$ let $P(z) := \prod\limits_{j=1}^{M} (z - \lambda_j) = \sum\limits_{k=0}^{M} p_k z^k$.

$$\sum_{k=0}^{M} p_k f(x_0 + h(k+m)) = \sum_{k=0}^{M} p_k (S_{h(k+m)} f)(x_0) = \sum_{k=0}^{M} p_k (S_h^{k+m} f)(x_0)$$

$$= \sum_{k=0}^{M} p_k S_h^{k+m} \Big(\sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cdot} \Big)(x_0) = \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j\, (S_h^{k+m} \mathrm{e}^{\alpha_j \cdot})(x_0)$$

$$= \sum_{j=1}^{M} c_j \sum_{k=0}^{M} p_k\, \lambda_j^{m+k}\, \mathrm{e}^{\alpha_j x_0} = \sum_{j=1}^{M} c_j \lambda_j^m \Big(\sum_{k=0}^{M} p_k\, \lambda_j^k \Big) \mathrm{e}^{\alpha_j x_0} = 0.$$

Generalized shift operators

A) Symmetric shift operator $S_{h,-h}$:

$$S_{h,-h}f(x) := \frac{1}{2}\Big(f(x-h) + f(x+h)\Big) = \frac{1}{2}(S_{-h} + S_h)f(x), \quad h > 0$$

B) Let $K \in C(\mathbb{R}^2)$ and

$$K(x, h_1 + h_2) = K(x, h_2)K(x + h_2, h_1) = K(x, h_1)K(x + h_1, h_2).$$

Generalized shift operator $S_{K,h}$:

$$S_{K,h}f(x) := K(x, h)\, f(x + h).$$

C) Let $G \in C(\mathbb{R})$ be strictly monotonous in $[a, b] \subseteq \mathbb{R}$.
Generalized shift operator $S_{G,h}$:

$$S_{G,h}f(x) := f(G^{-1}(G(x) + h)).$$

Generalized shift operators

Theorem

$$
\begin{aligned}
S_{h_2,-h_2}(S_{h_1,-h_1} f) &= S_{h_1,-h_1}(S_{h_2,-h_2} f) \\
&= \tfrac{1}{2}\left(S_{h_1+h_2,-(h_1+h_2)} f + S_{h_1-h_2,-(h_1-h_2)} f\right), \\
S_{G,h_1}(S_{G,h_2} f) &= S_{G,h_2}(S_{G,h_1} f) = S_{G,h_1+h_2} f, \\
S_{K,h_1}(S_{K,h_2} f) &= S_{K,h_2}(S_{K,h_1} f) = S_{K,h_1+h_2} f.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
S^k_{h,-h} f &= \frac{1}{2^{k-1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} (S_{(k-2l)h,-(k-2l)h} f + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} f, \\
S^k_{G,h} f &= S_{G,kh} f, \\
S^k_{K,h} f &= S_{K,kh} f,
\end{aligned}
$$

where $\delta_{k/2,\lfloor k/2 \rfloor} = 1$ *if* k *is even and vanishes otherwise.*

Recovery of sparse trigonometric expansions

We have

$$S_{h,-h} \cos(\alpha x) = \frac{1}{2} [\cos(\alpha(x+h)) + \cos(\alpha(x-h))] = \cos(\alpha h) \cos(\alpha x),$$

and

$$S_{h,-h} \sin(\alpha x) = \frac{1}{2} [\sin(\alpha(x+h)) + \sin(\alpha(x-h))] = \cos(\alpha h) \sin(\alpha x),$$

i.e., the symmetric shift operator $S_{h,-h}$ possesses the eigenfunctions $\cos(\alpha x)$ and $\sin(\alpha x)$ for all $\alpha \in \mathbb{R}$.

Recovery of cosine expansions

We want to recover

$$f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x).$$

Theorem

Assume that α_j are in the range $[0, K) \subset \mathbb{R}$.
Let $h = \frac{\pi}{K}$. Then, f in can be uniquely reconstructed using the $2M$ samples $f(kh)$, $k = 0, \ldots, 2M-1$.
More generally, for $x_0 \in \mathbb{R}$ satisfying $\alpha_j x_0 \neq (2k+1)\pi/2$ for $k \in \mathbb{Z}$ the $4M-1$ sample values $f(x_0 + hk)$, $k = -2M+1, \ldots, 2M-1$, are sufficient to reconstruct f.

Recovery of expansions of shifted Gaussians

We apply the generalized shift operator $S_{K,h}f(x) = K(x,h)f(x+h)$.

Let $g(x) := \mathrm{e}^{-\beta x^2}$ for some given $\beta \in \mathbb{C} \setminus \{0\}$.

We want to recover the parameters $c_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{R}$ of

$$f(x) = \sum_{j=1}^{M} c_j \, g(x - \alpha_j) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\beta(x-\alpha_j)^2}.$$

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Let $K(x,h) := \mathrm{e}^{\beta h(2x+h)}$. Then

$$(S_{K,h} \, \mathrm{e}^{-\beta(\cdot-\alpha_j)^2})(x) = \mathrm{e}^{\beta h(2x+h)} \, \mathrm{e}^{-\beta(x+h-\alpha_j)^2} = \mathrm{e}^{2\beta\alpha_j h} \, \mathrm{e}^{-\beta(x-\alpha_j)^2}.$$

Thus, $\mathrm{e}^{-\beta(\cdot-\alpha_j)^2}$ are eigenfunctions of $S_{K,h}$ to $\mathrm{e}^{2\beta\alpha_j h}$.

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$$f(x) = \sum_{j=1}^{M} c_j\, g(x-\alpha_j) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{-\beta(x-\alpha_j)^2}.$$

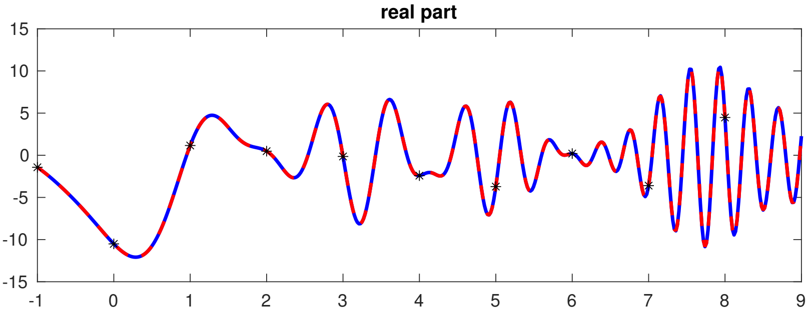
Theorem

If $\mathrm{Re}\,\beta \neq 0$, the stepsize $h \in \mathbb{R} \setminus \{0\}$ can be taken arbitrarily.
If $\mathrm{Re}\,\beta = 0$, we assume that $\alpha_j \in (-T,\ T)$ for $j = 1, \ldots, M$ for some given T and choose $0 < h \leq \frac{\pi}{2|\mathrm{Im}\,\beta|\,T}$.
Then, f can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.

Example: Recovery of shifts of Gaussians

$$f(x) = \sum_{j=1}^{5} c_j \, \mathrm{e}^{\mathrm{i}(x-\alpha_j)^2}$$

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$\mathrm{Re}\, c_j$	-2.37854	-4.55545	2.54933	-2.57214	-0.57597
$\mathrm{Im}\, c_j$	0.75118	-0.56308	0.94536	0.42117	0.73366
α_j	0.64103	-0.18125	-1.50929	-0.53137	-0.23778



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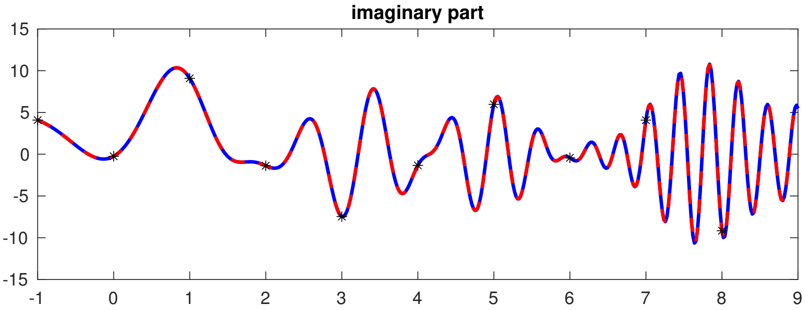

Recovery of sparse Gabor expansions

We want to recover the parameters α_j, c_j, $s_j \in \mathbb{R}$ of

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j),$$

with Gaussian window $g(x) := \mathrm{e}^{-\beta x^2}$ and known $\beta \in \mathbb{R} \setminus \{0\}$.

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Let $K(x,h) = \mathrm{e}^{\beta h(2x+h)}$ then

$$\begin{aligned} (S_{K,h} \, \mathrm{e}^{2\pi \mathrm{i} \alpha_j \cdot - \beta(\cdot - s_j)^2})(x) &= \mathrm{e}^{\beta h(2x+h)} \, \mathrm{e}^{2\pi \mathrm{i}(x+h)\alpha_j} \, \mathrm{e}^{-\beta(x+h-s_j)^2} \\ &= \mathrm{e}^{2h(\beta s_j + \pi \mathrm{i} \alpha_j)} \, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j - \beta(x-s_j)^2}. \end{aligned}$$

Thus, $\mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j) = \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} \, \mathrm{e}^{-\beta(x-s_j)^2}$ are eigenfunctions of $S_{K,h}$ to the eigenvalue $\mathrm{e}^{2h(\beta s_j + \pi \mathrm{i} \alpha_j)}$.

Recovery of Gabor expansions

We want to recover the parameters $\alpha_j\, c_j,\, s_j \in \mathbb{R}$ of

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Theorem

Assume that $\alpha_j \in (-K, K)$ for $j = 1, \ldots, M$ and let $0 < h \leq 1/2K$. Then, f can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.

Proof.

Define $$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{2h(\pi \mathrm{i} \alpha_j + \beta s_j)}) = \sum_{\ell=0}^{M} p_\ell \, z^\ell .$$

The zeros of $P(z)$ are complex, where the imaginary part covers the modulation parameters α_j and the real part the shift parameters s_j. Then for $m = 0, \ldots, M-1$,

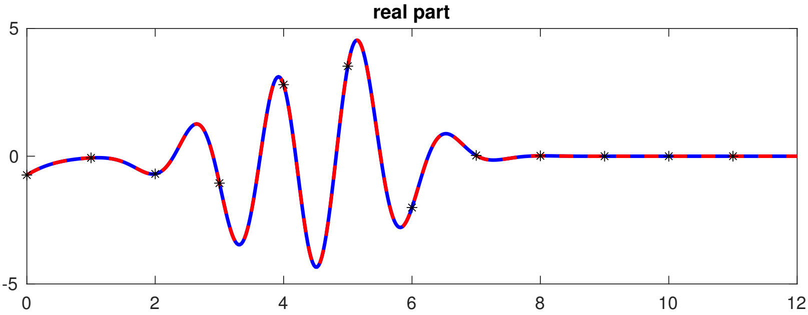
$$\begin{aligned}
& \sum_{\ell=0}^{M} p_\ell \, (S_{K,(\ell+m)h} f)(x_0) = \sum_{\ell=0}^{M} p_\ell \, \mathrm{e}^{\beta h(\ell+m)(2x_0+h(\ell+m))} f(x_0 + h(\ell+m)) \\
= & \sum_{\ell=0}^{M} p_\ell \, \mathrm{e}^{\beta h(\ell+m)(2x_0+h(\ell+m))} \sum_{j=1}^{M} c_j \, \mathrm{e}^{2\pi \mathrm{i}(x_0+h(m+\ell))\alpha_j} \mathrm{e}^{-\beta(x_0+h(\ell+m)-s_j)^2} \\
= & \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\beta(x_0+hm-s_j)^2} \mathrm{e}^{\beta hm(2x_0+hm)} \mathrm{e}^{2\pi \mathrm{i}(x_0+hm)\alpha_j} \sum_{\ell=0}^{M} p_\ell \, \mathrm{e}^{2\ell h(\pi \mathrm{i}\alpha_j + \beta s_j)} = 0.
\end{aligned}$$

Compute $P(z)$ and extract α_j and s_j from the zeros of $P(z)$. Compute c_j by solving the obtained linear system. $\square$

Example: Recovery of Gabor expansions

$$f(x) = \sum_{j=1}^{6} c_j \, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} \, \mathrm{e}^{-(x-s_j)^2/2}$$

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
c_j	0.0777	2.9361	−3.8450	−7.2255	−0.4885	−2.7508
s_j	−1.9918	−4.3941	4.8090	−2.1337	3.0082	3.9611
α_j	0.7881	0.7802	0.6685	0.1335	0.0215	0.5598



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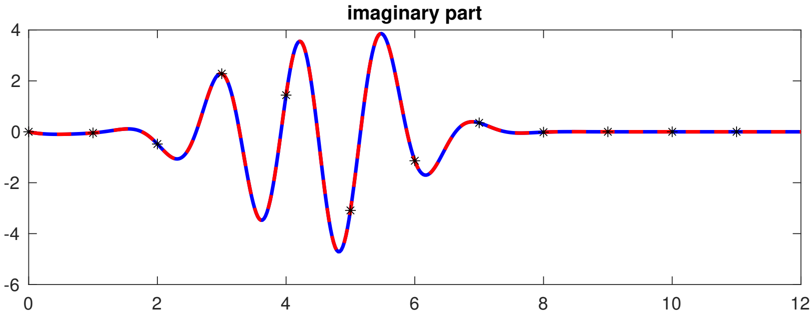

Recovery of signal models using the shift $S_{G,h}$

Let $S_{G,h}f(x) := f(G^{-1}(G(x)+h))$.

$G(x)$	$G^{-1}(x)$	$S_{G,h}f$	eigenfunctions
$\ln(x)$	e^x	$f(\mathrm{e}^{(\ln x)+h}) = f(x\,\mathrm{e}^h)$	$x^p,\ p \in \mathbb{C}$
x^2	$\sqrt{x}$	$f(\sqrt{x^2+h})$	$\mathrm{e}^{\alpha x^2},\ \alpha \in \mathbb{C}$
$x^p,\ p>0$	$\sqrt[p]{x}$	$f(\sqrt[p]{x^p+h})$	$\mathrm{e}^{\alpha x^p},\ \alpha \in \mathbb{C}$
$\cos(x)$	$\arccos(x)$	$f(\arccos(\cos(x)+h))$	$\mathrm{e}^{\alpha \cos x},\ \alpha \in \mathbb{C}$

Sparse expansions of Chebyshev polynomials

We want to recover
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Let $(S_{G,h,-h}f)(x) := \frac{1}{2}\Big(f(\cos(\arccos(x) + h)) + f(\cos(\arccos(x) - h))\Big)$.
Then

$$\begin{aligned}
(S_{G,h,-h}T_k)(x) &= \frac{1}{2}\Big(T_k(\cos(\arccos(x) + h)) + T_k(\cos(\arccos(x) - h))\Big) \\
&= \frac{1}{2}\Big(\cos k(\arccos(x) + h) + \cos k(\arccos(x) - h)\Big) \\
&= \cos(kh)\cos(k \arccos x) = \cos(kh)\, T_k(x).
\end{aligned}$$

Sparse expansions of Chebyshev polynomials

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Then

$$\begin{aligned}(S_{G,h,-h}T_k)(x) &= \frac{1}{2}\Big(T_k(\cos(\arccos(x)+h)) + T_k(\cos(\arccos(x)-h))\Big)\\ &= \frac{1}{2}\Big(\cos k(\arccos(x)+h) + \cos k(\arccos(x)-h)\Big)\\ &= \cos(kh)\cos(k\arccos x) = \cos(kh)\,T_k(x).\end{aligned}$$

Theorem

Let K be a bound of the degree of the polynomial f and let $0 < h \leq \frac{\pi}{K}$. Then the Chebyshev expansion $f(x)$ can be uniquely recovered from the samples $f(\cos(kh))$, $k = 0, \ldots, 2M-1$.

Recovery of non-stationary signals

We want to recover the parameters α_j, $c_j \in \mathbb{R}$, $\beta_j \in [0, 2\pi)$ of

$$f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x^p + \beta_j), \qquad p > 0 \text{ odd}$$

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Let

$$S_{x^p,h,-h} f(x) := \frac{1}{2} \left(f(\mathrm{sgn}(x^p + h) f\left(\sqrt[p]{|x^p + h|} \right) + f(\mathrm{sgn}(x^p + h) f \left(\sqrt[p]{|x^p - h|} \right) \right).$$

Then

$$S_{x^p,h,-h} \cos\left(\alpha_j x^p + \beta_j\right) = \quad \cos(\alpha_j h) \cos\left(\alpha_j x^p + \beta_j\right).$$

The eigenvalues $\cos(\alpha_j h)$ and $\cos(\alpha_k h)$ are pairwise different for $\alpha_j \neq \alpha_k$ if $\alpha_j, \alpha_k \in [0, \pi/h]$.

Recovery of non-stationary signals

We want to recover the parameters α_j, $c_j \in \mathbb{R}$, $\beta_j \in [0, 2\pi)$ of

$$f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x^p + \beta_j) \qquad \text{(with known odd } p > 0\text{).}$$

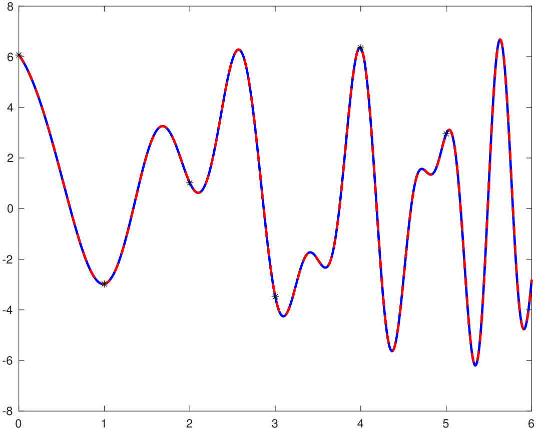
Theorem

Let $h := \pi/K$.
1. If the parameters β_j do not appear, then f can be uniquely recovered from its signal values $f\left(\sqrt[p]{hk}\right)$, $k = 0, \ldots, 2M-1$.
2. If the nonzero parameters β_j appear, then the α_j, $j = 1, \ldots, M$, can be recovered in a first step from signal values $f\left(\sqrt[p]{hk}\right)$, $k = 0, \ldots, 2M-1$, and the parameters c_j and β_j can be reconstructed, using in a second step additionally the signal values $f\left(\mathrm{sgn}(hk - \frac{\pi}{2\alpha_j})\sqrt[p]{|hk - \frac{\pi}{2\alpha_j}|}\right)$ for $k = -M+1, \ldots, M-1$.

Example: Recovery of non-stationary signals

$$f(x) = \sum_{j=1}^{3} c_j \cos(x^2 + \alpha_j x + \beta_j)$$

	$j=1$	$j=2$	$j=3$
c_j	-0.1835	4.2157	2.478
α_j	0.3132	2.2308	2.2181
β_j	0.3834	-0.4682	0.0416



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\thankyou