Spline Wavelets with Higher Defect

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Abstract. In this paper a generalized multiresolution analysis, generated by cardinal B-splines of degree $m$ and defect $r$, is considered. Using the cardinal Hermite fundamental splines of degree $2m+1$ and defect $r$ new spline wavelets with defect $r$ are represented. In contrast with other papers dealing with wavelets with higher defect (cf. [3, 4]) the two-scale symbol $Q_m^r$ of the wavelet vector can explicitly be given.

§1. Introduction

The subject of this paper is a natural generalization of the concept of interpolatory spline wavelets introduced in [2, pp. 177]. Let $\{V_j^m\} (j \in \mathbb{Z})$ be the multiresolution analysis of $L_2(\mathbb{R})$ generated by the cardinal B-spline $N_m$ of degree $m$. Further, with $\{W_j^m\} (j \in \mathbb{Z})$ we denote the sequence of wavelet spaces, in the sense that

$$V_{j+1}^m = V_j^m \oplus W_j^m,$$

where $\oplus$ indicates the orthogonal summation.

Let $T := \{z \in \mathbb{C}, |z| = 1\}$. With the help of the Euler–Frobenius polynomial of degree $2m+1$

$$\Phi_{2m+1}^1(z) := \sum_{l=\infty}^{\infty} N_{2m+1}(l)z^l \quad (z \in T)$$

we introduce the cardinal fundamental spline $L_{2m+1}$ of degree $2m+1$

$$L_{2m+1} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{N}_{2m+1}(u)}{\Phi_{2m+1}^1(e^{-2\pi i u})} e^{-i u \cdot \cdot} du$$

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satisfying

\[ L_{2m+1}(n) = \delta_0n \quad (n \in \mathbb{Z}) \]

with the Kronecker symbol \( \delta \). For \( m \in \mathbb{N} \) the interpolatory wavelet \( \psi_{I,m} \) is defined by

\[ \psi_{I,m} := D^{m+1}L_{2m+1}(2 \cdot -1), \]

where \( D \) denotes the differential operator. Then \( \psi_{I,m} \) generates the wavelet spaces \( W^m_j \) (\( j \in \mathbb{Z} \))

\[ W^m_j := \text{clos}_{L_2}(\text{span}\{\psi_{I,m}(2^j \cdot l); l \in \mathbb{Z}\}) \]

(cf. [2], p. 178).

We want to generalize this concept in the following way.
Let \( m \in \mathbb{N}_0 \) and \( r \in \mathbb{N} \) be given integers. We consider equidistant knots of multiplicity \( r \)

\[ x_r^i := \left\lfloor \frac{i}{r} \right\rfloor, \quad (1.1) \]

where \( \lfloor x \rfloor \) means the integer part of \( x \in \mathbb{R} \).

Let \( N^{m,r}_k \in C^{m-r}(\mathbb{R}) \) \((r \leq m, k \in \mathbb{Z})\) denote the normalized B-splines of degree \( m \) and defect \( r \) with the knots \( x_k, \ldots, x_{k+m+1} \). For \( r = m + 1 \)
the B-splines \( N^{m,m+1}_k \) \((k = 0, \ldots, m)\) coincide with the well-known Bernstein polynomials. According to the distribution theory, let \( N^{m,r}_k \) be defined for \( r > m + 1 \) and \( k = 0, \ldots, r - m - 2 \) as follows

\[ N^{m,r}_k := \frac{1}{r - 1 - k} D^{r-m-2-k} \delta_k \quad (1.2) \]

where \( \delta \) denotes the Dirac distribution.

Using the ideas in [3, 4] in Section 2, we shall consider the generalized multiresolution analysis \( \{V^{m,r}_j\} \) \((j \in \mathbb{Z})\) of multiplicity \( r \) of \( L_2(\mathbb{R}) \) generated by the linearly independent scaling functions \( N^{m,r}_k \) \((k = 0, \ldots, r - 1)\), that is

\[ V^{m,r}_j := \text{clos}_{L_2}(\text{span}\{N^{m,r}_k(2^j \cdot l); k = 0, \ldots, r - 1\}). \quad (1.3) \]

In particular, an explicit formula for the two-scale symbol \( \mathbf{P}^r_m \) of the B-spline vector \( \mathbf{N}^r_m := (N^{m,r}_k)_{k=0}^{r-1} \) can be given (cf. [7]).

Let \( \{W^{m,r}_j\} \) \((j \in \mathbb{Z})\) denote the sequence of wavelet spaces determined by

\[ V^{m,r}_{j+1} = V^{m,r}_j \oplus W^{m,r}_j. \]

In Section 3 we shall introduce the cardinal Hermite fundamental splines \( L^{2m+1,r}_k \) \((k = 0, \ldots, r - 1)\) satisfying for \( n \in \mathbb{Z} \) the interpolation conditions

\[ D^{\nu}L^{2m+1,r}_k(n) = \delta_{0n} \delta_{nk} \quad (\nu, k = 0, \ldots, r - 1). \quad (1.4) \]
We put
\[ \psi_k^{m,r} := \mathbb{D}^{m+1} L_k^{2m+1,r} (2 \cdot -1) \quad (k = 0, \ldots, r - 1). \]

Contrary to [3, 4] we can firstly give an explicit formula for the two-scale symbol \( Q_m^r \) of the wavelet vector \( \Psi_m^r := (\psi_k^{m,r})_{k=0}^{r-1} \). This two-scale symbol \( Q_m^r \) can be used for the computation of the wavelets \( \psi_k^{m,r} \ (k = 0, \ldots, r - 1) \) with defect \( r \) as well as for deriving the Riesz basis property in the wavelet space \( W_0^{m,r} \). For \( r \geq m + 1 \) the wavelets \( \psi_k^{m,r} \ (k = 0, \ldots, r - 1) \) are compactly supported, for \( r \leq m \) they have exponential decay.

Note that in [3] other wavelets are constructed, which are derived from special compactly supported splines, firstly introduced in [9].

In Section 4 we show the close connection between the wavelet space \( W_0^{m,r} \) and the subspace \( V_{1,0}^{2m+1,r} \subset V_1^{2m+1,r} \), which contains splines with degree \( 2m+1 \) and defect \( r \) satisfying some interpolation conditions. Analogous assertions for the simple case \( r = 1 \) can be found in [2].

Finally, in Section 5 the obtained formulas are applied to the case of cubic spline wavelets \( (m = 3) \) with defect \( r = 2 \).

\section{Multiresolution Analysis of Multiplicity \( r \)}

For a summary of basic properties of B-splines with multiple knots we refer to [1, 7]. Here we recall only the following important relations.

Let \( \hat{N}_m^r := (\hat{N}_k^{m,r})_{k=0}^{r-1} \) be the vector of Fourier transformed B-splines
\[ \hat{N}_k^{m,r} := \int_{-\infty}^{\infty} N_k^{m,r}(x) e^{-ixr} \, dx. \]

For the Fourier transformed B-spline vector \( \hat{N}_m^r \) of length \( r > m + 1 \geq 1 \) we find by (1.2)
\[ \hat{N}_m^r(u) = \left( \frac{(iu)^{r-m-2}}{r-1}, \ldots, \frac{(iu)^{0}}{m+1}, \hat{N}_m^{m+1} (u)^T \right)^T, \]
where \( \hat{N}_m^{m+1} \) denotes the vector of the \( m + 1 \) Bernstein polynomials of degree \( m \). Further, we put
\[ \hat{N}_{m-1}^r(u) := \left( \frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^{1}}{1}, 1 \right)^T \quad (u \in \mathbb{R}). \quad (2.1) \]

Then for \( m \in \mathbb{N}_0 \) and \( r \in \mathbb{N} \) the following recursion relation can be found:
\[ (iu) \hat{N}_m^r(u) = A_m^r (e^{-iu}) \hat{N}_{m-1}^r(u) \quad (u \in \mathbb{R}). \quad (2.2) \]
The \((r, r)\)-matrices \( A_m^r(z) \ (z \in \mathbb{T}) \) are defined for \( m > r - 1 \) by
where $x_m^r$ are given in (1.1). For $m = r-1 > 0$ let

$$A_m^{m+1}(z) := m \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -1 \\
-z & 0 & \cdots & 0 & 1
\end{pmatrix}, \quad (2.4)$$

and for $0 \leq m < r-1$

$$A_m^r(z) := \left( \begin{array}{cc}
I_{r-m+1} & 0 \\
0 & A_m^{m+1}(z)
\end{array} \right), \quad (2.5)$$

where $A_0(z) := 1 - z$. Further, $I_{r-m+1}$ denotes the $(r-m+1)$th unit matrix and $0$ a zero matrix (cf. [7]).

Note that $\det A_m^r(z) = c_m^r (1 - z)$. The Fourier transformed two-scale relation of $N_m^r$ is given by

$$\hat{N}_m^r = P_m^r(e^{-i\pi/2}) \hat{N}_m^r(e^{i\pi/2}) \quad (m \in \mathbb{N}_0, r \in \mathbb{N}).$$

The two-scale symbol (or refinement mask) of $N_m^r$

$$P_m^r(z) := \frac{1}{2} \sum_{n=-\infty}^{\infty} P_n z^n \quad (z \in \mathcal{T}) \quad (2.6)$$

is a finite sum and satisfies for $m \geq 0$ the following recursion formula

$$P_m^r(z) = \frac{1}{2} A_m^r(z^2) P_{m-1}^r(z) A_m^r(z)^{-1} \quad (z \in \mathcal{T}, z \neq 1)$$

$$P_m^r(1) = \frac{1}{2} \lim_{u \to 0} A_m^r(e^{-2iu}) P_{m-1}^r(e^{-iu}) A_m^r(e^{-iu})^{-1} \quad (u \in \mathbb{R}) \quad (2.7)$$

with $A_m^r(z)$ defined in (2.3) - (2.5) and

$$P_{m+1}^r(z) := \text{diag}(2^{r-1}, \ldots, 2^0)^T \quad (2.8)$$

(cf. [7]). In particular, the two-scale symbol $P_m^r(z)$ is a matrix polynomial in $z$ with

$$\det P_m^r(z) = 2^{-rm+r(r-3)/2} (1 + z)^{m+1} \quad (z \in \mathcal{T}),$$
The functions $N_{m,r}^{m,r}(\cdot - l)$ form a Riesz basis (or $L_2(\mathbb{R})$-stable basis) of $V_0^{m,r}$ (cf. [1]). The Riesz basis property is equivalent to the assertion that the autocorrelation symbol $\Phi_m^r$, defined by
\begin{equation}
\Phi_m^r(e^{-in}) := \sum_{n=\infty}^{\infty} \tilde{\Phi}_m^r(u + 2\pi n) \tilde{\Phi}_m^r(u + 2\pi n)^* \tag{2.9}
\end{equation}

with $\tilde{\Phi}_m^r(u)^* := \tilde{\Phi}_m^r(u)^T$ is positive definite (cf. [4, 5, 7]).

Further, we introduce the following Euler–Frobenius matrix
\begin{equation}
H_{2m+1}^r := (H_k^r)_{k=0}^{r-1} \tag{2.10}
\end{equation}

with
\begin{equation}
H_k^r(z) := \sum_{l=-\infty}^{\infty} D^\nu \chi^2_{k+1,r}(l) z^l \quad (k, \nu = 0, 1, \ldots, r - 1, \ z \in \mathcal{T}). \tag{2.11}
\end{equation}

For $2m + 1 - \nu \leq r - 1$, the functions $D^\nu \chi^2_{k+1,r}$ are understood according to the distribution theory. For $r = 1$ we obtain the well-known Euler–Frobenius polynomial
\begin{equation}
H^1_{2m+1}(z) = H^0_{2m+1}(z) = \sum_{l=-\infty}^{\infty} N_{2m+1}(l) z^l.
\end{equation}

By the Poisson summation formula the matrix $H_{2m+1}^r$ reads for $z = e^{-in}$ as follows
\begin{equation}
H_{2m+1}^r(e^{-in}) = \sum_{l=-\infty}^{\infty} \left((i(u + 2\pi l))^k\right)^{r-1} \tilde{\Phi}_m^r(u + 2\pi l)^T \quad (u \in \mathbb{R}). \tag{2.12}
\end{equation}

For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ we have the following relationship:
\begin{align}
\Phi_m^r(z) &= D_{m,0}^r(z) D_{r} H_{2m+1}^r(z) (D_{m,1}^r(z)^*)^{-1} \quad (z \in \mathcal{T}, \ z \neq 1), \\
\Phi_m^r(1) &= \lim_{u \to 0} D_{m,0}^r(e^{-iu}) D_{r} H_{2m+1}^r(e^{iu}) (D_{m,1}^r(e^{-iu})^*)^{-1} \quad (u \in \mathbb{R}) \tag{2.13}
\end{align}

with
\begin{align}
D_{m,0}^r(z) &= A_0^r(z) A_{m-1}^r(z) \ldots A_0^r(z), \\
D_{m,1}^r(z) &= A_{2m+1}^r(z) A_{m}^r(z) \ldots A_{m+1}^r(z), \tag{2.14}
\end{align}

where the $(r,r)$-matrices $A_k^r (k = 0, \ldots, 2m + 1)$ are defined in (2.3) – (2.5),
\begin{equation}
D_r := (-1)^{m+1} \begin{pmatrix}
0 & 0 & \cdots & 0 & (\frac{(-1)^{r-1}}{r}) \\
0 & 0 & \cdots & (\frac{(-1)^{r-2}}{r-2}) & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (-1)^1 & 0 \\
(-1)^0 & 0 & \cdots & 0 & 0
\end{pmatrix} \quad (r > 1)
\end{equation}
and \( D_1 := (-1)^{m+1} \). In particular, the invertibility of the autocorrelation matrix \( \Phi_m(z) \) for \( z \in \mathcal{T} \) causes the invertibility of the Euler–Frobenius matrix \( \mathbf{H}_{2m+1}^r(z) \) for \( z \in \mathcal{T} \) (cf. [7]).

Since \( V_{m,r}^j \) contains the space \( V_{m,1}^j \) generated by the cardinal B-spline \( N_m \) (cf. [2]), it follows that

\[
\text{clos}_{L_2} \bigcup_{j=-\infty}^{\infty} V_{j,m,r}^j = L_2(\mathbb{R}).
\]

The Riesz basis property and the partition of unity property for the B-splines \( N_{k,m,r}^j(\cdot - l) \) \( (k = 0, \ldots, r-1, \ l \in \mathbb{Z}) \) also lead to

\[
\bigcap_{j=-\infty}^{\infty} V_{j,m,r}^j = \{0\}.
\]

Thus, the sequence \( \{V_{j,m,r}^j\} \ (j \in \mathbb{Z}) \) generates a multiresolution analysis of \( L_2(\mathbb{R}) \) with multiplicity \( r \) (cf. [3,4]).

\[\text{§3. The Wavelet Space } W_{j,m,r}^r\]

In this section we want to find spline wavelets with defect \( r \)

\[
\psi_{k,m,r}^r \quad (k = 0, \ldots, r-1), \tag{3.1}
\]

such that the integer translates of (3.1) form a Riesz basis of the wavelet space \( W_{0,m,r}^r := V_{1,m,r}^j \cap V_{0,m,r}^j \).

First we want to introduce cardinal Hermite fundamental splines.

Let \( \hat{L}_{2m+1}^r := \left( \hat{L}_{k}^{2m+1,r} \right)_{k=0}^{r-1} \) be the Fourier transformed vector of spline functions \( L_{k}^{2m+1,r} \) \( (k = 0, \ldots, r-1) \) defined by

\[
\hat{L}_{2m+1}^r(u) := \left( \mathbf{H}_{2m+1}^r(e^{-iu})^T \right)_{k=0}^{r-1} \hat{N}_{2m+1}^r(u), \tag{3.2}
\]

and \( \mathbf{L}_{2m+1}^r := (\hat{L}_{k}^{m,r})_{k=0}^{r-1} \). Then we have:

\[\textbf{Theorem 3.1.} \text{ The spline functions } L_{k}^{2m+1,r} \text{ (} k = 0, \ldots, r-1 \text{) are cardinal Hermite fundamental splines in } V_{0}^{2m+1,r}, \ \text{i.e., for } n \in \mathbb{Z} \text{ the interpolation conditions}
\]

\[
D^\nu L_{k}^{2m+1,r}(n) = \delta_{0n} \delta_{nk} \quad (\nu, k = 0, \ldots, r-1) \tag{3.3}
\]

are satisfied.

\[\textbf{Proof:} \text{ By } \mathcal{W} \text{ we define the Wiener class. Since } \text{det} \mathbf{H}_{2m+1}^r(z) \in \mathcal{W}, \text{ it follows that there exists a representation}
\]

\[
\left[ \mathbf{H}_{2m+1}^r(e^{-iu}) \right]^{-1} = \sum_{n=-\infty}^{\infty} \mathbf{H}_n e^{-inu},
\]
where the elements of the \((r, r)\)-matrices \(H_n (n \in \mathbb{Z})\) lie in \(l_1\). Thus, (3.2) implies that the functions \(L_k^{2m+1, r} (k = 0, \ldots, r-1)\) are contained in \(V_0^{2m+1, r}\).

It remains to show that the interpolation conditions (3.3) hold. Putting

\[
[D^\nu \mathbf{L}_{2m+1}^r]^\sim := \sum_{n=-\infty}^{\infty} [D^\nu \mathbf{L}_{2m+1}^r](\cdot + 2\pi n)
\]

it follows by the Poisson summation formula that

\[
[D^\nu \mathbf{L}_{2m+1}^r]^\sim = \sum_{l=-\infty}^{\infty} D^\nu \mathbf{L}_{2m+1}^r(l) e^{-il}\cdot
\]

Therefore we have to show that \([D^\nu \mathbf{N}_{2m+1}^r]^\sim = \mathbf{e}_\nu\), where \(\mathbf{e}_\nu := (\delta_{\nu, k})_{k=0}^{r-1}\) are the unit vectors. By

\[
[D^\nu \mathbf{N}_{2m+1}^r]^\sim = \sum_{l=-\infty}^{\infty} D^\nu \mathbf{N}_{2m+1}^r(l) e^{-il}\cdot = (H_k^r)_{k=0}^{r-1}\]

the relation (3.2) leads to

\[
([D^0 \mathbf{L}_{2m+1}^r]^\sim(u), \ldots, [D^{r-1} \mathbf{L}_{2m+1}^r]^\sim(u)) = [H_{2m+1}^r(e^{-iu})^T]^{-1} ((H_k^0(e^{-iu}))_{k=0}^{r-1}, \ldots, (H_k^{r-1}(e^{-iu}))_{k=0}^{r-1})
\]

\[
= [H_{2m+1}^r(e^{-iu})^T]^{-1} H_{2m+1}^r(e^{-iu})^T
\]

\[
= I. \quad \blacksquare
\]

Now let

\[
\psi_k^{m, r} := D^{m+1} \mathbf{L}_k^{2m+1, r} (2 \cdot -1) \quad (k = 0, \ldots, r-1) \quad (3.4)
\]

and \(\Psi_m^r := (\psi_k^{m, r})_{k=0}^{r-1}\). We shall show that the spline wavelets \(\psi_k^{m, r} (k = 0, \ldots, r-1)\) and their integer translates form a Riesz basis of \(W_0^{m, r}\).

Using the relations (2.2) and (3.2) we obtain for the vector \(\hat{\Psi}_m^r := (\psi_k^{m, r})_{k=0}^{r-1}\) of Fourier transformed wavelets

\[
\hat{\Psi}_m^r(u) = [D^{m+1} \mathbf{L}_{2m+1}^r(2 \cdot -1)]^\sim(u)
\]

\[
= 1/2 \left((iu/2)^{m+1} e^{-iu/2} \mathbf{L}_{2m+1}^r(u/2) \right)
\]

\[
= 1/2 e^{-iu/2} [H_{2m+1}^r(e^{-iu/2})]^{-1} D_{m,1}^r(e^{-iu/2}) \hat{\mathbf{N}}_m^r(u/2)
\]

with \(D_{m,1}^r\) defined in (2.14). Thus, we have the two-scale relation

\[
\hat{\Psi}_m^r = \mathbf{Q}_m^r(e^{-iu/2}) \hat{\mathbf{N}}_m^r(u/2) \quad (3.5)
\]
with the two-scale symbol of $\Psi^r_m$

$$Q^r_m(z) := z/2[H^r_{2m+1}(z)^T]^{-1}D^r_{m,1}(z) \quad (z \in T).$$  \hspace{1cm} (3.6)

Observe that the elements of the matrix $Q^r_m$ belong to the Wiener class. The two-scale relation (3.5) implies that the functions $\psi^{m,r}_k$ ($k = 0, \ldots, r-1$) lie in $V_1^{m,r}$. The functions $\psi^{m,r}_k$ belong to $W_0^{m,r}$ if and only if for $k, \nu = 0, \ldots, r-1$ and $l \in \mathbb{Z}$,

$$\langle N_k^{m,r}(\cdot - l), \psi^{m,r}_\nu \rangle := \int_{-\infty}^{\infty} N_k^{m,r}(x - l)\psi^{m,r}_\nu(x)\, dx = 0,$$

i.e., if and only if the condition

$$P^r_m(z)\Phi^r_m(z)Q^r_m(z)^* + P^r_m(-z)\Phi^r_m(-z)Q^r_m(-z)^* = 0 \quad (z \in T)$$  \hspace{1cm} (3.7)

is satisfied (cf. [4]).

**Theorem 3.2.** The functions $\psi^{m,r}_k$ ($k = 0, \ldots, r-1$) belong to $W_0^{m,r}$.

**Proof:** From the recursion relation (2.7) it follows with (2.8) and (2.14)

$$P^r_m(z) = \frac{1}{2m+1} D^r_{m,0}(z^2)P^r_{-1}D^r_{m,0}(z)^{-1} \quad (z \in T),$$  \hspace{1cm} (3.8)

where for $z = 1$, (3.8) is understood according to (2.7). Using the relations (2.13) and (3.6) we obtain

$$P^r_m(z)\Phi^r_m(z)Q^r_m(z)^* = \frac{z}{2m+2} D^r_{m,0}(z^2)P^r_{-1}D_r,$$

where $P^r_{-1}$ and $D_r$ do not depend on $z$. Thus, the relation (3.7) is satisfied. \quad \blacksquare

We can even prove:

**Theorem 3.3.** The functions $\psi^{m,r}_k$ ($k = 0, \ldots, r-1$) form a Riesz basis of $W_0$, i.e., there exist Riesz bounds $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} |c^k_l|^2 \leq \| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c^k_l \psi^{m,r}_k(\cdot - l) \|^2_{L^2} \leq \beta \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} |c^k_l|^2$$

for any sequences $(c^k_l)_{l=-\infty}^{\infty} \in l_2$ ($k = 0, \ldots, r-1$).

**Proof:** For $(c^k_l)_{l=-\infty}^{\infty} \in l_2$ ($k = 0, \ldots, r-1$) let $C_k$ denote their $2\pi$-periodic symbols,

$$C_k := \sum_{l=-\infty}^{\infty} c^k_l e^{-in\cdot} \quad (k = 0, \ldots, r-1).$$
Put $C := (C_0, \ldots, C_{r-1})^T$. Then by the Parseval identity we find
\[
\left\| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_k \psi_k^{r}(\cdot - l) \right\|^2_{L_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |C(u)^T \Psi_m^r(u)|^2 \, du
\]
\[
= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} C(u)^T \Psi_m^r(u + 2\pi l) \Psi_m^r(u + 2\pi l)^* \overline{C(u)} \, du.
\]
Using the two-scale relation it follows
\[
\left\| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_k \psi_k^{r}(\cdot - l) \right\|^2_{L_2}
\]
\[
= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} C(u)^T Q_m^r(e^{-iu/2 + \pi l}) \Phi_m^r(u/2 + \pi l) \Phi_m^r(u/2 + \pi l)^* \overline{C(u)} \, du
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} C(u)^T \left( Q_m^r(e^{-iu/2}) \Phi_m^r(e^{-iu/2}) Q_m^r(e^{-iu/2})^* \right) \overline{C(u)} \, du
\]
\[
+ \frac{1}{2\pi} \int_0^{2\pi} C(u)^T \left( Q_m^r(-e^{-iu/2}) \Phi_m^r(-e^{-iu/2}) Q_m^r(-e^{-iu/2})^* \right) \overline{C(u)} \, du.
\]
(3.9)

Recall that $\Phi_m^r(z)$ is Hermitian and positive definite for $z \in T$. Thus, the matrix $Q_m^r(z) \Phi_m^r(z) Q_m^r(z)^*$ is Hermitian and positive definite for $z \in T$, $z \neq 1$ and positive semidefinite for $z = 1$. It follows that for $\|C\|^2_{L_2} := \sum_{k=0}^{r-1} \|C_k\|^2_{L_2} > 0$ the terms in (3.9) are nonnegative for all $z \in T$ and at least one term in (3.9) is positive. By the definitions of $Q_m^r$ and $\Phi_m^r$, the terms in (3.9) are bounded for fixed $m$ and $r$. ■

§4. The Spline Space $V_{1,0}^{2m+1,r}$

In view of the preceding result let us consider the subspace
\[
V_{1,0}^{2m+1,r} := \{ s \in V_1^{2m+1,r} : D^\nu s(n) = 0, \nu = 0, \ldots, r-1; n \in \mathbb{Z} \}
\]
of cardinal splines of degree $2m+1$ and defect $r$ with the knot sequence $2^{-1}\mathbb{Z}$. Then the functions
\[
\Lambda_k^{2m+1,r} := L_k^{2m+1,r}(2 \cdot -1)
\]
belong to $V_{1,0}^{2m+1,r}$ and we have:

**Theorem 4.1.** For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ the functions
\[
\Lambda_k^{2m+1,r}(\cdot - l) \quad (k = 0, \ldots, r-1; l \in \mathbb{Z})
\]
(4.1)
form a Riesz basis of $V_{1,0}^{2m+1,r}$.

**Proof:** The cardinal Hermite fundamental splines $L_k^{m+1,r}(2 \cdot -l) \ (k = 0, \ldots, r - 1; \ l \in \mathbb{Z})$ form a basis of $V_{1}^{2m+1,r}$, i.e., an arbitrarily chosen element $G \in V_{1,0}^{2m+1,r} \subset V_{1}^{2m+1,r}$ can be uniquely represented in the form

$$G = \sum_{k=0}^{r-1} \sum_{l=-\infty}^{\infty} a_k^{l} L_k^{2m+1,r}(2 \cdot -l).$$

The conditions $D^\nu G(n) = 0 \ (\nu = 0, \ldots, r - 1; \ n \in \mathbb{Z})$ imply that $a_{2l}^{k} = 0 \ (k = 0, \ldots, r - 1; \ l \in \mathbb{Z})$, i.e.,

$$G = \sum_{k=0}^{r-1} \sum_{l=-\infty}^{\infty} b_k^{l} L_k^{2m+1,r}(\cdot - l)$$

with $b_k^{l} := a_{2l+1}^{k} \ (k = 0, \ldots, r - 1; \ l \in \mathbb{Z})$. Thus, the functions in (4.1) form a basis of $V_{1,0}^{2m+1,r}$. To show the Riesz basis property we note that

$$L_k^{2m+1,r} := (L_k^{m+1,r})^{r-1} = R_{2m+1}^{r} (e^{-r/2}) \tilde{N}_{2m+1}^{r} (r/2)$$

with $R_{2m+1}^{r}(z) := z (H_{2m+1}^{r}(z)^\top)^{-1} (z \in \mathcal{T})$. Following the ideas in the proof of Theorem 3.3, we only have to consider the matrices

$$R_{2m+1}^{r}(z) \Phi_{2m+1}^{r}(z) R_{2m+1}^{r}(z)^* \ (z \in \mathcal{T}),$$

where $\Phi_{2m+1}^{r}(z)$ denotes the autocorrelation symbol for $\tilde{N}_{2m+1}^{r}$.

Since $H_{2m+1}^{r}(z)$ is invertible and $\Phi_{2m+1}^{r}(z)$ is positive definite for $z \in \mathcal{T}$, it follows that $R_{2m+1}^{r}(z) \Phi_{2m+1}^{r}(z) R_{2m+1}^{r}(z)^*$ is positive definite for $z \in \mathcal{T}$. Thus, the Riesz basis property is satisfied.

As a consequence of Theorems 4.1 and 3.3 we have the following result (cf. [2], p. 190 for $r = 1$).

**Theorem 4.2.** For $m \in \mathbb{N}$ and $r \in \mathbb{N}$ the $(m + 1)$-th order differential operator $D^{m+1}$ maps the spline space $V_{1,0}^{2m+1,r}$ one-to-one onto the wavelet space $W_{0}^{2m+1,r}$. Moreover, the Riesz basis $\{L_k^{2m+1,r}(\cdot - l) ; k = 0, \ldots, r - 1; \ l \in \mathbb{Z}\}$ of $V_{1,0}^{2m+1,r}$ corresponds to the Riesz basis $\{\psi_k^{m,r}(\cdot - l) ; k = 0, \ldots, r - 1; \ l \in \mathbb{Z}\}$ of $W_0^{m,r}$ via the relation $\psi_k^{m,r} = D^{m+1} L_k^{2m+1,r} (k = 0, \ldots, r - 1).$
§5. An Example

We want to apply the obtained formulas to the case \( m = 3, r = 2 \) of cubic spline wavelets with defect 2.

With \( \hat{N}_-^2 = (iu, 1)^T \) and

\[
\mathbf{D}_{3,0}^2(z) = \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \\
= 6 \begin{pmatrix}
1 & -1/2 \\
-1 & 1/2
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
0 & 1 - z
\end{pmatrix}
= 3 \begin{pmatrix}
2 + 4z & -5 + 4z + z^2 \\
-4z - 2z^2 & 1 + 4z - 5z^2
\end{pmatrix}
\]

it follows from (2.2) that

\[
\hat{N}_3(z) = \frac{3}{(iz)^4} \begin{pmatrix}
(2iu - 5) + 4(iu + 1)e^{-iu} + e^{-2iu} \\
1 + 4(-iu + 1)e^{-iu} + (-2iu - 5)e^{-2iu}
\end{pmatrix}.
\]

For the two-scale symbol satisfying \( \hat{N}_3^2 = \mathbf{P}_3^2(e^{-i/2}) \hat{N}_3^2(\cdot/2) \) we find with (2.8) and (3.8)

\[
\mathbf{P}_3^2(z) = \frac{1}{16} \mathbf{D}_{3,0}^2(z^2) \mathbf{P}_{-1}^2(z) \mathbf{D}_{3,0}^2(z)^{-1}
= \frac{1}{16} \begin{pmatrix}
2 + 6z + z^2 & 5 + 2z \\
2z + 5z^2 & 1 + 6z + 2z^2
\end{pmatrix}.
\]

The autocorrelation symbol reads

\[
\Phi_3(z) = \frac{1}{560} \begin{pmatrix}
9z^{-1} + 128 + 9z & 53z^{-1} + 80 + z \\
z^{-1} + 80 + 53z & 9z^{-1} + 128 + 9z
\end{pmatrix}.
\]

The Euler-Frobenius matrix \( \mathbf{H}_t^2 \) is given by

\[
\mathbf{H}_t^2(z) = \frac{1}{432} \begin{pmatrix}
37z + 176z^2 + 3z^3 & 3z + 176z^2 + 37z^3 \\
175z - 224z^2 - 21z^3 & 21z + 224z^2 - 175z^3
\end{pmatrix},
\]

such that (2.13) can simply be verified with

\[
\mathbf{D}_{3,1}^2(z) = \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \mathbf{A}_{3}^2(z) \\
= \frac{35}{9} \begin{pmatrix}
6 + 26z + 3z^2 & -17 - 18z \\
-18z - 17z^2 & 3 + 26z + 6z^2
\end{pmatrix}.
\]

The matrix \( \mathbf{H}_t^2(z) \) is invertible on the unit circle \( z \in \mathcal{T} \) and we have

\[
\mathbf{H}_t^2(z)^{-1} = \frac{12}{7z \Delta_t^4(z)} \begin{pmatrix}
21 + 224z - 175z^2 & -3 - 176z - 37z^2 \\
-175 + 224z + 21z^2 & 37 + 176z + 3z^2
\end{pmatrix}
\]

with
\[ \Delta_7^2(z) = 1 - 72z + 262z^2 - 72z^3 + z^4. \]

Thus, the two-scale symbol \( Q_3^2 \) of the wavelet vector \( \Psi_3^2 \) is given by

\[
Q_3^2(z) = \frac{z}{2} (H_7^2(z)^T)^{-1} D_{8,4}^2(z) = \frac{60}{\Delta_7^2(z)} \begin{pmatrix}
7(1 + 40z + 30z^2) & 7(-7 - 64z + 30z^2 + 40z^3 + z^4) \\
-64z^3 - 7z^4 & 9 + 252z + 478z^2 + 100z^3 + z^4
\end{pmatrix}.
\]

References


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