On the Impact of Prony's Method

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Outline

- Original Prony method: Reconstruction of Sparse Exponential Sums
- Maximum Likelihood Modification of Prony's Method
- Prony's Method Based on the Shift Operator
- Prony's Method Based on the Differential Operator
- Generalized Operator Based Prony Method

Collaborations

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Prony Method: Reconstruction of Sparse Exponential Sums

Function $$f(x) = \sum_{j=1}^{M} c_j \, e^{\alpha_j x}$$

We have M, $f(\ell)$, $\ell = 0, \ldots, 2M-1$

We want $c_j, \alpha_j \in \mathbb{C}$, where $-\pi \le \mathrm{Im}\, \alpha_j < \pi$, $j = 1, \ldots, M$.

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Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} (z - e^{\alpha_j}) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters α_j and $p_M = 1$.

$$\sum_{\ell=0}^{M} p_\ell f(\ell + m) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{\alpha_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{\alpha_j m} \sum_{\ell=0}^{M} p_\ell \, e^{\alpha_j \ell}$$

$$= \sum_{j=1}^{M} c_j e^{\alpha_j m} P(e^{\alpha_j}) = 0, \qquad m = 0, \ldots, M-1.$$

Reconstruction Algorithm

Input: $f(\ell)$, $\ell = 0, \ldots, 2M-1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \ldots & f(M-1) \\ f(1) & f(2) & \ldots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \ldots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^{M} p_\ell z^\ell$ and extract the parameters α_j from its zeros $z_j = e^{\alpha_j}$, $j = 1, \ldots, M$.
- Compute c_j solving the linear system

$$f(\ell) = \sum_{j=1}^{M} c_j e^{\alpha_j \ell}, \qquad \ell = 0, \ldots, 2M-1.$$

Output: Parameters α_j and c_j, $j = 1, \ldots, M$.

(Almost) Equivalent Models

If we can reconstruct

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x},$$

then we can also reconstruct

$$g(t) = \sum_{j=1}^{M} c_j \, \delta(t - t_j) \quad \Rightarrow \quad \widehat{g}(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i} t_j x}$$

$$g(t) = \sum_{j=1}^{M} c_j \, \phi(t - t_j) \quad \Rightarrow \quad \widehat{g}(x) = \Big(\sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i} t_j x} \Big) \widehat{\phi}(x)$$

$$g(t) = \sum_{j=1}^{M} \frac{c_j}{t - \alpha_j} \quad \Rightarrow \quad \mathcal{L}^{-1}(g)(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x}$$

Literature

[Prony] (1795):	Reconstruction of difference equations
[Schmidt] (1979):	**MUSIC** (Multiple Signal Classification)
[Roy, Kailath] (1989):	**ESPRIT** (Estimation of signal parameters via rotational invariance techniques)
[Hua, Sakar] (1990):	**Matrix-pencil method**
[Stoica, Moses] (2000):	**Annihilating filters**
[Vetterli, Marziliano, Blu (2002):	**Finite rate of innovation signals**
[Potts, Tasche] (2010, 2011):	**Approximate Prony method**
[Peter, Plonka] (2013):	**Generalized Prony Method**

Sidi ('75,'82,'85); Golub, Milanfar, Varah ('99); Maravić, Vetterli ('04);
Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10);
Andersson, Carlsson, de Hoop ('10), Berent, Dragotti, Blu ('10),
Batenkov, Sarg, Yomdin ('12,'13); Filbir, Mhaskar, Prestin ('12);
Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13);
Plonka, Tasche ('14); Kunis, Peter, Römer, von der Ohe ('16);
Wei, Dragotti ('16); Sauer ('17); Cuyt, Lee ('17), Mourrain ('17), ...
Very incomplete list !!!

Other Talks in this Conference

- A. Aldroubi, L. Huang, K. Kornelson, and **I. Krishtal**: A Prony-Laplace Method for Identifying Burst-like Forcing Terms
- G. Plonka, K. Stampfer and **I. Keller**: Reconstruction of Non-Stationary Signals by the Generalized Prony Method
- **Sui Tang**: Recovery of Linear Dynamics from Undersampled Time Series Data
- Z.M. Wu and **R. Zhang**: Learning Physics by Data for the Motion of a Sphere Falling in a Non-Newtonian Fluid

Maximum Likelihood Modification of Prony's Method

Let $\mathbf{y} = (y_k)_{k=0}^{L} \in \mathbb{C}^{L+1}$ be a given.

Goal: Approximate $\mathbf{y}$ by $\mathbf{f} = (f_k)_{k=0}^{L} \in \mathbb{C}^{L+1}$ where

$$f_k = \sum_{j=1}^{M} d_j \, \mathrm{e}^{\alpha_j k} = \sum_{j=1}^{M} d_j \, z_j^k, \quad k = 0, \ldots, L, \quad M \leq L/2$$

with d_j, $z_j = \mathrm{e}^{\alpha_j} \in \mathbb{C}$, $j = 1, \ldots, M$, such that

$$\|\mathbf{y} - \mathbf{f}\|_2^2 = \sum_{k=0}^{L} |y_k - f_k|^2$$

is minimal.

Survey: Zhang & Plonka '19

Maximum Likelihood Modification of Prony's Method

With $\mathbf{d} := (d_1, \ldots, d_M)^T$, $\mathbf{z} := (z_1, \ldots, z_M)^T$, and

$$\mathbf{V_z} := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_M \\ z_1^2 & z_2^2 & \ldots & z_M^2 \\ \vdots & \vdots & & \vdots \\ z_1^L & z_2^L & \ldots & z_M^L \end{pmatrix} \in \mathbb{C}^{(L+1)\times M}, \tag{1}$$

we have

$$\mathbf{f} = \mathbf{V_z}\, \mathbf{d}.$$

Thus, we want to solve the nonlinear least squares problem

$$\underset{\mathbf{z},\mathbf{d}\,\in\mathbb{C}^M}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{V_z d}\|_2^2 = \underset{\mathbf{z},\mathbf{d}\,\in\mathbb{C}^M}{\operatorname{argmin}} \sum_{k=0}^{L} |y_k - \sum_{j=1}^{M} d_j\, z_j^k|^2.$$

Maximum Likelihood Modification of Prony's Method

Variable projection gives

$$\mathbf{d} = \mathbf{V}_{\mathbf{z}}^{+}\mathbf{y} = [\mathbf{V}_{\mathbf{z}}^{*}\mathbf{V}_{\mathbf{z}}]^{-1}\,\mathbf{V}_{\mathbf{z}}^{*}\mathbf{y}.$$

Thus, solve

$$\underset{\mathbf{z}\in\mathbb{C}^M}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{V}_{\mathbf{z}}\mathbf{V}_{\mathbf{z}}^{+}\,\mathbf{y}\|_2^2 = \underset{\mathbf{z}\in\mathbb{C}^M}{\operatorname{argmax}} \left(\mathbf{y}^{*}\mathbf{V}_{\mathbf{z}}\mathbf{V}_{\mathbf{z}}^{+}\mathbf{y}\right).$$

Maximum Likelihood Modification of Prony's Method

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Let

$$\mathbf{X}_{\mathbf{p}}^{T} := \begin{pmatrix} p_0 & p_1 & \dots & p_M & & & \\ & p_0 & p_1 & \dots & p_M & & \\ & & \ddots & & & \ddots & \\ & & & p_0 & p_1 & \dots & p_M \end{pmatrix}$$

such that $\mathbf{X}_{\mathbf{p}}^{T}\mathbf{V}_{\mathbf{z}} = \mathbf{0}$. Then, we have to solve

$$\tilde{\mathbf{p}} := \underset{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|_2=1}}{\operatorname{argmin}}\,\mathbf{y}^*\overline{\mathbf{X}}_{\mathbf{p}}\overline{\mathbf{X}}_{\mathbf{p}}^{+}\mathbf{y}.$$

Maximum Likelihood Modification of Prony's Method

Theorem

For given $\mathbf{y} = (y_0, \ldots, y_L)^T$ *the vectors* $\tilde{\mathbf{z}}$ *and* $\tilde{\mathbf{d}}$ *solving*

$$\min_{\mathbf{z},\mathbf{d}\in\mathbb{C}^M} \|\mathbf{y} - \mathbf{V_z d}\|_2^2$$

are obtained by:

1. Solve $$\tilde{\mathbf{p}} = \operatorname*{argmin}_{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|_2=1}} \mathbf{y}^* \overline{\mathbf{X}}_\mathbf{p} \overline{\mathbf{X}}_\mathbf{p}^+ \mathbf{y} = \operatorname*{argmin}_{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|_2=1}} \mathbf{p}^* \mathbf{H}_\mathbf{y}^* [\mathbf{X}_\mathbf{p}^T \overline{\mathbf{X}}_\mathbf{p}]^{-1} \mathbf{H}_\mathbf{y} \mathbf{p}.$$

2. Compute the vector of zeros $\tilde{\mathbf{z}} = (\tilde{z}_1, \ldots, \tilde{z}_M)^T$ *of* $p(z) = \sum_{k=0}^{M} \tilde{p}_k z^k$ *with* $\tilde{\mathbf{p}} = (\tilde{p}_0, \ldots, \tilde{p}_M)^T$.

3. Compute

$$\tilde{\mathbf{d}} = \mathbf{V}_{\tilde{\mathbf{z}}}^+ \mathbf{y} = [\mathbf{V}_{\tilde{\mathbf{z}}}^* \mathbf{V}_{\tilde{\mathbf{z}}}]^{-1} \mathbf{V}_{\tilde{\mathbf{z}}}^* \mathbf{y}.$$

Maximum Likelihood Modification of Prony's Method

Solve
$$\tilde{\mathbf{p}} = \underset{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|_2=1}}{\operatorname{argmin}} \mathbf{p}^* \mathbf{H}_{\mathbf{y}}^* [\mathbf{X}_{\mathbf{p}}^T \overline{\mathbf{X}}_{\mathbf{p}}]^{-1} \mathbf{H}_{\mathbf{y}} \mathbf{p}.$$

Approaches:

- Pisarenko '73 method solves only $\underset{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|_2=1}}{\operatorname{argmin}} \mathbf{p}^* \mathbf{H}_{\mathbf{y}}^* \mathbf{H}_{\mathbf{y}} \mathbf{p}.$
- Levenberg-Marquardt Iteration (weighted structured low-rank approximation) (Markovsky & Usevich '14)
- Iterative Quadratic Maximum Likelihood (IQML): (Bressler & Macovski '86, Z. Dogan et al. '15)

$$\mathbf{p}_{j+1} = \underset{\substack{\mathbf{p}\in\mathbb{C}^{M+1}\\ \|\mathbf{p}\|=1}}{\operatorname{argmin}} \mathbf{p}^* \mathbf{H}_{\mathbf{y}}^* [\mathbf{X}_{\mathbf{p}_j}^T \overline{\mathbf{X}}_{\mathbf{p}_j}]^{-1} \mathbf{H}_{\mathbf{y}} \mathbf{p}.$$

- Gradient Condition Reweighting Algorithm (GRA): (Osborne & Smith '91,'95)
- Simultaneous Minimization (SIMI) (Zhang & Plonka '19)

Maximum Likelihood Modification of Prony's Method

Example: Consider

$$y_k = \exp(0.95kh) + \exp(0.5kh) + \exp(0.2kh) + \epsilon_k \quad k = 0, 1, \ldots, L$$

with $\epsilon_k \sim N(0, \sigma^2)$, $\sigma = 0.01$, $L = 69$, $h = 5/L$.
Results:

	APM	GRA	IQML	VARPRO	SIMI	IGRA
$\tilde{\alpha}_1$	12.13	0.957	0.939	0.939	0.953	0.957
$\tilde{\alpha}_2$	1.014	0.566	0.4+43.4 i	0.8+43.4 i	0.500	0.565
$\tilde{\alpha}_3$	0.532	0.214	0.338	0.338	0.165	0.214
2-error	4.646	0.0013	0.0014	0.0014	0.0013	0.0013

Using the correct parameters 0.95, 0.5, 0.2 we get a 2-error 0.0013. The parameter reconstruction is ill-posed but we get very good approximations.

Towards a Generalization of Prony's Method

Revisiting Prony's Method Using the Shift Operator

Let $S_h f := f(\cdot + h), \quad h \in \mathbb{R} \setminus \{0\}$. Then

$$(S_h \, \mathrm{e}^{\alpha \cdot})(x) = \mathrm{e}^{\alpha(h+x)} = \mathrm{e}^{\alpha h} \, \mathrm{e}^{\alpha x} \quad \text{(eigenfunction)}.$$

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For

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x} \quad \text{with} \quad P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{\alpha_j h}) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

we have

$$P(S_h) f = \sum_{\ell=0}^{M} p_\ell \, (S_h^\ell f) = \sum_{\ell=0}^{M} p_\ell \, S_{h\ell} \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j \cdot} = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \, S_{h\ell} \mathrm{e}^{\alpha_j \cdot}$$

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$$= \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j h \ell} \, \mathrm{e}^{\alpha_j \cdot} = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j \cdot} \sum_{\ell=0}^{M} p_\ell \, \mathrm{e}^{\alpha_j h \ell} = 0.$$

Revisiting Prony's Method Using the Shift Operator

Thus, $f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x}$ solves the difference equation $P(S_h)f = 0$.

Moreover,

$$S_h^k P(S_h) f = P(S_h) S_h^k f = \sum_{\ell=0}^{M} p_\ell \, S_h^{\ell+k} f = 0, \quad k \in \mathbb{Z}.$$

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With the point evaluation functional $F_0 f := f(0)$,

$$F_0(S_h^k P(S_h) f) = \sum_{\ell=0}^{M} p_\ell \, F_0(S_h^{\ell+k} f) = \sum_{\ell=0}^{M} p_\ell \, f(h(\ell+k)) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients p_ℓ of the Prony polynomial from M of these equations, i.e., using $f(hk)$, $k = 0, \ldots, 2M-1$.

Change the Sampling Scheme

Thus, $f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x}$ solves the difference equation $P(S_h)f = 0$.
Moreover, for each linear operator $T : C^{\infty}(\mathbb{R}) \mapsto C^{\infty}(\mathbb{R})$

$$T^k P(S_h)f = T^k P(S_h)f = \sum_{\ell=0}^{M} p_\ell \, T^k S_h^\ell f = 0, \qquad k \in \mathbb{Z}.$$

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With the linear functional $F : C^\infty(\mathbb{R}) \mapsto \mathbb{C}$

$$F(T^k P(S_h) f) = \sum_{\ell=0}^{M} p_\ell \, F(T^k S_h^\ell f) = \sum_{\ell=0}^{M} p_\ell F(T^k f(h(\ell + \cdot))) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients p_ℓ of the Prony polynomial from M of these equations, i.e., using $F(T^k S_h^\ell f)$, $\ell = 0, \ldots, M$, $k = 1, \ldots, M$.

Revisiting Prony's Method Using the Shift Operator

Example: Choose $h = 1$, $T = S_{h/2} = S_{1/2}$, $F = F_0$, then the linear system reads (for even M))

$$\begin{pmatrix} f(0) & f(1) & \dots & f(M-1) \\ f(\frac{1}{2}) & f(\frac{3}{2}) & \dots & f(\frac{M-1}{2}) \\ f(1) & f(2) & \dots & f(M) \\ \vdots & \vdots & & \vdots \\ f(\frac{M}{2}-1) & f(\frac{M}{2}) & \dots & f(\frac{3M}{2}-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+\frac{1}{2}) \\ f(M+1) \\ \vdots \\ f(\frac{3M}{2}-1) \end{pmatrix}$$

This matrix ist not longer of Hankel form.

Revisiting Prony's Method Using the Differential Operator

Let $\frac{\mathrm{d}}{\mathrm{d}x} : C^\infty(\mathbb{R}) \mapsto C^\infty(\mathbb{R}), \quad \frac{\mathrm{d}}{\mathrm{d}x} f := f'$. Then

$$(\tfrac{\mathrm{d}}{\mathrm{d}x}\, \mathrm{e}^{\alpha \cdot})(x) = \alpha\, \mathrm{e}^{\alpha x} \quad \text{(eigenfunction)}.$$

Revisiting Prony's Method Using the Differential Operator

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For

$$f(x) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j x} \quad \text{with} \quad \tilde{P}(z) := \prod_{j=1}^{M} (z - \alpha_j) = \sum_{\ell=0}^{M} \tilde{p}_\ell z^\ell$$

we have

$$\tilde{P}(\tfrac{\mathrm{d}}{\mathrm{d}x}) f = \sum_{\ell=0}^{M} \tilde{p}_\ell \left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^\ell f = \sum_{\ell=0}^{M} \tilde{p}_\ell \left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j \cdot} = \sum_{\ell=0}^{M} \tilde{p}_\ell \sum_{j=1}^{M} c_j \left(\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \mathrm{e}^{\alpha_j \cdot}$$

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$$= \sum_{\ell=0}^{M} \tilde{p}_\ell \sum_{j=1}^{M} c_j\, \alpha_j^\ell\, \mathrm{e}^{\alpha_j \cdot} = \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j \cdot} \sum_{\ell=0}^{M} \tilde{p}_\ell\, \alpha_j^\ell = 0.$$

Revisiting Prony's Method Using the Differential Operator

Thus, $f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x}$ solves the differential equation $\tilde{P}(\frac{\mathrm{d}}{\mathrm{d}x}) f = 0$.

Moreover,

$$(\tfrac{\mathrm{d}}{\mathrm{d}x})^k \, \tilde{P}(\tfrac{\mathrm{d}}{\mathrm{d}x}) f = \sum_{\ell=0}^{M} \tilde{p}_\ell \, (\tfrac{\mathrm{d}}{\mathrm{d}x})^{\ell+k} f = 0, \quad k \in \mathbb{Z}.$$

Revisiting Prony's Method Using the Differential Operator

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With the point evaluation functional $F_0 f := f(0)$,

$$F_0((\tfrac{\mathrm{d}}{\mathrm{d}x})^k \, \tilde{P}(\tfrac{\mathrm{d}}{\mathrm{d}x})f) = \sum_{\ell=0}^{M} \tilde{p}_\ell \, F_0((\tfrac{\mathrm{d}}{\mathrm{d}x})^{\ell+k} f) = \sum_{\ell=0}^{M} \tilde{p}_\ell \, f^{(\ell+k)}(0) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients $\tilde{p}_\ell$ of the Prony polynomial from M of these equations, i.e., using $f^{(k)}(0)$, $k = 0, \ldots, 2M-1$.

Switch Between Operators with the Same Eigenfunctions

What is the connection between Prony's method using the shift operator or the differential operator?
We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\alpha\cdot} = \alpha\,\mathrm{e}^{\alpha\cdot}, \qquad S_h\mathrm{e}^{\alpha\cdot} = \mathrm{e}^{\alpha h}\,\mathrm{e}^{\alpha\cdot}.$$

Obviously, the spectra are connected by the map $\exp(h\cdot) : \alpha \mapsto \mathrm{e}^{h\alpha}$.

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Moreover, for all monomials x^m,

$$\exp(h\tfrac{\mathrm{d}}{\mathrm{d}x})\,x^m = \sum_{k=0}^{\infty}\frac{h^k}{k!}(\tfrac{\mathrm{d}}{\mathrm{d}x})^k x^m = \sum_{k=0}^{m}\frac{h^k}{k!}\,\frac{m!}{(m-k)!}\,x^{m-k}$$

$$= \sum_{k=0}^{m}\binom{m}{k}\,h^k\,x^{m-k} = (x+h)^m = S_h x^m.$$

Thus,

$$\exp(h\tfrac{\mathrm{d}}{\mathrm{d}x})\,f(x) = S_h f(x).$$

Generalized Prony method (Peter & Plonka '13)

V: normed vector space
$\mathcal{A} : V \to V$ linear operator
$\{v_\lambda : \lambda \in \sigma(\mathcal{A})\}$ set of eigenfunctions of $\mathcal{A}$ to **pairwise different** eigenvalues $\lambda \in \sigma(\mathcal{A}) \subset \mathbb{C}$,

$$\mathcal{A}\, v_\lambda = \lambda\, v_\lambda.$$

Let

$$f = \sum_{\lambda \in \Lambda_f} c_\lambda\, v_\lambda, \qquad \Lambda_f \subset \sigma(\mathcal{A}) \qquad \text{with} \quad |\Lambda_f| = M,\ c_\lambda \in \mathbb{C}.$$

Let $G : V \to \mathbb{C}$ be a linear functional with $G(v_\lambda) \neq 0$ for all $\lambda \in \sigma(\mathcal{A})$.

We have $M,\ G(\mathcal{A}^\ell f)$ for $\ell = 0, \ldots, 2M-1$
We want $\Lambda_f \subset \sigma(\mathcal{A}),\ c_\lambda \in \mathbb{C}$ for $\lambda \in \Lambda_f$

Generalized Prony Method

Theorem (Peter & Plonka '13)

The expansion

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of eigenfunctions v_λ of the linear operator $\mathcal{A}$ can be uniquely recovered by $G(\mathcal{A}^\ell f)$, $\ell = 0, \ldots, 2M-1$, where $G : V \to \mathbb{C}$ is a linear functional with $G(v_\lambda) \neq 0$ for all $\lambda \in \sigma(\mathcal{A})$.

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Example: Let $V = C^\infty$, $\mathcal{A} := S_h$ with $S_h f = f(h + \cdot)$, and

$$f = \sum_{\lambda \in \Lambda_f} c_\lambda \, v_\lambda = \sum_{\lambda \in \Lambda_f} c_\lambda \, \mathrm{e}^{\lambda \cdot},$$

where $\Lambda_f \subset \mathbb{R} + \mathrm{i}[-\pi, \pi)$. Choose the functional $Gf := f(0)$, then f can be recovered from the samples

$$G(\mathcal{A}^\ell f) = S_h^\ell f(0) = f(h\ell), \qquad \ell = 0, \ldots, 2M-1.$$

Generalized Operator Based Prony Method (Stampfer & Plonka '19)

Assume, you want to recover a sparse expansion

$$f = \sum_{j=1}^{M} c_j \, v_j, \qquad v_j \in V, \; c_j \in \mathbb{C}.$$

Idea:

- Find a linear operator $\mathcal{A}$ such that v_j are eigenfunctions of $\mathcal{A}$ to pairwise different eigenvalues.
- Check, whether $G(\mathcal{A}^\ell f)$, $\ell = 0, \ldots, 2M-1$ can be computed from the given information.
- If not, transfer to a different operator $\mathcal{B} = \varphi(\mathcal{A})$ and suitable functionals G_k such that $G_k(\mathcal{B}^\ell f)$, $\ell = 0, \ldots, M$, $k = 1, \ldots, M$ that can be computed from the given information.
- Apply the generalized Prony method to recover v_j and c_j, $j = 1, \ldots, M$.

Generalized Operator Based Prony Method

Example:

$$f(x) = \sum_{j=1}^{M} c_j \, x^{\alpha_j}, \qquad \alpha_j \in \mathbb{R}, \, c_j \in \mathbb{R}.$$

- Find a linear operator $\mathcal{A}$ on $C^\infty(\mathbb{R})$:

$$\mathcal{A} \, f(x) := x \, f'(x) \qquad \Rightarrow \quad \mathcal{A} \, x^{\alpha_j} = \alpha_j \, x^{\alpha_j}.$$

However, $\mathcal{A}^\ell f$ involves higher order derivatives.

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- Choose $\varphi(z) := \exp(\tau z)$ with $\tau \in \mathbb{R} \setminus \{0\}$.

$$\exp(\tau x \tfrac{\mathrm{d}}{\mathrm{d}x}) x^m = \sum_{\ell=0}^{\infty} \frac{\tau^\ell}{\ell!} (x \frac{\mathrm{d}}{\mathrm{d}x})^\ell x^m = \sum_{\ell=0}^{\infty} \frac{\tau^\ell}{\ell!} m^\ell x^m = \mathrm{e}^{\tau m} x^m = (\mathrm{e}^\tau x)^m.$$

Thus $\mathcal{B} f(x) = \varphi(\mathcal{A}) f(x) = f(\mathrm{e}^\tau x)$ (dilation operator).

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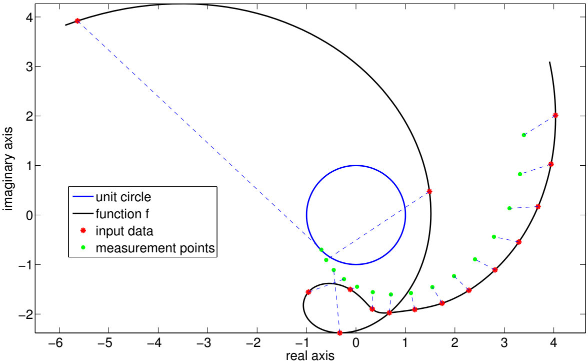
- Choose $f(\mathrm{e}^{\ell \tau} x_0)$, $\ell = 0, \ldots, 2M-1$ to recover f.

Example: dilation operator

Consider

$$f(x) = \frac{6}{x^9} + \frac{\sqrt{x}}{5} + 1.3\,x.$$

Choose $G(f) := f(x_0)$ with $x_0 = -0.7 - 0.7\mathrm{i}$ and $h = 1.1\mathrm{e}^{\mathrm{i}/5}$



Differential Operators of First Order and Generalized Shifts

Reconstruct

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j G(x)},$$

i.e., find c_j, α_j, $j = 1, \ldots, M$, where G is differentiable and strictly monotone on $[a, b]$.

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Find a linear operator with eigenfunctions $\mathrm{e}^{\alpha_j G(x)}$:
Let $g(x) := 1/G'(x)$ and

$$\mathcal{A}f(x) := g(x) \tfrac{\mathrm{d}}{\mathrm{dx}} f(x).$$

Then

$$\mathcal{A}\, \mathrm{e}^{\alpha G(x)} = g(x) \tfrac{\mathrm{d}}{\mathrm{dx}} \mathrm{e}^{\alpha G(x)} = \alpha \, \mathrm{e}^{\alpha G(x)}.$$

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Using the generalized Prony method, f can be recovered using

$$F((g(\cdot) \tfrac{\mathrm{d}}{\mathrm{d}x})^k f), \qquad k = 0, \ldots, 2M-1.$$

However, these may be difficult to provide.

Differential Operators of First Order and Generalized Shifts

Change the operator:

$$
\begin{aligned}
\exp(\tau\,\mathcal{A}) f(x) &= \exp\left(\tau\, g(x)\, \tfrac{\mathrm{d}}{\mathrm{d}x}\right) f(x) \\
&= \sum_{\ell=0}^{\infty} \frac{\tau^{\ell}}{\ell!} \left(g(x)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\right)^{\ell} \left(\sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j G(x)}\right) \\
&= \sum_{j=1}^{M} c_j \left(\sum_{\ell=0}^{\infty} \frac{\tau^{\ell}}{\ell!}\, \alpha_j^{\ell}\right) \mathrm{e}^{\alpha_j G(x)} \\
&= \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j \tau}\, \mathrm{e}^{\alpha_j G(x)} \\
&= \sum_{j=1}^{M} c_j\, \mathrm{e}^{\alpha_j G(G^{-1}(\tau + G(x)))} = f(G^{-1}(\tau + G(x))).
\end{aligned}
$$

Differential Operators of First Order and Generalized Shifts

Theorem (Stampfer & Plonka '19)

Let

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j G(x)},$$

where $G(x)$ is continuous and monotone on an interval $[a, b|$. Let $\tau k + G(x_0) \in G([a, b])$ for $k = 0, \ldots, 2M-1$. Then $f(x)$ can be uniquely reconstructed from the function samples

$$f(G^{-1}(\tau k + G(x_0))), \qquad k = 0, \ldots, 2M-1.$$

Differential Operators of First Order and Generalized Shifts

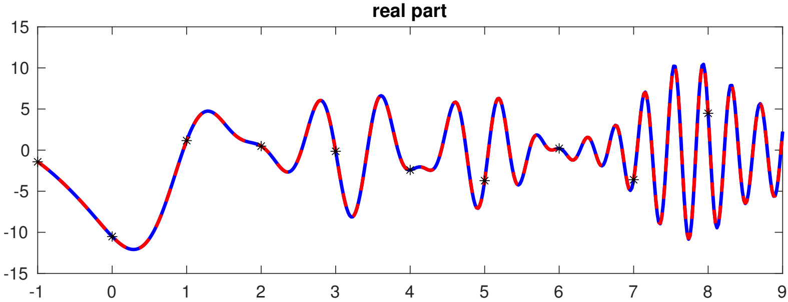
$g(x)$	$G(x)$	eigenfunctions	sampling values
$1/x$	$-\frac{1}{2}x^2$	$\exp\left(-\frac{\alpha}{2}x^2\right)$	$f\left(\sqrt{x_0 - k\tau}\right)$
1	x	$\exp(\alpha x)$	$f\left(\tau k + x_0\right)$
x	$\log(x)$	x^α	$f\left(e^{\tau k} x_0\right)$
$-\sqrt{1-x^2}$	$\arccos x$	$\exp(\alpha \arccos x)$	$f\left(\cos(k\tau + \arccos(x_0))\right)$
$\sqrt{1-x^2}$	$\arcsin x$	$\exp(\alpha \arcsin x)$	$f\left(\sin(k\tau + \arcsin(x_0))\right)$
$\frac{1}{\cos(x)}$	$\sin x$	$\exp(\alpha \sin x)$	$f\left(\arcsin(k\tau + \sin(x_0))\right)$
$-\frac{1}{\sin(x)}$	$\cos x$	$\exp(\alpha \cos x)$	$f\left(\arccos(k\tau + \cos(x_0))\right)$

Examples of operators $A = g(\cdot)\frac{\mathrm{d}}{\mathrm{d}x}$, corresponding eigenfunctions $\exp(\alpha\, G(\cdot))$ and sampling values for $k = 0, \ldots, 2M-1$ with sampling parameter τ to recover expansions f.

Example: Recovery of shifts of Gaussians (Plonka, Stampfer, Keller '19)

$$f(x) = \sum_{j=1}^{5} c_j \, \mathrm{e}^{\mathrm{i}(x-\alpha_j)^2}$$

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$\mathrm{Re}\, c_j$	-2.37854	-4.55545	2.54933	-2.57214	-0.57597
$\mathrm{Im}\, c_j$	0.75118	-0.56308	0.94536	0.42117	0.73366
α_j	0.64103	-0.18125	-1.50929	-0.53137	-0.23778



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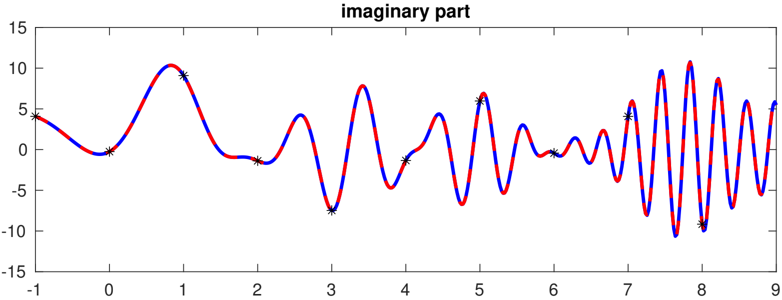

Summary

- Prony's method can be used in many contexts, since sparse representations can be often transformed to the form of exponential sums, e.g.,

$$g(t) = \sum_{j=1}^{M} c_j \, \phi(t - t_j) \quad \Rightarrow \quad \widehat{g}(x) = \Big(\sum_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i} t_j x} \Big) \widehat{\phi}(x)$$

- The underlying recovery problem is ill-posed.
- For noisy samples, one should use the modified Prony method.
- Prony's method can be generalized to recover sparse expansions of eigenfunctions of linear operators.
- One can use different operators with the same (sub)set of eigenfunctions.
- One can employ the generalized Prony method to find new more general sampling schemes.

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