Reconstruction of Stationary and Non-stationary Signals by the Generalized Prony Method

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Abstract. We employ the generalized Prony method to derive new reconstruction schemes for a variety of sparse signal models using only a small number of signal measurements. Introducing generalized shift operators, we study the recovery of sparse trigonometric and hyperbolic functions as well as sparse expansions of shifted Gaussians and Gabor functions with Gaussian windows. Furthermore, we show how to reconstruct sparse polynomial expansions and sparse non-stationary signals with structured phase functions.

Key words: generalized Prony method, exponential sums, trigonometric sums, Gabor expansions, shifted Gaussians, non-stationary signals, signal reconstruction, empirical mode decomposition.

Mathematics Subject Classification: 15A18, 39A10, 41A30, 45Q05, 65F15.

1 Introduction

The recovery of signals possessing a given structure is an important problem in various applications as e.g. wireless telecommunication [21], image super-resolution [23], nondestructive testing [5] and phase retrieval [3].

We assume that we can employ certain a priori knowledge about the underlying signal model, where a small number of parameters needs to be determined from given signal measurements in order to recover the structure of the signal. A prototype of such a signal is an exponential sum of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{x\alpha_j} \tag{1.1}$$

with unknown complex parameters c_j and α_j, $j = 1, \ldots, M$, which need to be recovered from measurement values of f. Here and in all other signal models we always assume that the parameters α_j are pairwise different and that all coefficients c_j are

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nonzero, otherwise, the exponential sum can be suitably simplified to a sum with less terms.

It is well-known, that this recovery problem can be solved by Prony's method using equidistant point evaluations $f(x_0 + kh)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ can be arbitrarily chosen, and $h \neq 0$ is a suitable step size. Indeed, the model (1.1) already covers many important applications. For example, taking $\alpha_j = -\mathrm{i}t_j$ with $t_j \in \mathbb{R}$, (1.1) is closely related to the model

$$g(t) = \sum_{j=1}^{M} c_j \, \delta(t - t_j), \tag{1.2}$$

a finite stream of Diracs, since the Fourier transform $\widehat{g}(x) := \int_{-\infty}^{\infty} g(t) \, \mathrm{e}^{-\mathrm{i}tx} \mathrm{d}t$ of g is equal to f in (1.1) . Therefore, g can be recovered from equidistant Fourier samples. Signals as given in (1.2) are said to have finite rate if innovation, [22]. More generally, considering a finite linear combination of arbitrary shifts of a given function ϕ,

$$g(t) = \sum_{j=1}^{M} c_j \, \phi(t - t_j) \tag{1.3}$$

with $t_j \in \mathbb{R}$ leads by Fourier transform to a product of an exponential sum $\sum\limits_{j=1}^{M} c_j \, \mathrm{e}^{-\mathrm{i}t_j x}$ of the form (1.1) with the Fourier transform $\widehat{\phi}(x)$, and can therefore also be recovered from suitable equidistant Fourier samples of g, see [9,17,19,22]. Applying the Laplace transform to (1.1) with the restriction $\alpha_j \in \mathbb{R}$, we find

$$h(s) = \mathcal{L}f(s) = \sum_{j=1}^{M} \frac{c_j}{s - \alpha_j}.$$

Therefore, the rational function $h(s)$ can be reconstructed from equidistant samples of its inverse Laplace transform using Prony's method. Also, sparse expansions of shifted Lorentzian functions with the Fourier transform $\mathrm{e}^{-\mathrm{i}t_j x - \alpha_j |x|/2}$, where t_j and α_j denote the shifts and the function width, can be recovered using the model (1.1), see [2].

However, many applications require more general signal models. In this paper we regard a signal as *stationary* if it can be e.g. written in the form

$$f(x) = \sum_{j=1}^{M} c_j(x) \, \mathrm{e}^{\mathrm{i}\phi_j(x)} \qquad \text{or} \qquad \sum_{j=1}^{M} c_j(x) \, \cos(\phi_j(x))$$

where the *amplitudes* $c_j(x)$ are constants and the *phase functions* $\phi_j(x)$ are linear polynomials. Thus the exponential sum in (1.1) is stationary for $\alpha_j = \mathrm{i}t_j$, $t_j \in \mathbb{R}$. We say that the signal is *non-stationary*, if these assumptions are not longer satisfied. In Section 6, we will consider expansions with phase functions being quadratic polynomials or of the form $\phi_j(x) = \alpha_j x^p + \beta_j$ with $p \in \mathbb{R}_+$.

Non-stationary signals play an important role in many applications in signal processing, since signals naturally change their behavior in time. Stationary signal analysis methods are usually not well suited for decomposing these signals, and some effort has been put into deriving new techniques for non-stationary signal decomposition. In [11] the empirical mode decomposition (EMD) has been proposed, a greedy non-parametric method to decompose non-stationary signals into so-called intrinsic mode functions. For applications and further investigation of EMD we refer to [12] and the references therein. Mathematically improved techniques include the synchrosqueezed wavelet transform [10] and the signal decomposition method using a signal separation operator [6]. Other recent attempts for signal reconstruction are based on hybrid methods employing wavelet analysis and the finite rate of innovation approach, [23].

Having more a priori information about the signal structure in a sparse model, we aim at a direct identification of the important signal components. In [4], the reconstruction of piecewise sinusoidal signals has been studied with finite rate of innovation methods based on the model (1.1), but this model is still restricted to linear phase functions. Recently, the generalized Prony method has been proposed in [15]. This approach allows the recovery of sparse sums of eigenfunctions of linear operators. It covers the reconstruction of exponential sums in (1.1) as a special case, where the exponentials are interpreted as eigenfunctions of the shift operator.

In this paper we want to exploit the generalized Prony method introduced in [15] and derive reconstruction procedures for different signal models that go essentially beyond the exponential sum in (1.1). Here, we particularly restrict ourselves to models that can be recovered just from direct measurement values, i.e., point evaluations of the signal. In Section 2.3 we will indicate how other sampling schemes can be also used instead.

Employing generalized shift operators we will present new recovery methods for a variety of signal models as e.g. linear combinations of Gaussians, Gabor expansions with Gaussian windows and non-stationary trigonometric expansions. For each of these new models, we will present a reconstruction method and show, which signal measurements are needed for the recovery.

The paper is organized as follows. In Section 2 we recall the Prony method and present its interpretation as a recovery method for sparse sums of eigenfunctions of the shift operator. We also describe some sampling schemes to reconstruct the exponential sum which we are allowed to use by exploiting the generalized Prony method. In Subsection 2.3, we introduce generalizations of the shift operator and show their basic properties.

Sections 3 − 6 are devoted to the investigation of various signal models that are based on the generalized shift operators. In Section 3 we consider sparse expansions of trigonometric and hyperbolic functions that can be reconstructed using the symmetric shift operator. In Section 4 we study the recovery of expansions of shifted Gaussians and Gabor expansions with shifted Gaussian windows.

In Section 5 we reconstruct sparse expansions of monomials and complex Gaussians with different scaling. Moreover, sparse expansions of Chebyshev polynomials and linear combinations of non-stationary exponentials as e.g. $\mathrm{e}^{\alpha_j \cos}$ can be recovered.

Section 6 is devoted to the recovery of further non-stationary signals whose components possess non-linear phase functions of the form $\alpha_j x^p + \beta_j$ with known real parameter $p > 0$ and unknown parameters α_j, $\beta_j \in \mathbb{R}$, or with phase functions of the form $x^2 + \alpha_j x + \beta_j$.

Finally, we illustrate the signal models by some numerical examples in Section 7. Stability issues of the numerical methods will be more closely considered in a forthcoming paper.

2 Prony's Method for Exponential Sums Revisited

2.1 The Prony method

Let us first recall Prony's method to reconstruct $f(x)$ in (1.1) using equidistant samples of f. The function $f(x)$ in (1.1) can be interpreted as the solution of a linear difference equation with constant coefficients. This observation is the key of this recovering method. The main idea is to reconstruct the parameters e^{α_j} in a first step, and the coefficients c_j in a second step. If $\alpha_j \in \mathbb{C}$, as we have assumed here, we need to put attention to the fact that the α_j may not be uniquely determined by e^{α_j} since $\mathrm{e}^{\mathrm{i}x}$ is 2π-periodic.

We assume therefore that we have an a priori known bound $|\mathrm{Im}\,\alpha_j| < T$. We choose a sampling size $h < \frac{\pi}{T}$ such that $|\alpha_j h| < \pi$ and will reconstruct the values $\mathrm{e}^{\alpha_j h}$. Then α_j can be uniquely extracted.

We define the characteristic polynomial (Prony polynomial)

$$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{\alpha_j h}) = \prod_{j=1}^{M} (z - \lambda_j) \tag{2.1}$$

with $\lambda_j := \mathrm{e}^{\alpha_j h}$. Assuming that $P(z)$ has the monomial representation

$$P(z) = \sum_{k=0}^{M} p_k\, z^k = z^M + \sum_{k=0}^{M-1} p_k\, z^k,$$

we consider the homogeneous linear difference equation of order M for f in (1.1),

$$\begin{aligned} \sum_{k=0}^{M} p_k\, f(h(k+m)) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j\, \lambda_j^{(k+m)} = \sum_{j=1}^{M} c_j \lambda_j^m \Big(\sum_{k=0}^{M} p_k\, \lambda_j^k \Big) \\ &= \sum_{j=1}^{M} c_j \lambda_j^m\, P(\lambda_j) = 0, \end{aligned}$$

which is satisfied for all $m \in \mathbb{Z}$. Exploiting $p_M = 1$ we derive the linear system

$$\sum_{k=0}^{M-1} p_k\, f(h(k+m)) = -f(h(M+m)), \qquad m \in \mathbb{Z} \tag{2.2}$$

from the given function values $f(hl)$, $l = 0, \ldots, 2M-1$. The coefficient matrix $\mathbf{H} = (f(h(k+m)))_{k,m=0}^{M-1}$ in (2.2) has Hankel structure, and the linear system in (2.2) is uniquely solvable, provided that the values $\lambda_j = \mathrm{e}^{\alpha_j h}$ in (1.1) are pairwise different and $c_j \neq 0$ for $j = 1, \ldots, M$. This can be easily seen from the factorization

$$\mathbf{H} = \mathbf{V}\, \mathrm{diag}\,(c_1, \ldots, c_M)\, \mathbf{V}^T,$$

where $\mathbf{V}$ denotes the Vandermonde matrix $\mathbf{V} = (\lambda_j^k)_{k,j=0}^{M-1}$.

Having found the coefficients p_k of the Prony polynomial $P(z)$ by solving (2.2), we can extract the zeros $\lambda_j = \mathrm{e}^{\alpha_j h}$, $j = 1, \ldots, M$ and finally determine the parameters c_j in (1.1) by solving the (overdetermined) linear system

$$f(l) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{l\alpha_j} = \sum_{j=1}^{M} c_j\, \lambda_j^l, \qquad l = 0, \ldots, 2M-1.$$

The described method has used the function values $f(hl)$, $l = 0, \dots, 2M-1$. But a careful inspection of this approach shows that there is no reason to start with $f(0)$. We can equivalently start with an arbitrary value $x_0 \in \mathbb{R}$ and apply the samples $f(x_0 + lh)$, $l = 0, \dots, 2M-1$ to reconstruct (1.1). As before, we get

$$\begin{aligned}\sum_{k=0}^{M} p_k f(x_0 + h(k+m)) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \mathrm{e}^{x_0 \alpha_j} \lambda_j^{(k+m)} = \sum_{j=1}^{M} c_j \mathrm{e}^{x_0 \alpha_j} \lambda_j^m \Big(\sum_{k=0}^{M} p_k \lambda_j^k \Big) \\ &= \sum_{j=1}^{M} c_j \, \mathrm{e}^{x_0 \alpha_j} \, \lambda_j^m \, P(\lambda_j) = 0, \end{aligned} \tag{2.3}$$

which leads to a similar Hankel system as in (2.2). For a recent survey on Prony's method and its applications we refer to [18].

2.2 Revisiting Prony's Method Using the Shift Operator

Following the ideas in [15], we now want to look at the exponential sum in (1.1) from a different point of view, i.e. as an expansion of eigenfunctions of a suitably chosen operator.

We consider the usual shift operator $S_h \colon C(\mathbb{R}) \to C(\mathbb{R})$ acting on the vector space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$, given by

$$S_h f := f(\cdot + h), \qquad h \in \mathbb{R} \setminus \{0\}. \tag{2.1}$$

Then, for each $\alpha \in \mathbb{C}$ we have

$$(S_h \mathrm{e}^{\alpha \cdot})(x) = \mathrm{e}^{\alpha(h+x)} = \mathrm{e}^{\alpha h} \, \mathrm{e}^{\alpha x},$$

i.e., $\mathrm{e}^{\alpha x}$ is an eigenfunction of S_h with the eigenvalue $\mathrm{e}^{\alpha h}$. Therefore, we can interpret the signal $f(x)$ in (1.1) as a sparse linear combination of eigenfunctions of the shift operator S_h and obtain

$$S_h f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j (x+h)} = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{\alpha_j x} = \sum_{j=1}^{M} c_j \, \lambda_j \, \mathrm{e}^{\alpha_j x}$$

with $\lambda_j := \mathrm{e}^{\alpha_j h}$, and the eigenvalues λ_j of the "active" eigenfunctions in the exponential sum are the zeros of the Prony polynomial $P(z)$ in (2.1).

As we have seen before, the eigenfunction $\mathrm{e}^{\alpha_j x}$ is only determined uniquely by its eigenvalue $\mathrm{e}^{\alpha_j h}$, if we have the a priori information that $\operatorname{Im} \alpha_j$ is contained in a fixed interval of length $\frac{2\pi}{h}$, since the eigenspace of $\mathrm{e}^{\alpha_j h}$ is very large. For $\alpha_j \in \mathbb{C}$ the relation

$$S_h \mathrm{e}^{\alpha_j x} = \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{\alpha_j x}$$

also implies

$$S_h \mathrm{e}^{(\alpha_j + \frac{2\pi \mathrm{i} k}{h})x} = \mathrm{e}^{(x+h)(\alpha_j + \frac{2\pi \mathrm{i} k}{h})} = \mathrm{e}^{\alpha_j h} \, \mathrm{e}^{(\alpha_j + \frac{2\pi \mathrm{i} k}{h})x}.$$

Therefore, for each eigenvalue $\mathrm{e}^{\alpha_j h}$ we find the eigenspace spanned by the eigenfunctions $\{\mathrm{e}^{x(\alpha_j + \frac{2\pi \mathrm{i} k}{h})} : k \in \mathbb{Z}\}$.

We can now reinterpret Prony's method for the reconstruction of the exponential sum in (1.1) as follows. Assume that all wanted parameters α_j satisfy $|\operatorname{Im} \alpha_j| < T$ and $h < \frac{\pi}{T}$.

Then, a given finite linear combination of M eigenfunctions of the shift operator S_h with the corresponding M pairwise different eigenvalues $\mathrm{e}^{\alpha_j h}$, $j = 1, \ldots, M$, and with nonzero complex coefficients c_j can be completely recovered from the values $S_h^l f(x_0) = S_{hl} f(x_0)$, $l = 0, \ldots, 2M-1$, with $x_0 \in \mathbb{R}$. Using the shift operator S_h in (2.1) the homogeneous difference equation in (2.3) can be simply rewritten as

$$\begin{aligned}\sum_{k=0}^{M} p_k (S_h^{k+m} f)(x_0) &= \sum_{k=0}^{M} p_k (S_{h(k+m)} f)(x_0) = \sum_{k=0}^{M} p_k S_{h(k+m)} \Big(\sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cdot} \Big)(x_0) \\ &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \, (S_{h(k+m)} \mathrm{e}^{\alpha_j \cdot})(x_0) = \sum_{j=1}^{M} c_j \sum_{k=0}^{M} p_k \, \lambda_j^{m+k} \, \mathrm{e}^{\alpha_j \, x_0} \\ &= \sum_{j=1}^{M} c_j \lambda_j^m \Big(\sum_{k=0}^{M} p_k \, \lambda_j^k \Big) \mathrm{e}^{\alpha_j x_0} = 0.\end{aligned}$$

As before, the coefficients p_k of the Prony polynomial can be computed by the Hankel system as in (2.2) with the coefficient matrix $\mathbf{H} = \Big((S_{h(k+l)} f)(x_0) \Big)_{k,l=0}^{M-1}$, and the procedure to evaluate all parameters in (1.1) is the same as before.

To make this procedure work, we have essentially used two properties of the shift operator, namely,

- S_h is a linear operator,
- $\mathrm{e}^{\alpha x}$ is the unique eigenfunction of S_h to the eigenvalue $\mathrm{e}^{\alpha h}$ for each $\alpha \in \mathbb{C}$ with $\operatorname{Im} \alpha \in (-\pi/h, \pi/h)$.

2.3 Sampling Schemes for Recovering Exponential Sums

As we have seen, the exponential sum in (1.1) can be completely reconstructed using the equidistant function values $f(x_0 + hl)$, $l = 0, \ldots, 2M-1$. These values can be understood as the application of a point evaluation functional F_{x_0} with

$$F_{x_0}(S_h^l f) = F_{x_0}(S_{hl} f) := f(hl + x_0).$$

However, the generalized Prony method in [15] allows a higher flexibility of sampling schemes. In [15], it has been shown that instead of using a point evaluation functional we can also employ another linear functional F to $S_{hl} f$ satisfying the assumption that all eigenfunctions of the shift operator (which may play an active role in the exponential sum to be recovered) do not vanish under the action of F.

Using this generalized approach, we replace our measurements $(S_h^l f)(x_0) = f(x_0 + hl)$ by $F(S_h^l f)(x)$. Then the recovery of the parameters α_j can be still achieved by evaluating the coefficients of the Prony polynomial $P(z)$ in the first step,

$$\begin{aligned}\sum_{k=0}^{M} p_k F(S_h^{k+m} f) &= \sum_{k=0}^{M} p_k F\Big(S_h^{k+m} \Big(\sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cdot} \Big) \Big) = \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \, F(S_h^{k+m} \mathrm{e}^{\alpha_j \cdot}) \\ &= \sum_{j=1}^{M} c_j \sum_{k=0}^{M} p_k \, \lambda_j^{m+k} \, F(\mathrm{e}^{\alpha_j \cdot}) = \sum_{j=1}^{M} c_j \lambda_j^m \Big(\sum_{k=0}^{M} p_k \, \lambda_j^k \Big) F(\mathrm{e}^{\alpha_j \cdot}) = 0.\end{aligned}$$

Therefore, we obtain the Hankel system

$$\sum_{k=0}^{M-1} p_k \, F(S_h^{k+m} f) = \sum_{k=0}^{M-1} p_k \, F(f(\cdot + h(k+m))) = -F(f(\cdot + h(M+m))), \tag{2.2}$$

for $m \in \mathbb{Z}$, where the Hankel matrix $\mathbf{H} = \Big(F(S_h^{k+m} f)\Big)_{k,m=0}^{M-1}$ is invertible, since we have

$$\mathbf{H} = \mathbf{V}\,\mathrm{diag}\,(c_1, \ldots, c_M)\,\mathrm{diag}\,(F(\mathrm{e}^{\alpha_1 \cdot}), \ldots, F(\mathrm{e}^{\alpha_M \cdot}))\,\mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (\lambda_j^k)_{k,j=0}^{M-1} = (\mathrm{e}^{\alpha_j hk})_{k,j=0}^{M-1}$.

To illustrate the variety of possible sampling schemes, we give two examples.

1. Assume that we know a priori that the parameters in (1.1) satisfy $\mathrm{Im}\,\alpha_j \in (0,T)$ and choose $0 < h < \frac{2\pi}{T}$. Then we can consider

$$Ff \coloneqq \int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\chi_{[0,h]}(x-x_0)\,\mathrm{d}x = \langle f, \chi_{[0,h]}(\cdot - x_0)\rangle,$$

where $\chi_{[0,h]}$ denotes the characteristic function on $[0,h]$, and the condition

$$F\mathrm{e}^{\alpha \cdot} = \int_{x_0}^{x_0+h} \mathrm{e}^{\alpha x}\mathrm{d}x \neq 0$$

is obviously satisfied for all $\alpha \in (0,T)$ since $\mathrm{e}^{\alpha h} \neq 1$. With this sampling functional, it is sufficient to take the values

$$F(S_h^l f) = \int_{x_0}^{x_0+h} f(x+hl)\,\mathrm{d}x = \int_{x_0+hl}^{x_0+h(l+1)} f(x)\,\mathrm{d}x, \qquad l = 0, \ldots, 2M-1,$$

to reconstruct f.

2. With the assumption $\alpha_j = -\mathrm{i}t_j$ and $t_j \in (-1,1)$, we consider the functional

$$Ff \coloneqq \int_{-\infty}^{\infty} f(x)\,\overline{\Phi(x)}\,\mathrm{d}x = \langle f, \Phi\rangle$$

with $\Phi(x) = \frac{1}{\pi}\mathrm{sinc}\,x = \frac{1}{\pi}\frac{\sin(x)}{x}$. The Fourier transform of the sinc kernel is the box function $\widehat{\Phi}(t) = \chi_{[-1,1]}(t)$ and we obtain by Parseval identity with $\widehat{g}(x) = f(x)$,

$$Ff(\cdot + hl) = \langle f(\cdot + hl), \Phi\rangle = \langle \widehat{g}(\cdot + hl), \Phi\rangle = \langle g, \widehat{\Phi}\,\mathrm{e}^{\mathrm{i}lh\cdot}\rangle.$$

In particular, since g in (1.2) is a stream of diracs, it follows that

$$Ff(\cdot + hl) \;=\; \sum_{j=1}^{M} c_j\,\langle \delta(\cdot - t_j), \widehat{\Phi}\,\mathrm{e}^{\mathrm{i}lh\cdot}\rangle = \sum_{j=1}^{M} c_j\,\chi_{[-1,1]}(t_j)\mathrm{e}^{\mathrm{i}lht_j} = \sum_{j=1}^{M} c_j\,\mathrm{e}^{\mathrm{i}lht_j}.$$

Therefore, also the samples $\langle f(\cdot + hl), \Phi\rangle$, $l = 0, \ldots, 2M-1$, are sufficient to recover f in (1.1).

2.4 Generalized Shift Operators

We want to generalize the shift operator in order to be able to recover many more signal models beyond exponential sums. In particular, we consider the following linear operators.

A) Let $S_{h,-h}\colon C(\mathbb{R}) \to C(\mathbb{R})$ denote the *symmetric shift operator* for given $h > 0$ by

$$S_{h,-h} f(x) \coloneqq \frac{1}{2}\Big(f(x-h) + f(x+h)\Big) = \frac{1}{2}(S_{-h} + S_h)f(x). \tag{2.3}$$

B) Let $K\colon \mathbb{R}^2 \to \mathbb{C}$ be a given continuous function satisfying the property

$$K(x, h_1 + h_2) = K(x, h_2)K(x + h_2, h_1) = K(x, h_1)K(x + h_1, h_2). \tag{2.4}$$

We define for $h \neq 0$ the shift operator $S_{K,h}\colon C(\mathbb{R}) \to C(\mathbb{R})$ by

$$S_{K,h} f(x) := K(x, h)\, f(x + h). \tag{2.5}$$

C) Let the function $G\colon [a, b] \to \mathbb{R}$ be continuous and strictly monotonous in the sampling domain $[a, b] \subseteq \mathbb{R}$ and let G^{-1} denote its inverse function. We introduce the shift operator $S_{G,h}\colon C([a, b]) \to C(\mathbb{R})$ for $h \neq 0$ by

$$S_{G,h} f(x) := f(G^{-1}(G(x) + h)). \tag{2.6}$$

These operators have the following properties.

Theorem 2.1. *Let the operators $S_{h,-h}$, $S_{G,h}$ and $S_{K,h}$ be defined as above. Then the following holds,*

$$\begin{aligned} S_{h_2,-h_2}(S_{h_1,-h_1} f) &= S_{h_1,-h_1}(S_{h_2,-h_2} f) \\ &= \tfrac{1}{2}\left(S_{h_1+h_2,-(h_1+h_2)} f + S_{h_1-h_2,-(h_1-h_2)} f\right), \end{aligned} \tag{2.7}$$

$$S_{G,h_1}(S_{G,h_2} f) = S_{G,h_2}(S_{G,h_1} f) = S_{G,h_1+h_2} f, \tag{2.8}$$

$$S_{K,h_1}(S_{K,h_2} f) = S_{K,h_2}(S_{K,h_1} f) = S_{K,h_1+h_2} f. \tag{2.9}$$

In particular, we have

$$\begin{aligned} S^k_{h,-h} f &= \frac{1}{2^{k-1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} (S_{(k-2l)h,-(k-2l)h} f + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} f, \\ S^k_{G,h} f &= S_{G,kh} f, \qquad S^k_{K,h} f = S_{K,kh} f, \end{aligned}$$

where $\delta_{k/2,\lfloor k/2 \rfloor} = 1$ if k is even and vanishes otherwise.

Proof. We find for the symmetric shift

$$\begin{aligned} & S_{h_2,-h_2}(S_{h_1,-h_1} f)(x) = S_{h_2,-h_2}\left(\frac{1}{2}(f(x + h_1) + f(x - h_1))\right) \\ = {} & \frac{1}{4}\left(f(x + h_1 + h_2) + f(x - h_1 + h_2) + f(x + h_1 - h_2) + f(x - h_1 - h_2)\right) \\ = {} & \frac{1}{2}\left((S_{h_1+h_2,-(h_1+h_2)} f)(x) + (S_{h_1-h_2,-(h_1-h_2)} f)(x)\right). \end{aligned}$$

Repeated application of the operator $S_{h,-h}$ yields

$$\begin{aligned} S^k_{h,-h} f &= \frac{1}{2^k}(S_{-h} + S_h)^k(f) = \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} S^l_{-h} S^{k-l}_h f \\ &= \frac{1}{2^k} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} (S_{(k-2l)h} + S_{-(k-2l)h}) f + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} f \\ &= \frac{1}{2^{k-1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} S_{(k-2l)h,-(k-2l)h} f + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} f. \end{aligned}$$

Let the continuous function G be strictly monotonous in $[a,b] \subset \mathbb{R}$, and let $x, x+h_1, x+h_2, x+h_1+h_2 \in [a,b]$. Then

$$\begin{aligned} S_{G,h_2}(S_{G,h_1}f)(x) &= S_{G,h_2}(f(G^{-1}(G(\cdot)+h_1)))(x) \\ &= f(G^{-1}(G(G^{-1}(G(x)+h_1))+h_2)) \\ &= f(G^{-1}(G(x)+h_1+h_2)) = (S_{G,h_1+h_2}f)(x) \\ &= S_{G,h_1}(S_{G,h_2}f)(x). \end{aligned}$$

Finally, using the special properties (2.4) of the function K, it follows that

$$\begin{aligned} S_{K,h_2}(S_{K,h_1}f)(x) &= S_{K,h_2}(K(\cdot,h_1)\, f(\cdot+h_1))(x) \\ &= K(x,h_2)K(x+h_2,h_1)f(x+h_1+h_2) \\ &= K(x,h_1)K(x+h_1,h_2)f(x+h_1+h_2) \\ &= S_{K,h_1}(S_{K,h_2}f)(x). \end{aligned}$$

□

Remark 2.2. 1. Operators of the form (2.6) are well established as generalized operators in the field of quantum mechanics and are used to solve evolution operator equations in quantum field theory, see [7,8].
2. Besides using the generalized shift operators introduced in (2.3), (2.4) and (2.5), we can also combine these shift operators to generate further operators. For example, we will consider

$$S_{G,h,-h}f(x) := \frac{1}{2}\left(f(G^{-1}(G(x)-h)) + f(G^{-1}(G(x)+h))\right)$$

in Sections 5.3 and 6.1. Similarly, a combination of $S_{K,h}$ and $S_{h,-h}$ can be applied.
3. By Theorem 2.1, the iterated symmetric shift $S_{h,-h}^k$ can be presented as a linear combination of shifts $S_{hl,-hl}$ for $l = 0,\ldots,k$. Applying the symmetric shift operator we will therefore always use these shifts instead of $S_{h,-h}^l$, $l = 0,\ldots,k$. Since the Chebyshev polynomials $T_l(z) := \cos(l \arccos z)$ satisfy the relation

$$x^k = \frac{1}{2^{k-1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{l} T_{k-2l}(x) + \delta_{k/2,\lfloor k/2 \rfloor} \frac{1}{2^k} \binom{k}{k/2} T_{k/2},$$

it will be advantageous to write the Prony polynomial $P(z)$ as an expansion of Chebyshev polynomials.

3 Reconstruction of Expansions of Trigonometric and Hyperbolic Functions

First we employ the symmetric shift operator in order to derive a new method to reconstruct expansions of trigonometric functions. We observe that for each $h \in \mathbb{R}$, we have

$$S_{h,-h}\cos(\alpha x) = \frac{1}{2}\left[\cos(\alpha(x+h)) + \cos(\alpha(x-h))\right] = \cos(\alpha h)\,\cos(\alpha x),$$

and

$$S_{h,-h}\sin(\alpha x) = \frac{1}{2}\left[\sin(\alpha(x+h)) + \sin(\alpha(x-h))\right] = \cos(\alpha h)\,\sin(\alpha x),$$

i.e., the symmetric shift operator $S_{h,-h}$ possesses the eigenfunctions $\cos(\alpha x)$ and $\sin(\alpha x)$ for all $\alpha \in \mathbb{R}$.

3.1 Reconstruction of Cosine Expansions

We want to reconstruct an expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(\alpha_j x), \tag{3.10}$$

where we need to recover the unknown coefficients $c_j \in \mathbb{R} \setminus \{0\}$ and frequency parameters $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, M$.

Theorem 3.1. *Assume that all parameters α_j are in the range $[0, K) \subset \mathbb{R}$ and let $h = \frac{\pi}{K}$. Then, f in (3.10) can be uniquely reconstructed using the $2M$ samples $f(kh)$, $k = 0, \ldots, 2M-1$. More generally, for $x_0 \in \mathbb{R}$ satisfying $\alpha_j x_0 \neq (2k+1)\pi/2$ for $k \in \mathbb{Z}$ the $4M-1$ sample values $f(x_0 + hk)$, $k = -2M+1, \ldots, 2M-1$, are sufficient to reconstruct f in (3.10).*

Proof. We define the Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \cos(h\alpha_j))$$

which can be written as

$$P(z) = \sum_{l=0}^{M} p_l \, T_l(z),$$

where $T_l(z) := \cos(l \arccos z)$ denotes the Chebyshev polynomial of first kind of degree l. In particular, $p_M = 2^{-M+1}$ since for $l \geq 1$ the leading coefficient of T_l is 2^{l-1}. As the first step we compute the coefficients p_l of the polynomial $P(z)$ using the sample values. By definition of the Prony polynomial and Theorem 2.1 we have for f in (3.10),

$$\begin{aligned}
& \sum_{l=0}^{M} p_l \Big((S_{lh,-lh}) S_{mh} f(x_0) \Big) \\
= \; & \frac{1}{2} \sum_{l=0}^{M} p_l \left(f(x_0 + (m+l)h) + f(x_0 + (m-l)h) \right) \\
= \; & \frac{1}{2} \sum_{l=0}^{M} p_l \sum_{j=1}^{M} c_j \left[\cos(\alpha_j (x_0 + (m+l)h)) + \cos(\alpha_j (x_0 + (m-l)h)) \right] \\
= \; & \sum_{j=1}^{M} c_j \cos(\alpha_j (x_0 + mh) \sum_{l=0}^{M} p_l \cos(\alpha_j l h) \\
= \; & \sum_{j=1}^{M} c_j \cos(\alpha_j (x_0 + mh)) \sum_{l=0}^{M} p_l T_l(\cos(\alpha_j h)) = 0
\end{aligned}$$

for all $m = 0, \ldots, M-1$. Similarly, it follows that

$$\sum_{l=0}^{M} p_l \Big(S_{lh,-lh} (S_{-mh} f)(x_0) \Big) = \sum_{j=1}^{M} c_j \cos(\alpha_j (x_0 - mh)) \sum_{l=0}^{M} p_l T_l(\cos(\alpha_j h)) = 0$$

for all $m = 0, \ldots, M-1$. For $x_0 = 0$, we obtain the linear system

$$\sum_{l=0}^{M-1} p_l (f((m+l)h) + f((m-l)h) = -\frac{2}{2^M} (f((m+M)h) + f((m-M)h) \tag{3.11}$$

for $m = 0, \dots, M-1$. Since f is an even function, it suffices to know the signal values $f(kh)$, $k = 0, \dots, 2M-1$ to build this system. The quadratic coefficient matrix has Toeplitz-plus-Hankel structure,

$$\begin{aligned} \mathbf{H} &= \Big((f((m+l)h) + f((m-l)h) \Big)_{m,l=0}^{M-1} \\ &= 2\Big(\sum_{j=1}^{M} c_j \cos(\alpha_j mh) \cos(\alpha_j lh) \Big)_{m,l=0}^{M-1} = 2\, \mathbf{V}\, \mathrm{diag}(c_j)_{j=1}^{M}\, \mathbf{V}^T, \end{aligned} \tag{3.12}$$

with the generalized Vandermonde matrix

$$\mathbf{V} = \Big(T_k(\cos(\alpha_j h)) \Big)_{k=0,j=1}^{M-1,M}. \tag{3.13}$$

The matrix $\mathbf{V}$ is always invertible, since the terms $\cos(\alpha_j h)$ are nonzero and pairwise distinct by assumption. Therefore, $\mathbf{H}$ is invertible if $c_j \neq 0$ for $j = 1, \dots, M$. For $x_0 \neq 0$, we need to take all values $S_{lh,-lh} S_{mh,-mh} f(x_0)$ into account. Here, we apply

$$\begin{aligned} &\sum_{l=0}^{M-1} p_l (f(x_0 + (m+l)h) + f(x_0 - (m+l)h) + f(x_0 + (m-l)h) + f(x_0 - (m-l)h)) \\ = &-\frac{2}{2^M} (f(x_0 + (m+M)h) + f(x_0 - (m+M)h) + f(x_0 + (m-M)h) + f(x_0 - (m-M)h)), \end{aligned} \tag{3.14}$$

and, similarly as in (3.12), the coefficient matrix factorizes in the form

$$\begin{aligned} \mathbf{H} &= \Big((f(x_0+(m+l)h) + f(x_0-(m+l)h) + f(x_0+(m-l)h) + f(x_0-(m-l)h) \Big)_{l,m=0}^{M-1} \\ &= 2\Big(\sum_{j=1}^{M} c_j \cos(\alpha_j x_0) \cos(\alpha_j mh) \cos(\alpha_j lh) \Big)_{l,m=0}^{M-1} = 2\mathbf{V} \mathrm{diag}(c_j \cos(\alpha_j x_0))_{j=1}^{M}\, \mathbf{V}^T. \end{aligned}$$

The diagonal matrix $\mathrm{diag}(c_j \cos(\alpha_j x_0))_{j=1}^{M}$ is invertible if we have $c_j \neq 0$ and $\alpha_j x_0 \neq (2k+1)\pi/2$ for all $k \in \mathbb{Z}$ and all $j = 1, \dots, M$. Having found the coefficients of the Prony polynomial, we can extract the zeros $\cos(h\alpha_j)$, $j = 1, \dots, M$.
In the second step, we solve the linear system

$$f(x_0 + hk) = \sum_{j=1}^{M} c_j \, \cos(\alpha_j (x_0 + hk)), \qquad k = 0, \dots 2M-1$$

in order to compute c_j, $j = 1, \dots M$. □

3.2 Reconstruction of Sine Expansions

The symmetric shift operator can also be applied for the reconstruction of sparse linear combination of sines of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \sin(\alpha_j x), \tag{3.15}$$

with unknown coefficients $c_j \in \mathbb{R} \setminus \{0\}$ and $\alpha_j \in \mathbb{R} \setminus \{0\}$. This recovery problem is closely related to the problem in (3.10), but we need to pay attention at some details. For example, $f(0)$ does not give us any information here, since the function f in (3.15) is odd, and a frequency $\alpha_j = 0$ does not occur.

Theorem 3.2. *Assume that all parameter α_j are in the range $(0, K)$ and let $h = \frac{\pi}{K}$. Then, f in (3.15) can be reconstructed using the $4M-1$ sample values $f(x_0 + hk)$, $k = -2M+1, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ satisfies $\sin(\alpha_j x_0) \neq 0$ for $j = 1, \ldots, M$. In particular, the $2M$ samples $f(kh)$, $k = 1, \ldots, 2M$ are sufficient to reconstruct f in (3.15).*

Proof. We proceed similarly as in the last proof. We define the Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \cos(h\alpha_j)) = \sum_{l=0}^{M} p_l T_l(z),$$

with coefficients p_l in the Chebyshev expansion and $p_M = 2^{-M+1}$. To compute the coefficients of $P(z)$, we obtain a linear system as in (3.14), this time with the coefficient matrix

$$\begin{aligned}
\mathbf{H} &= \Big((f(x_0 + (m+l)h) + f(x_0 - (m+l)h) + f(x_0 + (m-l)h) + f(x_0 - (m-l)h)) \Big)_{l,m=0}^{M-1} \\
&= 4 \Big(\sum_{j=1}^{M} c_j \sin(\alpha_j x_0) \cos(\alpha_j mh) \cos(\alpha_j lh) \Big)_{l,m=0}^{M-1} \\
&= 4\, \mathbf{V}\, \mathrm{diag}(c_j \sin(\alpha_j x_0))_{j=1}^{M}\, \mathbf{V}^T
\end{aligned}$$

with $\mathbf{V}$ as in (3.13). Invertibility follows if $\sin(\alpha_j x_0) \neq 0$. This is satisfied for $x_0 = \frac{\pi}{K} = h$. Thus, the function values $f(x_0 + hl) = f(h(l+1))$, $l = 0, \ldots, 2M-1$, are already sufficient for the reconstruction, since we have

$$f(x_0 - hl) = f(h(1-l)) = \begin{cases} 0 & l = 1, \\ -f(h(l-1)) & l \geq 2. \end{cases}$$

Having found $P(z)$ we obtain the zeros $\cos(h\alpha_j)$ and can compute the coefficients c_j in (3.15) by solving a linear system using the sample values. □

Remark 3.3. 1. The symmetric shift operator possesses also the eigenfunctions $\sinh(\alpha x)$ and $\cosh(\alpha x)$ with $\alpha \in \mathbb{R}$. Therefore, sparse expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \cosh(\alpha_j x), \tag{3.16}$$

and

$$f(x) = \sum_{j=1}^{M} c_j \sinh(\alpha_j x), \tag{3.17}$$

can be reconstructed using at most $4M-1$ consecutive sample values $f(x_0 + kh)$. Taking the samples $f(hl)$, $l = 0, \ldots, 2M-1$ for (3.16) or $f(hl)$, $l = 1, \ldots, 2M$ for (3.17) is also sufficient for reconstructing these expansions.

2. Obviously, the considered expansions can also be studied using the well-known exponential sums by expanding the trigonometric and hyperbolic functions into sums of exponentials. But in this case, the number of terms in the sparse sums is doubled from M to $2M$.

3. Using the Laplace transform with $\mathcal{L}(\cos \alpha \cdot)(s) = \frac{s}{s^2+\alpha^2}$ and $\mathcal{L}(\sin \alpha \cdot)(s) = \frac{\alpha}{s^2+\alpha^2}$, we can also reconstruct signals of the form

$$f(s) = \sum_{j=1}^{M} \frac{c_j s}{s^2 + \alpha_j^2} \qquad \text{or} \qquad f(s) = \sum_{j=1}^{M} \frac{c_j \alpha_j}{s^2 + \alpha_j^2}.$$

Analogously, models arising for the Laplace transform of cosh and sinh can be recovered.

4 Reconstruction of Expansions Using the Operator $S_{K,h}$

In this section, we study signal models that arise using the generalized shift operator

$$S_{K,h} f(x) = K(x,h)\, f(x+h)$$

in (2.5).

4.1 Reconstruction of Expansions of shifted Gaussians

Let $g(x) := \mathrm{e}^{-\beta x^2}$ for some given $\beta \in \mathbb{C} \setminus \{0\}$. We want to reconstruct an expansion of shifted Gaussians of the form

$$f(x) = \sum_{j=1}^{M} c_j\, g(x - \alpha_j) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{-\beta(x-\alpha_j)^2}, \tag{4.18}$$

and need to recover the $2M$ coefficients $c_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, M$.

Let $K_1(x,h) := \mathrm{e}^{\beta h(2x+h)}$ such that for all $h_1, h_2 \in \mathbb{R}$ the relation

$$K_1(x, h_1 + h_2) = K_1(x, h_1) K_1(x + h_1, h_2) = K_1(x, h_2) K_1(x + h_2, h_1)$$

is satisfied. Then, the functions $\mathrm{e}^{-\beta(\cdot - \alpha_j)^2}$ are eigenfunctions of $S_{K_1,h}$ for all $\alpha_j \in \mathbb{R}$, since

$$(S_{K_1,h}\, \mathrm{e}^{-\beta(\cdot-\alpha_j)^2})(x) = \mathrm{e}^{\beta h(2x+h)}\, \mathrm{e}^{-\beta(x+h-\alpha_j)^2} = \mathrm{e}^{2\beta\alpha_j h}\, \mathrm{e}^{-\beta(x-\alpha_j)^2}.$$

Theorem 4.1. *If $\operatorname{Re}\beta \neq 0$, the stepsize $h \in \mathbb{R} \setminus \{0\}$ can be taken arbitrarily. If $\operatorname{Re}\beta = 0$, we assume that $\alpha_j \in (-T, T)$ for $j = 1, \ldots, M$ for some given T and choose $0 < h \leq \frac{\pi}{2|\operatorname{Im}\beta|\, T}$. Then, f in (4.18) can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.*

Proof. We define the Prony polynomial

$$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{2h\beta\alpha_j}) = \sum_{l=0}^{M} p_l\, z^l,$$

where the parameters p_l denote the coefficients of $P(z)$ in monomial representation with $p_M = 1$. Then we find

$$\begin{aligned}
&\sum_{l=0}^{M} p_l\, (S_{K_1,(l+m)h} f)(x_0) = \sum_{l=0}^{M} p_l\, \mathrm{e}^{\beta h(l+m)(2x_0 + h(l+m))} f(x_0 + h(l+m)) \\
&= \sum_{l=0}^{M} p_l\, \mathrm{e}^{\beta h(l+m)(2x_0+h(l+m))} \sum_{j=1}^{M} c_j \mathrm{e}^{-\beta(x_0 + h(l+m) - \alpha_j)^2} \\
&= \sum_{j=1}^{M} c_j \mathrm{e}^{-\beta(x_0+hm-\alpha_j)^2} \mathrm{e}^{\beta hm(2x_0+hm)} \sum_{l=0}^{M} p_l\, \mathrm{e}^{-\beta(h^2 l^2 + 2hl(x_0+hm-\alpha_j))} \mathrm{e}^{\beta hl(2x_0+h(l+2m))}
\end{aligned}$$

$$= \sum_{j=1}^{M} c_j \mathrm{e}^{-\beta(x_0+hm-\alpha_j)^2} \mathrm{e}^{\beta hm(2x_0+hm)} \sum_{l=0}^{M} p_l \, \mathrm{e}^{-\beta(-2hl\alpha_j)} = 0$$

for $m = 0, \ldots, M-1$. The coefficients $p_0, \ldots, p_{M-1}$ of $P(z)$ can therefore be computed by the linear system

$$\begin{aligned} & \sum_{l=0}^{M-1} p_l \, \mathrm{e}^{\beta h(l+m)(2x_0+h(l+m))} f(x_0 + h(l+m)) \\ = \quad & -\mathrm{e}^{\beta h(M+m)(2x_0+h(M+m))} \, f(x_0 + h(M+m)), \quad m = 0, \ldots, M-1. \end{aligned} \tag{4.19}$$

This system matrix has Hankel structure and its invertibility follows from the factorization

$$\begin{aligned} \mathbf{H} \; &\coloneqq \; (K_1(x_0, h(l+m)) f(x_0 + h(l+m)))_{l,m=0}^{M-1} \\ &= \; \left(\mathrm{e}^{\beta h(l+m)(2x_0+h(l+m))} \sum_{j=1}^{M} c_j \, \mathrm{e}^{-\beta(x_0+h(l+m)-\alpha_j)^2} \right)_{l,m=0}^{M-1} \\ &= \; \left(\sum_{j=1}^{M} c_j \, \mathrm{e}^{-\beta(x_0-\alpha_j)^2} \, \mathrm{e}^{2\beta h(l+m)\alpha_j} \right)_{l,m=0}^{M-1} \\ &= \; \mathbf{V}_h \, \mathrm{diag} \, (c_j \mathrm{e}^{-\beta(\alpha_j - x_0)^2}) \, \mathbf{V}_h^T \end{aligned}$$

with the Vandermonde matrix

$$\mathbf{V}_h \coloneqq \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \mathrm{e}^{2\beta\alpha_1 h} & \mathrm{e}^{2\beta\alpha_2 h} & \ldots & \mathrm{e}^{2\beta\alpha_M h} \\ \vdots & \vdots & & \vdots \\ \mathrm{e}^{2(M-1)\beta\alpha_1 h} & \mathrm{e}^{2(M-1)\beta\alpha_2 h} & \ldots & \mathrm{e}^{2(M-1)\beta\alpha_M h} \end{pmatrix}.$$

Having solved (4.19), we can find the zeros $\mathrm{e}^{2h\beta\alpha_j}$ of the Prony polynomial and extract the parameters α_j, $j = 1, \ldots, M$. This is always possible using the supposed restrictions on h. Finally, we solve the linear system

$$f(x_0 + hk) = \sum_{j=1}^{M} c_j \mathrm{e}^{-\beta(x_0+hk-\alpha_j)^2}$$

in order to compute the coefficients c_j in (4.18). □

Remark 4.2. 1. The recovery of sums of Gaussians has also been considered in in [14] and in a short note in the multivariate case, see [16], but without using the property, that the Gaussian is an eigenfunction of a suitable generalized shift operator.

2. The model (4.18) is also of the form (1.3) and can therefore be reconstructed using equidistant Fourier values, as it has been done e.g. in [19].

4.2 Reconstruction of Gabor Expansions with the Gaussian Window Function

Similarly as in the previous subsection, we can even consider modulated Gaussians. We now want to recover a Gabor expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j), \tag{4.20}$$

where $g(x) \coloneqq \mathrm{e}^{-\beta x^2}$ is the Gaussian window with known $\beta \in \mathbb{R} \setminus \{0\}$, and where we have to recover the parameters c_j, $\alpha_j \in \mathbb{R}$, and the shifts $s_j \in \mathbb{R}$, $j = 1, \ldots, M$. Using again the shift operator $S_{K_1,h}$ in (2.5) with $K_1(x,h) = \mathrm{e}^{\beta h(2x+h)}$ we observe that indeed $\mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j) = \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} \, \mathrm{e}^{-\beta(x-s_j)^2}$ are eigenfunctions of $S_{K_1,h}$,

$$\begin{aligned} (S_{K_1,h} \, \mathrm{e}^{2\pi \mathrm{i} \alpha_j \cdot - \beta(\cdot - s_j)^2})(x_0) &= \mathrm{e}^{\beta h(2x_0+h)} \, \mathrm{e}^{2\pi \mathrm{i}(x_0+h)\alpha_j} \, \mathrm{e}^{-\beta(x_0+h-s_j)^2} \\ &= \mathrm{e}^{2h(\beta s_j + \pi \mathrm{i} \alpha_j)} \, \mathrm{e}^{2\pi \mathrm{i} x_0 \alpha_j - \beta(x_0 - s_j)^2}. \end{aligned}$$

Theorem 4.3. *Assume that all parameters α_j are in the range $(-K, K)$ for $j = 1, \ldots, M$ and let $0 < h \leq \frac{1}{2K}$. Then, f in (4.20) can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R}$ is an arbitrary real number.*

Proof. This time, we define the Prony polynomial in the form

$$P(z) \coloneqq \prod_{j=1}^{M} (z - \mathrm{e}^{2h(\pi \mathrm{i} \alpha_j + \beta s_j)}) = \sum_{l=0}^{M} p_l \, z^l,$$

where p_l denote the coefficients of $P(z)$ in monomial representation with $p_M = 1$. We observe that the zeros of the Prony polynomial are complex, where the imaginary part covers the modulation parameters α_j and the real part the shift parameters s_j. Then we find

$$\begin{aligned} &\sum_{l=0}^{M} p_l \, (S_{K_1,(l+m)h} f)(x_0) = \sum_{l=0}^{M} p_l \, \mathrm{e}^{\beta h(l+m)(2x_0+h(l+m))} f(x_0 + h(l+m)) \\ = \; &\sum_{l=0}^{M} p_l \, \mathrm{e}^{\beta h(l+m)(2x_0+h(l+m))} \sum_{j=1}^{M} c_j \, \mathrm{e}^{2\pi \mathrm{i}(x_0+h(m+l))\alpha_j} \, \mathrm{e}^{-\beta(x_0+h(l+m)-s_j)^2} \\ = \; &\sum_{j=1}^{M} c_j \, \mathrm{e}^{-\beta(x_0+hm-s_j)^2} \mathrm{e}^{\beta hm(2x_0+hm)} \mathrm{e}^{2\pi \mathrm{i}(x_0+hm)\alpha_j} \sum_{l=0}^{M} p_l \, \mathrm{e}^{2lh(\pi \mathrm{i} \alpha_j + \beta s_j)} = 0 \end{aligned}$$

for $m = 0, \ldots, M-1$. The coefficients $p_0, \ldots, p_{M-1}$ of $P(z)$ can therefore be computed by the same system as in (4.19), and we can extract the parameters α_j and s_j from the zeros of the Prony polynomial. Finally, the coefficients c_j are determined by inserting the function values $f(x_0 + hk)$ into the model (4.20) and by solving the obtained linear system. □

4.3 Reconstruction of Generalized Exponential Sums

We want to reconstruct expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \, (x\alpha_j)^r \, \mathrm{e}^{x\alpha_j}, \tag{4.21}$$

where $r \in \mathbb{R}$ is known and where we have to recover c_j, $\alpha_j \in \mathbb{C}$. Here we employ the shift operator $S_{K_2,h}$ with $K_2(x,h) = (\frac{x}{h+x})^r$ satisfying

$$\Big(\frac{x}{h_1+h_2+x}\Big)^r = \Big(\frac{x}{h_1+x}\Big)^r \Big(\frac{x+h_1}{h_1+h_2+x}\Big)^r.$$

Then $(x\alpha_j)^r \, \mathrm{e}^{x\alpha_j}$ are eigenfunctions of $S_{K_2,h}$ for each $\alpha_j \in \mathbb{C}$, since

$$(S_{K_2,h}(\alpha_j \cdot)^r \, \mathrm{e}^{\alpha_j \cdot})(x_0) = \Big(\frac{x_0}{h+x_0}\Big)^r (\alpha_j(x_0+h))^r \, \mathrm{e}^{\alpha_j(x_0+h)} = \mathrm{e}^{\alpha_j h} \, (\alpha_j x_0)^r \, \mathrm{e}^{\alpha_j x_0}.$$

For pairwise different α_j it follows that the eigenvalues $\mathrm{e}^{\alpha_j h}$ are pairwise different, if the imaginary part of $\alpha_j h$ is in a fixed interval of length 2π. Therefore, we will assume that h is chosen such that $\operatorname{Im} \alpha_j \in [-\pi/h, \pi/h)$ holds.

Theorem 4.4. *Let $h \in \mathbb{R} \setminus \{0\}$ be such that $\operatorname{Im} \alpha_j \in [-\pi/h, \pi/h)$. Then, f in (4.21) can be reconstructed using the $2M$ sample values $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R} \setminus \{0\}$ is an arbitrary real number.*

Proof. We employ the Prony polynomial $P(z) = \prod_{j=1}^{M}(z - \mathrm{e}^{\alpha_j h}) = \sum_{l=0}^{M} p_l z^l$ with $p_M = 1$. We simply observe that

$$\begin{aligned}
&\sum_{l=0}^{M} p_l \, (S_{K_2,(l+m)h} f)(x_0) \\
= \; &\sum_{l=0}^{M} p_l \left(\frac{x_0}{x_0+h(l+m)}\right)^r \sum_{j=1}^{M} c_j \, ((x_0+h(l+m))\alpha_j)^r \, \mathrm{e}^{(x_0+h(l+m))\alpha_j} \\
= \; &\sum_{j=1}^{M} c_j \, (x_0\alpha_j)^r \mathrm{e}^{(x_0+hm)\alpha_j} \left(\sum_{l=0}^{M} p_l \mathrm{e}^{h\alpha_j l}\right) = 0,
\end{aligned}$$

and the coefficients p_l of $P(z)$ can be computed by the system

$$\sum_{l=0}^{M-1} p_l \Big(\frac{x_0}{x_0+h(l+m)}\Big) f(x_0+h(l+m)) = -\Big(\frac{x_0}{x_0+h(M+m)}\Big) f(x_0+h(M+m))$$

for $m = 0, \ldots, M-1$. The system matrix has Hankel structure with the factorization

$$\Big(\Big(\frac{x_0}{x_0+h(l+m)}\Big) f(x_0+h(l+m))\Big)_{l,m=0}^{M-1} = \mathbf{V} \operatorname{diag}\left(c_j \, (x_0\alpha_j)^r \mathrm{e}^{x_0\alpha_j}\right)_{j=1}^{M} \mathbf{V}^T$$

where the Vandermonde matrix $\mathbf{V} = (\mathrm{e}^{h\alpha_j k})_{k=0,j=1}^{M-1,M}$ is generated by the knots $\mathrm{e}^{h\alpha_j}$, $j = 1, \ldots, M$. Invertibility is ensured since $\mathrm{e}^{h\alpha_j}$ are pairwise different, $c_j \neq 0$ and $x_0 \neq 0$. Using p_l to construct the Prony polynomial we firstly compute its roots, recover $\mathrm{e}^{\alpha_j h}$ and afterwards the coefficients c_j in (4.21) by solving a linear system. $\square$

Remark 4.5. The expansion in (4.21) is equivalent to

$$f(x) = \sum_{j=1}^{M} \tilde{c}_j \, x^r \, \mathrm{e}^{x\alpha_j}$$

if we take $\tilde{c}_j = c_j \alpha_j^r$. We observe that also more general expansions of the form

$$f(x) = \sum_{j=1}^{M} \tilde{c}_j \, H(x) \, \mathrm{e}^{x\alpha_j}$$

can be recovered, if we assume that the function $H(x)$ is known and that $K_2(x,h) = H(x)/H(x+h)$ is well-defined. Then, obviously

$$(S_{K_2,h}(H(\cdot)\,\mathrm{e}^{\alpha_j \cdot})(x_0) = \left(\frac{H(x_0)}{H(h+x_0)}\right) H(x_0+h)\,\mathrm{e}^{\alpha_j(x_0+h)} = \mathrm{e}^{\alpha_j h}\, H(x_0)\,\mathrm{e}^{\alpha_j x_0}.$$

Alternatively, we can also consider $\tilde{f}(x) = f(x)/H(x)$ which is again a simple exponential sum.

5 Reconstruction of Expansions Using the Operator $S_{G,h}$

Now we consider the generalized shift operator $S_{G,h}$ in (2.6) with

$$S_{G,h} f(x) = f(G^{-1}(G(x)+h)).$$

This operator gives us a lot of freedom to generate generalized shifts.

5.1 Reconstruction of Expansions of Monomials

If we consider for example $G(x) = \ln x$ and $G^{-1}(x) = \mathrm{e}^x$, we obtain

$$S_{\ln,h} f(x) = f(\mathrm{e}^{(\ln x)+h}) = f(x\mathrm{e}^h) = f(xa)$$

with $a \coloneqq \mathrm{e}^h > 0$. With other words, $S_{\ln,h}$ is the dilation operator with the dilation factor $a = \mathrm{e}^h$. In particular, all functions of the form x^{p_k} with $p_k \in \mathbb{C}$ are eigenfunctions of this operator,

$$S_{\ln,h}((\cdot)^{p_k})(x) = (\mathrm{e}^{(\ln x)+h})^{p_k} = x^{p_k}\,\mathrm{e}^{hp_k} = x^{p_k}\,a^{p_k}.$$

If the p_k are pairwise distinct and if $\operatorname{Im} p_k \in [-\pi/h, \pi/h)$, then the eigenvalues e^{hp_k} are pairwise distinct. We now want to recover an expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j\, x^{p_j}, \tag{5.22}$$

where $c_j \in \mathbb{C}$ and $p_j \in \mathbb{C}$ with $\operatorname{Im} p_j \in [-\pi/h,\, \pi/h)$ for all $j = 1, \ldots, M$.

Theorem 5.1. *Let $h \in \mathbb{C} \setminus \{0\}$ and $a \coloneqq \mathrm{e}^h \notin \{\mathrm{e}^{2\pi \mathrm{i}k/N} :\ k \in \mathbb{Z}\}$ for all $N < 2M$ be such that a^k, $k = 0, \ldots, 2M-1$ are pairwise distinct values. Then f in (5.22) can be reconstructed by the $2M$ sample values $f(a^k x_0)$, where $x_0 \in \mathbb{C} \setminus \{0\}$ can be chosen arbitrarily.*

Proof. We define

$$P(z) \coloneqq \prod_{k=1}^{M} (z - a^{p_j}) = \sum_{l=0}^{M} p_l z^l$$

and observe that

$$\sum_{l=0}^{M} p_l f(a^{l+m} x_0) = \sum_{l=0}^{M} p_l \sum_{j=1}^{M} c_j a^{(l+m)p_j} x_0^{p_j} = \sum_{j=1}^{M} c_j (a^m x_0)^{p_j} \sum_{l=0}^{M} p_l\, a^{p_j l} = 0$$

for $m = 0, \ldots, M-1$, leading to the system

$$\sum_{l=0}^{M-1} f(a^{l+m}x_0)p_l = -f(a^{M+m}x_0), \qquad m = 0, \ldots, M-1.$$

The invertibility of the Hankel matrix $\mathbf{H} = (f(a^{l+m}x_0))_{l,m=0}^{M-1}$ follows from

$$\mathbf{H} = \mathbf{V} \, \mathrm{diag} \, (c_j x_0^{p_j})_{j=1}^{M} \, \mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (a^{p_j k})_{k=0,j=1}^{M-1,M}$ generated by the N pairwise different knots a^{p_j}, $j = 1, \ldots, M$. Once the coefficients of the Prony polynomial $P(z)$ are found, we compute a^{p_j} as the zeros of $P(z)$, extract p_j and finally compute c_j in (5.22) by solving the system

$$f(x_0 a^k) = \sum_{j=1}^{M} c_j (x_0 a^k)^{p_j}, \qquad k = 0, \ldots, 2M-1.$$

□

Remark 5.2. This example was already discussed in [15] using the dilation operator D_a with $D_a f(x) := f(ax)$. Moreover, using the substitution $x = \mathrm{e}^x$, the model (5.22) can be transferred to the original model (1.1).

If the parameters p_j are positive integers, then f in (5.22) is a sparse polynomial, and its reconstruction can be performed using the Ben-Or and Tiwari Algorithm, see e.g. [1, 13].

5.2 Expansions of Complex Gaussians with Different Scaling

Let now $G(x) = x^2$ and $G^{-1}(x) = \sqrt{x}$ for $x \geq 0$. We consider the corresponding generalized shift operator

$$(S_{x^2,h} f)(x) = f\left(\sqrt{x^2 + h}\right)$$

and observe that the (complex) Gaussians $\mathrm{e}^{\alpha x^2}$ with $\alpha \in \mathbb{C}$ are eigenfunctions of this operator,

$$(S_{x^2,h} \, \mathrm{e}^{\alpha (\cdot)^2})(x) = \mathrm{e}^{\alpha (\sqrt{x^2+h}^2)} = \mathrm{e}^{\alpha h} \mathrm{e}^{\alpha x^2}.$$

The parameter α can be uniquely recovered from the eigenvalues $\mathrm{e}^{\alpha h}$ if $\mathrm{Im}\, \alpha h \in [-\pi, \pi)$, i.e. if $\mathrm{Im}\, \alpha \in [-\pi/h, \pi/h)$. We now consider the reconstruction of expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x^2} \tag{5.23}$$

where we need to recover $c_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$.

Theorem 5.3. *Let* $\mathrm{Im}\, \alpha_j \in (-K, K)$ *for some* $K > 0$ *for all* $j = 1, \ldots, M$. *We choose* $h := 1/K$. *Then the expansion in* (5.23) *can be uniquely recovered from the samples* $f\left(\sqrt{x_0^2 + kh}\right)$, $k = 0, \ldots, 2M-1$, *where* $x_0 \in \mathbb{R}$ *can be chosen arbitrarily.*

Proof. We define

$$P(z) := \prod_{j=1}^{M} (z - \mathrm{e}^{\alpha_j h}) = \sum_{l=0}^{M} p_l \, z^l$$

and observe for the monomial coefficients p_l of $P(z)$,

$$\begin{aligned} \sum_{l=0}^{M} p_l f\left(\sqrt{x_0^2 + (m+l)h}\right) &= \sum_{l=0}^{M} p_l \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j (x_0^2 + (m+l)h)} \\ &= \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j (x_0^2 + mh)} \sum_{l=0}^{M} p_l \, \mathrm{e}^{\alpha_j h l} = 0 \end{aligned}$$

for all $m = 0, \ldots, M-1$. Therefore this system can be used to compute the coefficients p_l (using $p_M = 1$), since the corresponding matrix $\left(f\left(\sqrt{x_0^2 + (m+l)h}\right)\right)_{l,m=0}^{M-1}$ can be factorized in the form

$$\left(f\left(\sqrt{x_0^2 + (m+l)h}\right)\right)_{l,m=0}^{M-1} = \mathbf{V} \, \mathrm{diag}(c_j \mathrm{e}^{\alpha_j x_0^2})_{j=1}^{M} \, \mathbf{V}^T$$

with the Vandermonde matrix $\mathbf{V} = (\mathrm{e}^{h\alpha_j k})_{k=0,j=1}^{M-1,M}$ generated by the pairwise different knots $\mathrm{e}^{h\alpha_j}$, $j = 1, \ldots, M$. Having found $P(z)$, we can extract its zeros, recover p_j and finally also c_j by solving a linear system. $\square$

Remark 5.4. Similarly, using $G(x) = x^p$ and $G^{-1}(x) = \sqrt[p]{x}$ with given $p > 0$ we can recover expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\alpha_j x^p}$$

by reconstructing c_j, $\alpha_j \in \mathbb{C}$ from the samples $f\left(\sqrt[p]{x_0^p + hk}\right)$, $k = 0, \ldots, 2M-1$.

Taking e.g. $G(x) = \cos(x)$ for $x \in [0, \pi]$ and $G^{-1}(x) = \arccos(x)$ we obtain the operator

$$S_{\cos,h} f(x) = f(\arccos(\cos(x) + h))$$

which has the eigenfunctions $\mathrm{e}^{\alpha_k \cos x}$ with

$$(S_{\cos,h} \mathrm{e}^{\alpha_k \cos \cdot})(x) = \mathrm{e}^{\alpha_k \cos(\arccos(\cos x + h))} = \mathrm{e}^{\alpha_k (\cos x + h)}.$$

In this way, we can also recover expansions of the form

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{\alpha_j \cos x}$$

with parameters $c_j, \alpha_j \in \mathbb{C}$ using the samples $f(\arccos(\cos x + kh))$, $k = 0, \ldots 2M-1$ with suitably chosen h.

5.3 Sparse Expansions of Chebyshev Polynomials

Let now $x \in [-1,1]$ and $G(x) = \arccos x$. Then G is monotonous in $[-1,1]$ and $G^{-1}(y) = \cos y$ for $y \in [0,\pi]$. We apply a combination of the symmetric shift operator $S_{h,-h}$ and $S_{G,h}$,

$$(S_{G,h,-h}f)(x) := \frac{1}{2}\left(f(\cos(\arccos(x)+h)) + f(\cos(\arccos(x)-h))\right),$$

which is also called Chebyshev shift operator, see also [15, 20]. Then the Chebyshev polynomials $T_k(x) = \cos(k \arccos x)$ of degree $k \geq 0$ are eigenfunctions of this operator,

$$\begin{aligned}(S_{G,h,-h}T_k)(x) &= \frac{1}{2}\left(T_k(\cos(\arccos(x)+h)) + T_k(\cos(\arccos(x)-h))\right)\\ &= \frac{1}{2}\left(\cos k(\arccos(x)+h) + \cos k(\arccos(x)-h)\right)\\ &= \cos(kh)\ \cos(k\arccos x) = \cos(kh)\, T_k(x).\end{aligned}$$

We want to reconstruct a sparse Chebyshev expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j\, T_{n_j}(x), \tag{5.24}$$

where we need to recover the unknown indices $0 \leq n_1 < n_2 < \ldots < n_M$ and the coefficients $c_j \in \mathbb{R}$. We assume that an upper bound K of the degree n_M of the polynomial in (5.24) is a priori known.

Theorem 5.5. *Let K be a bound of the degree of the polynomial f in (5.24) and let $0 < h \leq \frac{\pi}{K}$. Then the Chebyshev expansion in (5.24) can be uniquely recovered from the samples $f(\cos(kh))$, $k = 0,\ldots,2M-1$.*

Proof. Let $P(z) = \prod_{j=1}^{M}(z - \cos(n_j h)) = \sum_{l=0}^{M} p_l T_l(z)$, where p_l are the coefficients of $P(z)$ in the Chebyshev expansion, and $p_M = 2^{-M+1}$. Then, since the cosine is even, we obtain

$$\begin{aligned}&\sum_{l=0}^{M} p_l\left(S_{G,(m+l)h,-(m+l)h}f(\cos(0)) + S_{G,(m-l)h,-(m-l)h}f(\cos(0))\right)\\ =\ &\sum_{l=0}^{M} p_l\left(f(\cos(m+l)h) + f(\cos(m-l)h)\right)\\ =\ &\sum_{l=0}^{M} p_l \sum_{j=1}^{M} c_j\left(T_{n_j}(\cos(m+l)h) + T_{n_j}(\cos(m-l)h)\right)\\ =\ &\sum_{j=1}^{M} c_j \sum_{l=0}^{M} p_l(2\,\cos(n_j ml)\,\cos(n_j lh))\\ =\ &2\sum_{j=1}^{M} c_j \cos(n_j mh) \sum_{l=0}^{M} p_l\, T_l\ \cos(n_j h) = 0.\end{aligned}$$

This observation leads to the linear system

$$\left(f(\cos(m+l)h) + f(\cos(m-l)h)\right)_{l,m=0}^{M-1} \mathbf{p} = -2^{-M+1}\ \left(f(\cos(m+M)h)\right)_{m=0}^{M-1}$$

to evaluate the vector $\mathbf{p} = (p_0, \ldots, p_{M-1})^T$ of Prony polynomial coefficients. It can be simply shown that the coefficient matrix of this system is invertible, provided that $\cos(n_j h)$ are pairwise distinct. Having $P(z)$ it is simple to recover the indices n_j and afterwards the coefficients c_j. □

Remark 5.6. 1. Similarly as in Sections 3.1 and 3.2, Theorem 5.5 can be generalized by taking samples $f(\cos(x_0 + kh))$, $k = -2M + 1, \ldots, 2M - 1$, and the choice of x_0 governs the number of needed function values taking into account that cosine is even.

2. A numerical algorithm for the reconstruction of sparse expansions of Chebyshev polynomials can be already found in [20]. The approach can also be transferred to Chebyshev polynomials of second, third and fourth kind, see [20]. However, in [20] the connection to shift operators with Chebyshev polynomials as eigenfunctions has not been explicitly used.

6 Reconstruction of Non-stationary Signals

Within the last years, more efforts have been made to reconstruct non-stationary signals of the form

$$f(x) = \sum_{j=1}^{M} c_j(x) \, \cos(\phi_j(x)).$$

The empirical mode decomposition method described in [11,12] is a heuristic iterative method to decompose a given non-stationary signal into certain signal components. However, this algorithm does not always provide the wanted signal components in a suitable way. If there is more a priori information on the envelope functions $c_j(x)$ and the phase functions $\phi_j(x)$ available we may be able to exploit it in a direct way. Using the Prony method with generalized shifts we will consider some models of non-stationary signals. In particular, we will study constant envelope functions and polynomial phase functions with a priori known polynomial structure.

6.1 Non-stationary Signals with Phase Functions $\phi_j(x) = \alpha_j x^p + \beta_j$

We want to recover signals of the form

$$f(x) = \sum_{j=1}^{M} c_j \, \cos(\alpha_j x^p + \beta_j), \tag{6.25}$$

where the odd integer $p > 0$ is a priori known, and where the coefficients $c_j, \alpha_j \in \mathbb{R}$ and $\beta_j \in [-\pi, \pi)$ need to be recovered. We assume here that the α_j are pairwise different and nonnegative. This last assumption is not a restriction since $\cos x$ is an even function.

First, we construct a generalized shift operator that possesses the eigenfunctions $\cos(\alpha_j x^p + \beta_j)$. We employ the operator $S_{x^p,h,-h} \colon C(\mathbb{R}) \to C(\mathbb{R})$ with $h > 0$, which is a combination of the symmetric shift operator $S_{h,-h}$ and the operator $S_{G,h}$ with $G(x) = x^p = \operatorname{sgn}(x)|x|^p$ and $G^{-1}(x) = \operatorname{sgn}(x) \sqrt[p]{|x|}$, given by

$$S_{x^p,h,-h} f(x) := \frac{1}{2} \left(f \left(\operatorname{sgn}(x^p + h) \sqrt[p]{|x^p + h|} \right) + f \left(\operatorname{sgn}(x^p - h) \sqrt[p]{|x^p - h|} \right) \right).$$

Here $\operatorname{sgn}(x)$ denotes the sign of x and is 1 for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$. We find

$$S_{x^p,h,-h} \cos(\alpha_j x^p + \beta_j) \quad = \quad \frac{1}{2} \cos \left(\alpha_j \left(\operatorname{sgn}(x^p + h) \sqrt[p]{|x^p + h|} \right)^p + \beta_j \right)$$

$$
\begin{aligned}
&+\frac{1}{2}\cos\Big(\alpha_j\left(\operatorname{sgn}(x^p-h)\sqrt[p]{|x^p-h|}\right)^p+\beta_j\Big)\\
=\;& \frac{1}{2}\Big(\cos\left(\alpha_j(x^p+h)+\beta_j\right)+\cos\left(\alpha_j(x^p-h)+\beta_j\right)\Big)\\
=\;& \cos\left(\alpha_j x^p+\beta_j\right)\,\cos(\alpha_j h).
\end{aligned}
$$

The eigenvalues $\cos(\alpha_j h)$ and $\cos(\alpha_k h)$ are pairwise different for $\alpha_j \neq \alpha_k$ if $\alpha_j, \alpha_k \in [0, \pi/h]$. We conclude

Theorem 6.1. *Let f be of the form (6.25) with known odd integer $p > 0$, and with unknown parameters $c_j \in \mathbb{R}$, $\beta_j \in [0, 2\pi)$ and pairwise different $\alpha_j \in [0, K)$ for some $K > 0$ for all $j = 1, \ldots, M$. Let $h := \pi/K$.*

1. *If the parameters β_j do not appear in the model (6.25), then f can be uniquely recovered from its signal values $f\left(\operatorname{sgn}(x_0+hk)\sqrt[p]{|x_0+hk|}\right)$, $k = -2M+1, \ldots, 2M-1$, where $x_0 \geq 0$ only needs to satisfy $\cos(\alpha_j x_0) \neq 0$ for $j = 1, \ldots, M$. Taking $x_0 = 0$, the function values $f\left(\sqrt[p]{hk}\right)$, $k = 0, \ldots, 2M-1$ are already sufficient for the reconstruction of the parameters α_j, c_j, $j = 1, \ldots, M$.*
2. *If the nonzero parameters β_j appear in (6.25), then the parameters α_j, $j = 1, \ldots, M$, can be recovered from signal values $f\left(\operatorname{sgn}(x_0+hk)\sqrt[p]{|x_0+hk|}\right)$, $k = -2M+1, \ldots, 2M-1$ in a first step. Using in a second step additionally the signal values $f\left(\operatorname{sgn}(x_0+hk-\pi/(2\alpha_j)\sqrt[p]{|x_0+hk-\pi/(2\alpha_j)|}\right)$ for $k = -M+1, \ldots, M-1$, the parameters c_j and β_j can be reconstructed. The value x_0 needs to be chosen such that $\cos(\alpha_j x_0 + \beta_j) \neq 0$ for $j = 1, \ldots, M$.*

Proof. We consider the Prony polynomial of the form

$$
P(z) := \prod_{j=1}^{M}(z-\cos(\alpha_j h)) = \sum_{l=0}^{M} p_l\, T_l(z),
$$

where p_l denote the coefficients in the representation of $P(z)$ using the Chebyshev polynomials $T_l(z) = \cos(l \arccos z)$ with $p_M = 2^{-M+1}$. Then we observe that

$$
\begin{aligned}
&\sum_{l=0}^{M} p_l\Big(S_{x^p,h(m+l),-h(m+l)}f(\sqrt[p]{x_0})+S_{x^p,h(m-l),-h(m-l)}f(\sqrt[p]{x_0})\Big)\\
=\;& \sum_{l=0}^{M-1} p_l\frac{1}{2}\Big(f(\sqrt[p]{x_0+h(m+l)})+f(\operatorname{sgn}(x_0-h(m+l))\sqrt[p]{|x_0-h(m+l)|})\\
&\quad +f(\operatorname{sgn}(x_0+h(m-l))\sqrt[p]{|x_0+h(m-l)|})\\
&\quad +f(\operatorname{sgn}(x_0-h(m-l))\sqrt[p]{|x_0-h(m-l)|})\Big)\\
=\;& \sum_{l=0}^{M} p_l\sum_{j=1}^{M} c_j\frac{1}{2}\Big(\cos\alpha_j((x_0+h(m+l))+\beta_j)+\cos(\alpha_j(x_0-h(m+l))+\beta_j)\\
&\quad +\cos\alpha_j((x_0+h(m-l))+\beta_j)+\cos(\alpha_j(x_0-h(m-l))+\beta_j)\Big)\\
=\;& 2\sum_{j=1}^{M} c_j\cos(\alpha_j x_0+\beta_j)\cos(\alpha_j hm)\sum_{l=0}^{M} p_l\cos(\alpha_j hl)\\
=\;& 2\sum_{j=1}^{M} c_j\cos(\alpha_j x_0+\beta_j)\cos(\alpha_j hm)\sum_{l=0}^{M} p_l T_l(\cos(\alpha_j h)) = 0
\end{aligned}
$$

for all $m = 0, \ldots, M-1$. We obtain the linear system

$$\mathbf{Hp} = -2^{-M+1}\,\mathbf{f}_M$$

with $\mathbf{p} = (p_0, \ldots, p_{M-1})^T$ and with

$$\begin{aligned}\mathbf{H} = \Big(& f\Big(\sqrt[p]{x_0+h(m+l)}\Big)+f\Big(\operatorname{sgn}(x_0-h(m+l))\sqrt[p]{|x_0-h(m+l)|}\Big)\\ &+f\Big(\operatorname{sgn}(x_0+h(m-l))\sqrt[p]{|x_0+h(m-l)|}\Big)+f\Big(\operatorname{sgn}(|x_0-h(m-l)|)\sqrt[p]{|x_0-h(m-l)|}\Big)\Big)_{m,l=0}^{M-1},\\ \mathbf{f}_M = \Big(& f\Big(\sqrt[p]{x_0+h(m+M)}\Big)+f\Big(\operatorname{sgn}(x_0-h(m+M))\sqrt[p]{|x_0-h(m+M)|}\Big)\\ &+f\Big(\operatorname{sgn}(x_0+h(m-M))\sqrt[p]{|x_0+h(m-M)|}\Big)+f\Big(\operatorname{sgn}(x_0-h(m-M))\sqrt[p]{|x_0-h(m-M)|}\Big)\Big)_{m=0}^{M-1},\end{aligned}$$

to compute the coefficients p_l of the Prony polynomial in Chebyshev representation. The coefficient matrix of this system has the form

$$\mathbf{H} = \mathbf{V}\,\mathrm{diag}(c_j\cos(\alpha_j x_0+\beta_j))_{j=1}^{M}\,\mathbf{V}^T \tag{6.26}$$

with the generalized Vandermonde matrix $\mathbf{V} = (\cos(\alpha_j hl))_{l,j=0}^{M-1} = (T_l(\cos(\alpha_j h)))_{l,j=0}^{M-1}$ as in (3.13). These Vandermonde matrices are invertible if the values $\cos(\alpha_j h)$ are pairwise different, which is ensured by the choice of h. The invertibility of the diagonal matrix is ensured if $c_j \neq 0$ and if $\alpha_j x_0 + \beta_j$ is not of the form $\pi(k+1/2)$ for some $k \neq 0$. In particular, for vanishing β_j we can simply use $x_0 = 0$ and need only the function values $f(lh)$, $l = 0, \ldots 2M-1$ to recover α_j, since f in (6.25) is an even function.

Having found the parameters α_j by the described procedure, in the case of vanishing β_j we can simply obtain the values $c_j\cos(\alpha_j x_0)$ by solving the linear system

$$\begin{aligned}&\sum_{j=1}^{M} c_j(\cos(\alpha_j(x_0+lh))+\cos(\alpha_j(x_0-lh))\\ =\;& 2\sum_{j=1}^{M} c_j\cos(\alpha_j x_0)\cos(\alpha_j lh) = f(\sqrt[p]{x_0+lh}) + f(\operatorname{sgn}(x_0-lh)\sqrt[p]{|x_0-lh|})\end{aligned}$$

for $l = 0, \ldots, M-1$, where the coefficient matrix is the same generalized Vandermonde matrix $\mathbf{V}$ as in (6.26).

If the model contains nonvanishing parameters β_j, then we have to solve the system

$$\begin{aligned}&\sum_{j=1}^{M} c_j(\cos(\alpha_j(x_0+lh)+\beta_j)+\cos(\alpha_j(x_0-lh)+\beta_j)\\ =\;& 2\sum_{j=1}^{M} c_j\cos(\alpha_j x_0+\beta_j)\cos(\alpha_j lh) = f(\sqrt[p]{x_0+lh}) + f(\operatorname{sgn}(x_0-lh)\sqrt[p]{|x_0-lh|})\end{aligned}$$

to obtain $d_j := c_j\cos(\alpha_j x_0+\beta_j)$. In addition, we have to solve

$$\begin{aligned}&\sum_{j=1}^{M} c_j\Big(\cos(\alpha_j(x_0+lh)-\frac{\pi}{2}+\beta_j)+\cos(\alpha_j(x_0-lh)-\frac{\pi}{2}+\beta_j)\Big)\\ =\;& 2\sum_{j=1}^{M} c_j\cos(\alpha_j x_0-\frac{\pi}{2}+\beta_j)\cos(\alpha_j lh)\end{aligned}$$

$$= f\left(\mathrm{sgn}(x_0 - \tfrac{\pi}{2\alpha_j} + lh)\sqrt[p]{|x_0 - \tfrac{\pi}{2\alpha_j} + lh|}\right) + f\left(\mathrm{sgn}(x_0 - \tfrac{\pi}{2\alpha_j} - lh)\sqrt[p]{|x_0 - \tfrac{\pi}{2\alpha_j} - lh|}\right)$$

for $l = 0, \ldots, M-1$, to find $\tilde{d}_j := c_j \cos(\alpha_j x_0 - \frac{\pi}{2} + \beta_j) = \sin(\alpha_j x_0 + \beta_j)$ for $j = 1, \ldots, M$. Thus, we conclude

$$c_j = \sqrt{d_j^2 + \tilde{d}_j^2}, \qquad \beta_j = \arg(d_j + \mathrm{i}\tilde{d}_j) - \alpha_j x_0 \bmod 2\pi.$$

□

6.2 Non-stationary Signals with Quadratic Phase Functions

Finally, we consider signals of the form

$$f(x) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j) \tag{6.27}$$

with parameters $c_j \in \mathbb{R}$, $\alpha_j \in (-T, T)$ for some $T > 0$, and $\beta_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $j = 1, \ldots, M$. This model can be rewritten as

$$\begin{aligned} f(x) &= \sum_{j=1}^{M} (c_j \cos\beta_j)\cos(x^2 + \alpha_j x) - (c_j \sin\beta_j)\sin(x^2 + \alpha_j x) \\ &= \sum_{j=1}^{M} c_{j,1}\cos(x^2 + \alpha_j x) - c_{j,2}\sin(x^2 + \alpha_j x) \\ &= \sum_{j=1}^{M} \left(\frac{c_{j,1} + \mathrm{i}\, c_{j,2}}{2}\right) \mathrm{e}^{\mathrm{i}(x^2 + \alpha_j x)} + \left(\frac{c_{j,1} - \mathrm{i}\, c_{j,2}}{2}\right) \mathrm{e}^{-\mathrm{i}(x^2 + \alpha_j x)} \\ &= \sum_{j=1}^{M} b_j\, \mathrm{e}^{\mathrm{i}((x+\alpha_j/2)^2 - \alpha_j^2/4)} + \overline{b}_j\, \mathrm{e}^{-\mathrm{i}((x+\alpha_j/2)^2 - \alpha_j^2/4)} \\ &= 2\,\mathrm{Re} \sum_{j=1}^{M} b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}\, \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2} \\ &= 2\,\mathrm{Re} \sum_{j=1}^{M} d_j \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2}, \end{aligned} \tag{6.28}$$

where we have used the substitutions $c_{j,1} := c_j \cos\beta_j$, $c_{j,2} := c_j \sin\beta_j$, $b_j := \left(\frac{c_{j,1} + \mathrm{i}c_{j,2}}{2}\right)$, and $d_j := b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}$. Similarly, we observe that

$$\begin{aligned} \tilde{f}(x) &= \sum_{j=1}^{M} c_j \sin(x^2 + \alpha_j x + \beta_j) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j - \frac{\pi}{2}) \\ &= 2\,\mathrm{Im} \sum_{j=1}^{M} d_j \mathrm{e}^{\mathrm{i}(x+\alpha_j/2)^2} \end{aligned}$$

with $d_j = b_j \mathrm{e}^{-\mathrm{i}\alpha_j^2/4}$. The model is therefore closely related to the model in (4.18) (with $\beta = -\mathrm{i}$). We conclude

Theorem 6.2. *Assume that* $\beta_j \in [-\frac{\pi}{2}, \frac{\pi}{2})$ *and that* $\alpha_j \in (-T, T)$, $j = 1, \ldots, M$, *for some* $T > 0$ *and let* $0 < h \leq \frac{\pi}{T}$. *Then,* f *in* (6.27) *can be reconstructed using the* $2M$ *sample values* $f(x_0 + hk)$, $k = 0, \ldots, 2M-1$ *and the* $2M$ *sample values* $\tilde{f}(x_0 + hk)$, *where* $x_0 \in \mathbb{R}$ *is an arbitrary real number.*

Proof. Considering the function $h(x) = f(x) + \mathrm{i}\,\tilde{f}(x)$, we can apply Theorem 4.1 to recover the parameters d_j and α_j for $j = 1, \ldots, M$. The original parameters c_j and β_j, $j = 1, \ldots, M$, are then obtained using the relations

$$b_j = d_j \mathrm{e}^{\mathrm{i}\alpha_j^2/4},\; c_{j,1} = 2\mathrm{Re}\, b_j,\; c_{j,2} = 2\mathrm{Im}\, b_j,\; |c_j| = 2|b_j|,\; \beta_j = \arg(b_j),\; \operatorname{sgn} c_j = \operatorname{sgn} c_{j,1}.$$

□

7 Numerical examples

In this section we want to illustrate the recovery method for non-stationary signals with some examples.

Example 7.1. We start with considering the recovery of an expansion of complex shifted Gaussians,

$$f(x) = \sum_{j=1}^{M} c_j\, g(x - \alpha_j) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{-\beta(x-\alpha_j)^2},$$

with $M = 5$, $g(x) = \mathrm{e}^{\mathrm{i}x^2}$, i.e., $\beta = -\mathrm{i}$, and with complex coefficients c_j and real shifts α_j given in Table 1. The coefficients have been obtained by applying a uniform random choice from the intervals $(-5, 5) + \mathrm{i}(-2, 2)$ for c_j and from $(-\pi, \pi)$ for α_j. For reconstruction, we have used the 10 signal values $f(j)$, $j = -1, \ldots, 8$, indicated by $*$ in Figure 1 (left). The maximal error for recovering the parameters is given by

$$\max_j |c_j - \tilde{c}_j| = 1.5 \cdot 10^{-10}, \qquad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.5 \cdot 10^{-12},$$

where $\tilde{c}_j$ and $\tilde{\alpha}_j$ denote the computed parameters.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$\mathrm{Re}\, c_j$	-2.37854	-4.55545	2.54933	-2.57214	-0.57597
$\mathrm{Im}\, c_j$	0.75118	-0.56308	0.94536	0.42117	0.73366
α_j	0.64103	-0.18125	-1.50929	-0.53137	-0.23778

Table 1 *Coefficients* $c_j \in \mathbb{C}$ *and* $\alpha_j \in \mathbb{R}$ *for the expansion of shifted Gaussians in Example 7.1.*

Example 7.2. Next, we consider the recovery of a Gabor expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j\, \mathrm{e}^{2\pi \mathrm{i} x \alpha_j} g(x - s_j),$$

with $g(x) = \mathrm{e}^{-x^2/2}$, $M = 6$, real coefficients c_j, α_j and s_j as given in Table 2. The coefficients have been obtained by applying a uniform random sampling from

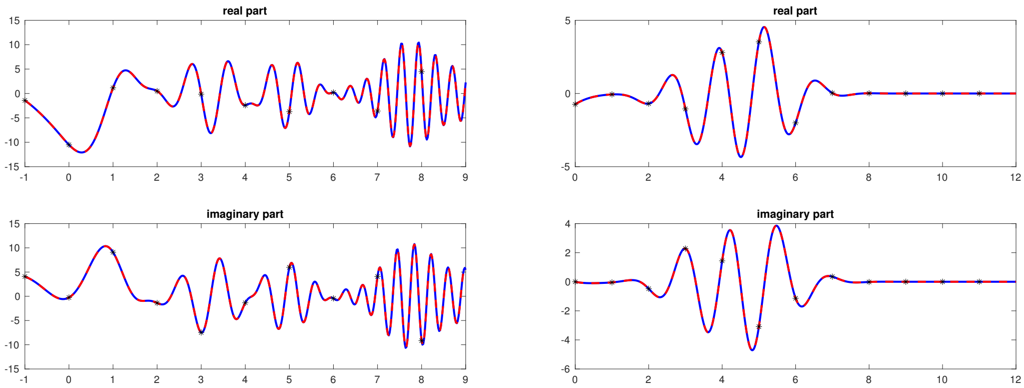

Figure 1 *Left: Real and imaginary part of the signal f(x) consisting of shifted Gaussians given in Example 7.1. Right: Real and imaginary part of the Gabor expansion considered in Example 7.2. Stars indicate the used signal values.*

the intervals $(-10, 10)$ for c_j, from $(-5, 5)$ for s_j and from $(0, 1)$ for α_j. For the reconstruction we have used the 12 signal values $f(l)$, $l = 0, \ldots, 11$ indicated by $*$ in Figure 1 (right). For the errors we obtain in this example

$$\max_j |c_j - \tilde{c}_j| = 1.3 \cdot 10^{-6}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| < 3.3 \cdot 10^{-7}, \quad \max_j |s_j - \tilde{s}_j| < 3.1 \cdot 10^{-6},$$

where $\tilde{c}_j$, $\tilde{\alpha}_j$, $\tilde{s}_j$ denote the parameters computed by the numerical procedure.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
c_j	0.0777	2.9361	-3.8450	-7.2255	-0.4885	-2.7508
s_j	-1.9918	-4.3941	4.8090	-2.1337	3.0082	3.9611
α_j	0.7881	0.7802	0.6685	0.1335	0.0215	0.5598

Table 2 *Coefficients c_j, α_j, $s_j \in \mathbb{R}$ for the Gabor expansion in Example 7.2.*

Example 7.3. Finally, we consider two examples for the model with quadratic phase function

$$f(x) = \sum_{j=1}^{M} c_j \cos(x^2 + \alpha_j x + \beta_j)$$

in (6.27). In Figure 2 (left), we display a signal with $M = 3$ components with corresponding parameters given in Table 3. For reconstruction, we have used the signal values $f(l)$, $l = 0, \ldots, 5$. In Figure 2 (right), we give a second example with coefficients given in Table 4. Here, $M = 6$, and we have used the signal values $f(-1 + \frac{5l}{12})$, $l = 0, \ldots, 11$ for reconstruction. The coefficients have been obtained by applying a uniform random sampling from the intervals $(-1, 5)$ for c_j in the first and from $(0, 5)$ in the second example, from $(-\pi, \pi)$ for α_j and from $(-\pi/2, \pi/2)$ for β_j (for both examples). The reconstruction errors in the first example with $M = 3$ terms are

$$\max_j |c_j - \tilde{c}_j| = 1.3 \cdot 10^{-8}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.3 \cdot 10^{-11}, \quad \max_j |\beta_j - \tilde{\beta}_j| = 1.7 \cdot 10^{-9}.$$

For the second example with $M = 6$ we obtain

$$\max_j |c_j - \tilde{c}_j| = 7.7 \cdot 10^{-5}, \quad \max_j |\alpha_j - \tilde{\alpha}_j| = 3.6 \cdot 10^{-6}, \quad \max_j |\beta_j - \tilde{\beta}_j| = 5.5 \cdot 10^{-5}.$$

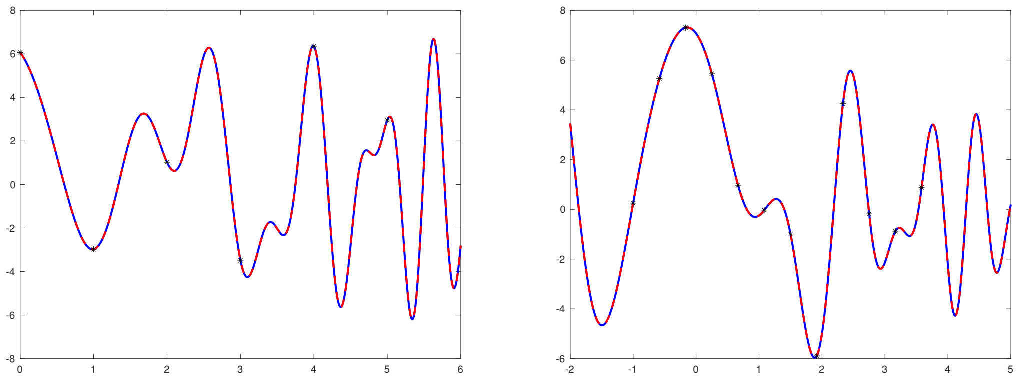

Figure 2 *Left: non-stationary signal f(x) with quadratic phase function with parameters given in Table 3. Right: non-stationary signal f(x) with quadratic phase function with parameters given in Table 4. Stars indicate the used signal values.*

	$j=1$	$j=2$	$j=3$
c_j	-0.1835	4.2157	2.478
α_j	0.3132	2.2308	2.2181
β_j	0.3834	-0.4682	0.0416

Table 3 *Coefficients c_j, $\alpha_j, \beta_j \in \mathbb{R}$ for the non-stationary signal in Figure 2 (left).*

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
c_j	3.8940	2.117	0.4541	1.3323	0.7682	1.4050
α_j	-0.3764	0.1705	-0.2675	2.3585	0.1134	2.7873
β_j	0.4326	1.4378	-0.8145	0.5533	-0.6626	0.5397

Table 4 *Coefficients c_j, α_j, $s_j \in \mathbb{R}$ for the non-stationary signal in Figure 2 (right).*

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