A general solution algorithm for mixed continuous and combinatorial optimization problems

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Abstract

Geometric branch-and-bound techniques are popular solution algorithms for non-convex global optimization problems. Although several quite similar approaches can be found in the literature, they differ in the bounds they use. In a recent work we introduced the rate of convergence which allows theoretical results about the different bounds, see Schöbel and Scholz (2010b).

The aim of this paper is to extend the geometric branch-and-bound methods to mixed continuous and combinatorial optimization problems, i.e. to objective functions with some continuous and some combinatorial variables. In particular, we derive several bounding operations and theoretical results about their rate of convergence. The suggested techniques are demonstrated on some facility location problems in which we succeed in finding exact optimal solutions.

Keywords: global optimization, combinatorial optimization, non-convex optimization, branch-and-bound methods, facility location problems.

1 Introduction

Geometric branch-and-bound methods are popular solution algorithms for continuous and non-convex optimization problems with a small number of variables, see
e.g. Horst et al. (2000) or Tuy (1998). These techniques find applications for example in facility location problems, see Plastria (1992), Drezner and Suzuki (2004), Blanquero and Carrizosa (2009), or Schöbel and Scholz (2010a) for general branch-and-bound solution approaches in location theory and e.g. Drezner and Drezner (2007), Fernández et al. (2007), and Blanquero et al. (2009) among plenty of other references for some specific location problems solved by these techniques.

The most important task throughout the branch-and-bound algorithm is the calculation of lower bounds on the objective function for some smaller rectangles or boxes. Different techniques to do so are collected in Schöbel and Scholz (2010b). Therein, the rate of convergence is introduced that allows to evaluate the quality of some well-known bounding operations.

All the above mentioned techniques are dealing with pure continuous objective functions. The contribution of this paper is to extend the method to mixed continuous and combinatorial optimization problems. To this end, we derive some general bounding operations and we present theoretical results about the rate of convergence similar to Schöbel and Scholz (2010b). Moreover, we discuss an extension of the method which leads to exact optimal solutions under certain conditions given below. We implemented the approach and applied it to some facility location problems. The numerical results show that we succeeded in finding exact optimal solutions.

The remainder of the paper is organized as follows. In the next section we will summarize notations and basic concepts which we will use throughout the paper. Section 3 presents the geometric branch-and-bound prototype algorithm for mixed continuous and combinatorial optimization problems before we prove the convergence of the suggested algorithm in Section 4. In Section 5 we discuss some general bounding operations and results concerning the rate of convergence are given. Next, in Section 6 we suggest an extension of the algorithm which leads to exact optimal solutions under certain conditions. In the following two sections (Sections 7 and 8) we apply the proposed techniques to some facility location problems and numerical results are given. Finally, a brief conclusion can be found in Section 9.

2 Notations and basic concepts

Throughout the paper, we will use the following notations.

Notation 1. A box or hyperrectangle with sides parallel to the axes is denoted by

\[ X = [x_1, \pi_1] \times \ldots \times [x_n, \pi_n] \subset \mathbb{R}^n. \]

The diameter of a box \( X \subset \mathbb{R}^n \) is

\[ \delta(X) = \max\{\|x - x'\|_2 : x, x' \in X\} = \sqrt{(\pi_1 - x_1)^2 + \ldots + (\pi_n - x_n)^2} \]
and the **center** of a box $X \subset \mathbb{R}^n$ is defined by

$$c(X) = \left( \frac{1}{2}(x_1 + \pi_1), \ldots, \frac{1}{2}(x_n + \pi_n) \right).$$

Our goal is to minimize a mixed continuous and combinatorial function

$$f : \mathbb{R}^n \times \mathbb{Z}^m \to \mathbb{R}.$$  

We assume a feasible area $X \times \Pi$ where $X \subset \mathbb{R}^n$ is a box with sides parallel to the axes and $\Pi \subset \mathbb{Z}^m$ with $|\Pi| < \infty$. In order to apply the algorithm presented in the next section, we further need the following definition which is an extension of the bounding operation defined in Schöbel and Scholz (2010b) for mixed continuous and combinatorial functions.

**Definition 2.** Let $X \subset \mathbb{R}^n$ be a box, $\Pi \subset \mathbb{Z}^m$ with $|\Pi| < \infty$, and consider

$$f : X \times \Pi \to \mathbb{R}.$$  

A **bounding operation** is a procedure to calculate for any subbox $Y \subset X$ a lower bound $LB(Y) \in \mathbb{R}$ with

$$LB(Y) \leq f(x, \pi) \quad \text{for all} \quad x \in Y \text{ and } \pi \in \Pi$$

and to specify a point $r(Y) \in Y$ and a point $\kappa(Y) \in \Pi$.

### 3 The prototype algorithm

The algorithm suggested in this section is a generalization of the big cube small cube method presented in Schöbel and Scholz (2010a). Here, we extend the problem to mixed integer minimization problems, i.e. to problems which contain continuous and integer variables.

Given $X \subseteq \mathbb{R}^n$ and $\Pi \subseteq \mathbb{Z}^m$ the goal of our approach is to minimize

$$f : X \times \Pi \to \mathbb{R}.$$  

This is done using the following algorithmic scheme with an absolute accuracy of $\varepsilon > 0$. 

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3
(1) Let \( \mathcal{X} \) be a list of boxes and initialize \( \mathcal{X} := \{X\} \).

(2) Apply the bounding operation to \( X \) and set \( UB := f(r(X), \kappa(X)) \).

(3) If \( \mathcal{X} = \emptyset \), the algorithm stops. Else set
\[
\delta_{\text{max}} := \max\{\delta(Y) : Y \in \mathcal{X}\}.
\]

(4) Select a box \( Y \in \mathcal{X} \) with \( \delta(Y) = \delta_{\text{max}} \) and split it into \( s \) subboxes \( Y_1 \) to \( Y_s \) such that \( Y = Y_1 \cup \ldots \cup Y_s \).

(5) Set \( \mathcal{X} = (\mathcal{X} \setminus Y) \cup \{Y_1, \ldots, Y_s\} \).

(6) Apply the bounding operation to \( Y_1 \) to \( Y_s \) and set
\[
UB = \min\{UB, f(r(Y_1), \kappa(Y_1)), \ldots, f(r(Y_s), \kappa(Y_s))\}.
\]

(7) For all \( Z \in \mathcal{X} \), if \( LB(Z) + \epsilon \geq UB \) set \( \mathcal{X} = \mathcal{X} \setminus Z \). If \( UB \) has not changed it is sufficient to check only the subboxes \( Y_1 \) to \( Y_s \).

(8) Whenever possible, apply some further discarding test, see Section 6.

(9) Return to Step (3).

We remark that it is a non-trivial task to calculate the lower bound \( LB(Y) \) as required throughout the algorithm. We will address this question in Section 5.

### 4 Theoretical results

In order to evaluate the quality of bounding operations, we extend the definition for the rate of convergence given in Schöbel and Scholz (2010b).

**Definition 3.** Let \( X \subset \mathbb{R}^n \) be a box, let \( \Pi \subset \mathbb{Z}^m \) with \( |\Pi| < \infty \), and \( f : X \times \Pi \rightarrow \mathbb{R} \). Furthermore, consider the minimization problem
\[
\min_{x \in X, \pi \in \Pi} f(x, \pi).
\]

We say a bounding operation has the **rate of convergence** \( p \in \mathbb{N} \) if there exists a fixed constant \( C > 0 \) such that
\[
f(r(Y), \kappa(Y)) - LB(Y) \leq C \cdot \delta(Y)^p
\]
for all boxes \( Y \subset X \).
As shown in Schöbel and Scholz (2010b) for pure continuous objective functions, the larger the rate of convergence the smaller the number of iterations needed throughout the algorithm.

The next theorem shows that the proposed algorithm terminates after a finite number of iterations if the bounding operation has a rate of convergence of at least one.

**Theorem 1.** Let \( X \subset \mathbb{R}^n \) be a box, let \( \Pi \subset \mathbb{Z}^m \) with \( |\Pi| < \infty \), and \( f : X \times \Pi \rightarrow \mathbb{R} \). Furthermore, consider the minimization problem

\[
\min_{x \in X} \min_{\pi \in \Pi} f(x, \pi)
\]

and assume a bounding operation with a rate of convergence of \( p \geq 1 \).

Then the proposed algorithm terminates after a finite number of iterations.

**Proof.** Define the objective function

\[
g(x) := \min_{\pi \in \Pi} f(x, \pi)
\]

and consider the global optimization problem

\[
\min_{x \in X} g(x)
\]

with a function \( g \) containing only continuous variables. Note that this problem is equivalent to the minimization of \( f(x, \pi) \) on \( X \times \Pi \). Since \( f(r(Y), \kappa(Y)) - LB(Y) \leq C \cdot \delta(Y)^p \) for any subbox \( Y \subset X \), we also find

\[
g(r(Y)) - LB(Y) = \min_{\pi \in \Pi} f(r(Y), \pi) - LB(Y) \leq f(r(Y), \kappa(Y)) - LB(Y) \leq C \cdot \delta(Y)^p.
\]

This yields a bounding operation for the continuous function \( g : X \rightarrow \mathbb{R} \) with a rate of convergence of \( p \geq 1 \). From Schöbel and Scholz (2010b) we hence know that the algorithm terminates after a finite number of iterations. \( \square \)

### 5 Bounding operations

Our general scheme is applicable to all mixed-integer programming problems. However, example problems that we have in mind contain a combinatorial part which can be easily solved once the continuous variables are fixed. Such problems include the uncapacitated facility location problem as a classical example for a mixed-integer
programming problem (see ?) in which not only the location of the facilities but also
the assignment between the facilities and its customers has to be determined. Other
typical problems of this type are hub-location problems (see ?) or the Ordered Weber
problem (see Nickel and Puerto (2005)). This type of problem also appears in
robust statistics, for example in trimmed regression problems where outliers should
be neglected and the quality of the estimation is only measured to a fixed percentage
of the (best fitted) data points.

Our main goal in this section is to derive some bounding operations for mixed integer
optimization problems with a rate of convergence of \( p \geq 1 \). Therefore, we assume
that some combinatorial problems can be easily solved as discussed above.

### 5.1 Concave bounding operation

For the concave bounding operation, we assume that we are in a position to solve
the combinatorial problem

\[
\min_{\pi \in \Pi} f(x, \pi)
\]

for any fixed \( x \in X \). Moreover, for any \( \pi \in \Pi \) we assume that the function

\[
f_\pi(x) := f(x, \pi)
\]

is concave on \( X \). Next, define

\[
g(x) := \min_{\pi \in \Pi} f_\pi(x) = \min_{\pi \in \Pi} f(x, \pi)
\]

and note that \( g \) is concave since the minimum of a finite number of concave functions
is concave. Denoting by \( V(Y) \) the \( 2^n \) vertices of \( Y \), we then find the concave
bounding operation

\[
LB(Y) = \min_{x \in Y} g(x) = \min_{v \in V(Y)} g(v),
\]

\[
r(Y) \in \arg \min_{v \in V(Y)} g(v),
\]

\[
\kappa(Y) \in \arg \min_{\pi \in \Pi} f(r(Y), \pi).
\]

Formally, we obtain the following result.

**Theorem 2.** Consider \( f : X \times \Pi \to \mathbb{R} \) and assume that for any fixed \( \pi \in \Pi \) the
function \( f_\pi(x) = f(x, \pi) \) is concave.

Then the concave bounding operation has a rate of convergence of \( p = \infty \).

**Proof.** Since

\[
f(r(Y), \kappa(Y)) = \min_{x \in Y, \pi \in \Pi} f(x, \pi) = \min_{x \in Y} g(x) = LB(Y)
\]
we have
\[ f(r(Y), \kappa(Y)) - LB(Y) = 0 \]
for all subboxes \( Y \subset X \).

5.2 Location bounding operation

The idea of the location bounding operation for pure continuous location problems is given in Plastria (1992).

Assume that the objective function can be written as
\[ f(x, \pi) = h(\pi, d(a_1, x), \ldots, d(a_m, x)) \]
where \( a_1, \ldots, a_m \in \mathbb{R}^n \) are some given demand points and \( d(a, x) \) is a given distance function. Furthermore, we assume that we can solve problems of the form
\[
\min \{ h(\pi, z) : \pi \in \Pi \text{ and } l_k \leq z_k \leq u_k \text{ for } k = 1, \ldots, m \}.
\]
where \( l_k, u_k \in \mathbb{R} \) are some given constants for \( k = 1, \ldots, m \). In order to calculate a lower bound \( LB(Y) \) for an arbitrary box \( Y \subset X \), suppose that the values
\[
d_{k}^{\min}(Y) = \min \{ d(a_k, x) : x \in Y \},
\]
\[
d_{k}^{\max}(Y) = \max \{ d(a_k, x) : x \in Y \}
\]
for \( k = 1, \ldots, m \) are easily derived. This is the case if \( d \) is a monotone norm or a polyhedral gauge, see Plastria (1992). We then have the location bounding operation
\[
LB(Y) = \min \{ h(\pi, z) : \pi \in \Pi \text{ and } d_{k}^{\min}(Y) \leq z_k \leq d_{k}^{\max}(Y) \text{ for } k = 1, \ldots, m \},
\]
\[
r(Y) = c(Y),
\]
\[
\kappa(Y) \in \arg \min_{\pi \in \Pi} f(c(Y), \pi).
\]

Note that this lower bound results in exactly the same bound if we are using tools from interval analysis for the continuous variables \( x \), namely the natural interval extension.

Theorem 3. Assume that
\[
f(x, \pi) = h(\pi, d(a_1, x), \ldots, d(a_m, x))
\]
where \( h : \Pi \times \mathbb{R}^m \to \mathbb{R} \) is for all \( \pi \in \Pi \) a Lipschitzian function in the last \( m \) variables with constant \( L_\pi \leq L_{\text{max}} \) and assume that \( d \) is a norm.

Then the bounding operation for location problems has a rate of convergence of \( p = 1 \).
Proof. Using the function
\[ g(x) = \min_{\pi \in \Pi} f(x, \pi) \]
the proof is analogous to the result in Plastria (1992) and therefore omitted here. □

5.3 D.c. bounding operation

The idea of the d.c. bounding operation is to reduce the problem to the concave bounding operation as follows, see e.g. Tuy (1998) or Tuy and Horst (1988) for d.c. decompositions of continuous functions.

We assume that for all \( \pi \in \Pi \) a d.c. decomposition of \( f_\pi(x) = f(x, \pi) \) can be constructed, i.e.
\[ f_\pi(x) = g_\pi(x) - h_\pi(x) \]
where \( g_\pi \) and \( h_\pi \) are convex functions. Consider an arbitrary subbox \( Y \subset X \) and \( c = c(Y) \). Then, for any subgradient \( \xi_\pi \in \mathbb{R}^n \) of \( g_\pi \) at \( c \) we obtain
\[ a_\pi(x) := g_\pi(c) + \xi^T_\pi \cdot (x - c) - h_\pi(x) \leq f_\pi(x) \]
for all \( x \in Y \) and \( a_\pi \) is concave. Hence,
\[ q(x) := \min_{\pi \in \Pi} a_\pi(x) \leq \min_{\pi \in \Pi} f_\pi(x) = f(x, \pi) \]
for all \( x \in Y, \pi \in \Pi \) and also \( q \) is concave. We now apply the concave bounding operation on \( q \) and we obtain the d.c. bounding operation
\[ LB(Y) := \min_{v \in V(Y)} q(v), \]
\[ r(Y) \in \arg \min_{v \in V(Y)} q(v), \]
\[ \kappa(Y) \in \arg \min_{\pi \in \Pi} a_\pi(r(Y)) \]
where \( V(Y) \) is again the set of the \( 2^n \) vertices of \( Y \). Note that
\[ LB(Y) = \min_{\pi \in \Pi} a_\pi(r(Y)) = a_{\kappa(Y)}(r(Y)). \]

In contrast to the concave bounding operation, we assume for the d.c. bounding operation that for any subbox \( Y \subset X \) and for all \( x \in V(Y) \) we are able to solve the combinatorial problem
\[ \min_{\pi \in \Pi} a_\pi(x). \]

Theorem 4. Consider \( f : X \times \Pi \to \mathbb{R} \) such that for all \( \pi \in \Pi \) a d.c. decomposition of
\[ f_\pi(x) = g_\pi(x) - h_\pi(x) \]
is known and assume that \( g_\pi \) is twice continuously differentiable on \( X \) for all \( \pi \in \Pi \). Then the d.c. bounding operation has a rate of convergence of \( p = 2 \).

**Proof.** For any fixed \( \pi \in \Pi \) we know from Schöbel and Scholz (2010b) that

\[
LB(Y) := \min_{v \in V(Y)} a_\pi(v),
\]

\[
r(Y) \in \arg\min_{v \in V(Y)} q(v)
\]

is a bounding operation for \( f_\pi \), i.e.

\[
f_\pi(x) - a_\pi(x) \leq C_\pi \cdot \delta(Y)^2
\]

for some \( C_\pi > 0 \) which do not depend on \( Y \). Defining

\[
C_{\text{max}} := \max\{C_\pi : \pi \in \Pi\},
\]

we obtain

\[
f(r(Y), \kappa(Y)) - LB(Y) = f_{\kappa(Y)}(r(Y)) - a_{\kappa(Y)}(r(Y)) \leq C_{\kappa(Y)} \cdot \delta(Y)^2 \leq C_{\text{max}} \cdot \delta(Y)^2
\]

which proves the theorem.

\[\square\]

### 6 An exact solution method: If the continuous problem can be solved

For a given mixed integer optimization problem, we now want to extend the prototype algorithm in such a way that we obtain an exact global minimum. This will be done by the following further discarding test, see Step (8) in the prototype algorithm in Section 3.

We assume that for any fixed \( \pi \in \Pi \) we are in a position to solve the pure continuous problem

\[
\min_{x \in X} f(x, \pi).
\]

**Definition 4.** Let \( Y \subset X \) be a subbox of \( X \). Then a set \( \Omega(Y) \subset \Pi \) is said to be a **combinatorial dominating set** for \( Y \) if

\[
\min_{x \in Y} f(x, \pi) = \min_{\pi \in \Omega(Y)} f(x, \pi).
\]
In other words, if we know that \( x \in Y \), we only have to consider \( \pi \in \Omega(Y) \) and can neglect all \( \pi \in \Pi \setminus \Omega(Y) \). Examples for sets \( \Omega(Y) \) are given in the next section for some example problems.

If a combinatorial dominating set is known for every subbox \( Y \subset X \), we suggest to use the following further discarding test for a given parameter \( M \in \mathbb{N} \).

\[
(8) \quad \text{For } i = 1, \ldots, s, \text{ if } |\Omega(Y_i)| \leq M \text{ set } X = X \setminus Y_i \text{ and for all } \pi \in \Omega(Y_i) \text{ solve the minimization problem}
\]

\[
t_{i,\pi} = \min_{x \in X} f(x, \pi)
\]

and set \( UB = \min\{UB, t_{i,\pi}\} \).

Note that the choice of the parameter \( M \in \mathbb{N} \) strongly depends on the given objective function. If \( M \) is too small, the condition \(|\Omega(Y_i)| \leq M\) might be satisfied only rarely. On the other hand, if \( M \) is too large, it could be too expensive to solve all \(|\Omega(Y_i)|\) minimization problems. Therefore, we suggest to start the algorithm with a small value for \( M \) and increase \( M \) throughout the algorithm if the method does not terminate.

Many practical mixed continuous and combinatorial optimization problems admit the property that both, the combinatorial and the continuous part can be solved exactly if the respective other variables are known. This is the case for the truncated Weber problem of the next section, for many Ordered Weber problems, and also for many applications in robust statistics. In such a case we can use our results to find an exact global minimum by setting \( \varepsilon = 0 \), i.e. we only delete subboxes \( Z \in X \) with \( LB(Z) \geq UB \) in Step (7) of the prototype algorithm. Formally, we obtain the following result.

**Lemma 5.** Let \( X \subset \mathbb{R}^n \) be a box, let \( \Pi \subset \mathbb{Z}^m \) with |\( \Pi \)\| < \( +\infty \), and \( f : X \times \Pi \rightarrow \mathbb{R} \). Moreover, assume that for any fixed \( \pi \in \Pi \) the pure continuous problem

\[
\min_{x \in X} f(x, \pi)
\]

can be solved exactly and consider an \( M \geq 1 \). Finally, assume that there exists a fixed constant \( \tau > 0 \) such that

\[
|\Omega(Y)| \leq M
\]

(2)

for all boxes \( Y \subset X \) with \( \delta(Y) \leq \tau \).

Then the geometric branch-and-bound algorithm for mixed combinatorial problems using \( \varepsilon = 0 \) and the previously presented discarding test finds an exact optimal
solution for
\[
\min_{x \in X} \min_{\pi \in \Pi} f(x, \pi).
\]

Proof. Since in every iteration of the algorithm a box with largest diameter is selected for a split into some smaller subboxes, we obtain
\[
\delta(Y) \leq \tau
\]
for all \(Y \in X\) after a finite number of iterations. Thus, the previously presented discarding test ensures the termination of the algorithm after a finite number of iterations.

Moreover, since \(\varepsilon = 0\), no box \(Y\) which contains an optimal solution will be discarded throughout the algorithm. Hence, the lemma is shown since the pure continuous problems can be solved exactly. \(\square\)

Note that Lemma 5 holds independent from the bounding operation chosen.

7 Example problems

In this section, two example problems are discussed, namely the truncated Weber problem and the multisource Weber problem. For both problems, consider \(m\) demand points \(a_1, \ldots, a_m \in \mathbb{R}^2\).

7.1 The truncated Weber problem

The truncated Weber problem is to find a new facility location \(x \in \mathbb{R}^2\) such that the sum of the weighted distances between the new facility location \(x\) and the nearest \(1 < K < m\) demand points is minimized. Therefore, assume some weights \(w_1, \ldots, w_m \geq 0\), let \(X \subset \mathbb{R}^2\) be a box, and consider
\[
\Pi = \{\pi = (\pi_1, \ldots, \pi_m) \in \{0, 1\}^m : \pi_1 + \ldots + \pi_m = K\}.
\]

Thus, the truncated Weber problem is to minimize the objective function
\[
f(x, \pi) = \sum_{k=1}^{m} \pi_k \cdot w_k \cdot d(a_k, x) \quad \text{for} \quad x \in X, \ \pi \in \Pi
\]
where \(d\) is a given distance function. Note that this problem is a special case of the general ordered median problem, see Nickel and Puerto (2005), and moreover a special case of the general location-allocation problem introduced by Plastria and
Elomani (2008). Solution algorithms can be found in Drezner and Nickel (2009b) and Drezner and Nickel (2009a). But note that these methods do not provide an exact optimal solution.

Obviously, for any fixed \( x \in X \) we can easily solve the problem

\[
\min_{\pi \in \Pi} f(x, \pi)
\]

by sorting \( w_k \cdot d(a_k, x) \) for \( k = 1, \ldots, m \). Moreover, for any fixed \( \pi \in \Pi \) the functions \( f_\pi(x) = f(x, \pi) \) are classical Weber problem objective functions and therefore convex. Hence, we can apply the d.c. bounding operation.

Next, denote by

\[
\begin{align*}
\alpha_{k}^{\min}(Y) &= \min \{ w_k \cdot d(a_k, x) : x \in Y \}, \\
\alpha_{k}^{\max}(Y) &= \max \{ w_k \cdot d(a_k, x) : x \in Y \}
\end{align*}
\]

the minimal and the maximal weighted distances between a subbox \( Y \) and the demand point \( a_k \), respectively. Defining

\[
\Omega_k(Y) := \begin{cases} 
\{0\} & \text{if } \alpha_{k}^{\min}(Y) > \alpha_{j}^{\max}(Y) \text{ for at least } K \text{ indices } j \in \{1, \ldots, m\} \\
\{1\} & \text{if } \alpha_{k}^{\max}(Y) > \alpha_{j}^{\min}(Y) \text{ for at most } K \text{ indices } j \in \{1, \ldots, m\} \\
\{0,1\} & \text{else}
\end{cases}
\]

for \( k = 1, \ldots, m \), we find

\[
\Omega(Y) := \Pi \cap (\Omega_1(Y) \times \ldots \times \Omega_m(Y)).
\]

**Example 1.** As an example with \( m = 7 \) demand points, see Figure 1, consider the values of \( \alpha_{k}^{\min}(Y) \) and \( \alpha_{k}^{\max}(Y) \) for \( k = 1, \ldots, 7 \) as depicted in Figure 2.

![Figure 1: Input data for the truncated Weber problem discussed in Example 1.](image)

For the case \( K = 3 \), we obtain the following sets:

\[
\begin{align*}
\Omega_1 &= \{1\}, & \Omega_2 &= \{0\}, & \Omega_3 &= \{0\}, & \Omega_4 &= \{0\}, \\
\Omega_5 &= \{1\}, & \Omega_6 &= \{0,1\}, & \Omega_7 &= \{0,1\}.
\end{align*}
\]

Hence, for the given box \( Y \) only the variables \( \pi_6 \) and \( \pi_7 \) are not assigned to a fixed value. Furthermore, note that \( |\Omega(Y)| = 2 \).
Lemma 6. Consider the truncated Weber problem with objective function \( f(x, \pi) \), let \( Y \subset X \) be an arbitrary subbox, and assume that \( (x^*, \pi^*) \in Y \times \Pi \) is an optimal solution for

\[
\min_{x \in Y, \, \pi \in \Pi} f(x, \pi).
\]

Then we have \( \pi^* \in \Omega(Y) \), i.e. \( \Omega(Y) \) is a combinatorial dominating set for \( Y \).

Proof. Assume that \( \pi^* \not\in \Omega(Y) \). We now construct for all \( x \in Y \) a \( \mu = (\mu_1, \ldots, \mu_m) \in \Omega(Y) \) such that \( f(x, \pi^*) > f(x, \mu) \).

To this end, let \( I \subset \{1, \ldots, m\} \) such that \( i \in I \) if and only if \( \pi^*_i \not\in \Omega_i(Y) \). Next, define

\[
\nu_i = \begin{cases} 
0 & \text{if } \pi^*_i = 1, \\
1 & \text{if } \pi^*_i = 0 
\end{cases}
\]

for all \( i \in I \) and find a \( \mu \in \Omega(Y) \) such that \( \mu_i = \nu_i \) for all \( i \in I \). Note that such a \( \mu \) exists and note that

\[
\sum_{k=1}^{m} \pi^*_k = \sum_{\substack{k=1 \\mu_k \neq \pi^*_k}}^{m} \mu_k.
\]

Moreover, by construction of \( \Omega(Y) \) we obtain for all \( x \in Y \)

\[
\sum_{\substack{k=1 \\mu_k \neq \pi^*_k}}^{m} \pi^*_k \cdot w_k \cdot d(a_k, x) > \sum_{\substack{k=1 \\mu_k \neq \pi^*_k}}^{m} \mu_k \cdot w_k \cdot d(a_k, x),
\]

see also Example 1. Hence, for all \( x \in Y \) we have

\[
f(x, \pi^*) > f(x, \mu),
\]

a contradiction to the optimality of \( (x^*, \pi^*) \). \( \square \)
Unfortunately, in general we cannot expect

$$|\Omega(Y)| \leq M$$

for all boxes $$Y \subset X$$ with $$\delta(Y) \leq \tau$$ for a $$\tau \geq 0$$ and an $$M < |\Pi|$$ as the following counterexample shows.

**Example 2.** Consider the $$m$$ demand points $$a_k = \left(\frac{k}{m}, 1 - \frac{k}{m}\right) \in \mathbb{R}^2$$ for $$k = 1, \ldots, m$$ and consider the objective function

$$f(x, \pi) = \sum_{k=1}^{m} \pi_k \cdot \|x - a_k\|_1.$$  

Then, for all $$1 < K < m$$ and $$Y = [0,0] \times [0,0]$$ we have

$$d_k^{\min}(Y) = d_k^{\max}(Y) = 1$$ for $$k = 1, \ldots, m.$$  

Hence, $$\Omega(Y) = \Pi$$ although $$\delta(Y) = 0.$$  

### 7.2 The multisource Weber problem

The **multisource Weber problem** is to locate $$p$$ new facilities $$x_1, \ldots, x_p \in \mathbb{R}^2$$ assuming that each demand point is served by its nearest new facility. To be more precise, with $$X \subset \mathbb{R}^{2p}$$, with $$P = \{1, \ldots, p\}$$, and with $$\Pi = P^m \subset \mathbb{Z}^m$$ the problem is to minimize the objective function

$$f(x, \pi) = f(x_1, \ldots, x_p, \pi_1, \ldots, \pi_m) = \sum_{k=1}^{m} w_k \cdot d(a_k, x_{\pi_k})$$

where $$w_1, \ldots, w_m \geq 0$$ are some given non-negative weights.

The multisource Weber problem is one of the most studied facility location problems. For example, Drezner (1984) presented an exact algorithm for $$p = 2$$ using the Euclidean norm and some more general global optimization approaches can be found in Chen et al. (1998) and Schöbel and Scholz (2010a). Furthermore, the variable neighborhood search heuristic was applied in Brimberg et al. (2004) and Brimberg et al. (2006).

Here, we can make use of the d.c. bounding operation again since the objective function is convex for all fixed $$\pi \in \Pi$$.

Next, let $$Y = Y_1 \times \ldots \times Y_p \subset \mathbb{R}^2$$ such that $$x_i \in Y_i$$ for the new facilities $$i = 1, \ldots, p$$ and define the sets $$\Omega_k(Y) \subset P$$ for $$k = 1, \ldots, m$$ as follows:

$$\Omega_k(Y) := P \setminus \left\{ i \in P : \text{ there is a } j \in P \text{ with } d_k^{\min}(Y_i) > d_k^{\max}(Y_j) \right\}$$

where $$d_k^{\min}(Y_i)$$ and $$d_k^{\max}(Y_i)$$ are defined as before, see also Example 3.
Example 3. As an example with $p = 7$, consider the values of $d_{1}^{\text{min}}(Y_i)$ and $d_{1}^{\text{max}}(Y_i)$ for $i = 1, \ldots, 7$ as depicted in Figure 3.

In this case, we obtain $\Omega_1(Y) = \{2, 3, 6\}$, i.e. for all $x = (x_1, \ldots, x_7) \in Y$ the demand point $a_1$ will never be served by $x_1, x_4, x_5, \text{or } x_7$.

This leads to

$$\Omega(Y) := \Omega_1(Y) \times \ldots \times \Omega_m(Y)$$

and we obtain the following result.

Lemma 7. Consider the multisource Weber problem with objective function $f(x, \pi)$, let $Y \subset X$ be an arbitrary subbox, and assume that $(x^*, \pi^*) \in Y \times \Pi$ is an optimal solution for

$$\min_{x \in Y} \min_{\pi \in \Pi} f(x, \pi).$$

Then we have $\pi^* \in \Omega(Y)$, i.e. $\Omega(Y)$ is a combinatorial dominating set for $Y$.

Proof. Assume that $\pi^* \notin \Omega(Y)$. Hence, our goal is again to construct for all $x \in Y$ a $\mu = (\mu_1, \ldots, \mu_m) \in \Omega(Y)$ such that $f(x, \pi^*) > f(x, \mu)$.

To this end, assign $\mu_k = \pi_k^*$ for all $k \in \{1, \ldots, m\}$ with $\pi_k^* \in \Omega_k(Y)$, i.e.

$$d(a_k, x_{\pi_k^*}) = d(a_k, x_{\mu_k})$$

for all $x = (x_1, \ldots, x_p) \in Y$. Moreover, note that for all $x = (x_1, \ldots, x_p) \in Y$ and all $k \in \{1, \ldots, m\}$ with $\pi_k^* \notin \Omega_k(Y)$ there exists by construction of $\Omega_k(Y)$ a $\mu_k \in \Omega_k(Y)$ such that

$$d(a_k, x_{\pi_k^*}) > d(a_k, x_{\mu_k}),$$

see also Example 3. To sum up, for all $x \in Y$ it exists a $\mu \in \Omega(Y)$ such that

$$f(x, \pi^*) > f(x, \mu),$$

a contradiction to the optimality of $(x^*, \pi^*)$. □
8 Numerical results

In this section we present some numerical results for the example problems introduced in the previous section.

The proposed algorithm was implemented in JAVA, compiled by JAVA 2 SDK 1.4, using double precision arithmetic. All tests were run on a 3.0 GHz computer with 4 GB of memory. For both example problems, we generated up to \( m = 10,000 \) demand points \( a_1, \ldots, a_m \) uniformly distributed in \( X = \{-500, -499, \ldots, 499, 500\}^2 \) and weights \( w_k \in \{1, 2, \ldots, 10\} \) for \( k = 1, \ldots, m \). Ten problems were run for different values of \( m \). As distance measure, we applied the rectilinear norm, i.e.

\[
d(a_k, x) = \|x - a_k\|_1
\]

for \( k = 1, \ldots, m \). Hence, the pure facility location problems could be solved easily up to an exact optimal solution using the algorithm given in Drezner et al. (2001) with a complexity of \( O(m \log m) \). Moreover, we used \( \varepsilon = 0 \) and \( M = 4 \) throughout the algorithm and note that we found an exact optimal solution in all problem instances.

8.1 The truncated Weber problem

In our first example problem, we set \( K = m/5 \) gerundet and all boxes were split into \( s = 4 \) congruent subboxes. Our results are illustrated in Table 1. Therein, the run times, the number of iterations throughout the branch-and-bound algorithm, and the number of solved single facility location problems in Step (8) are reported.

<table>
<thead>
<tr>
<th>( m )</th>
<th>Run time (sec.)</th>
<th>Iterations</th>
<th>Location problems</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Min</td>
<td>Max</td>
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<tr>
<td>10</td>
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<td>0.16</td>
<td>0.05</td>
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<td>0.04</td>
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<td>0.30</td>
<td>0.05</td>
</tr>
<tr>
<td>100</td>
<td>0.04</td>
<td>0.53</td>
<td>0.10</td>
</tr>
<tr>
<td>200</td>
<td>0.10</td>
<td>0.42</td>
<td>0.16</td>
</tr>
<tr>
<td>500</td>
<td>0.30</td>
<td>0.91</td>
<td>0.44</td>
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<tr>
<td>1,000</td>
<td>0.78</td>
<td>1.37</td>
<td>1.07</td>
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<tr>
<td>2,000</td>
<td>2.37</td>
<td>5.47</td>
<td>3.04</td>
</tr>
<tr>
<td>5,000</td>
<td>6.14</td>
<td>15.93</td>
<td>9.43</td>
</tr>
<tr>
<td>10,000</td>
<td>20.03</td>
<td>32.35</td>
<td>27.38</td>
</tr>
</tbody>
</table>

Table 1: Numerical results for the truncated Weber problem.

Observe that the average number of solved facility location problems is for all values of \( m \) almost constant.
8.2 The p-median problem

The $p$-median problem was solved for $p = 2$ and $p = 3$. Here, every selected box $Y$ throughout the algorithm was split into $s = 2$ subboxes, i.e. $Y$ was bisected in two subboxes perpendicular to the direction of the maximum width component. Our results are collected in Table 2.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$p$</th>
<th>Run time (sec.)</th>
<th>Iterations</th>
<th>Location problems</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Min  Max  Ave.</td>
<td>Min  Max  Ave.</td>
<td>Min  Max  Ave.</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>0.01  0.17  0.05</td>
<td>223  788  463.1</td>
<td>11  261  115.4</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>0.02  0.36  0.08</td>
<td>319  1,728  926.1</td>
<td>48  501  200.3</td>
</tr>
<tr>
<td>50</td>
<td>2</td>
<td>0.08  0.38  0.23</td>
<td>637  2,450  1,463.5</td>
<td>9   379  119.4</td>
</tr>
<tr>
<td>100</td>
<td>2</td>
<td>0.21  1.11  0.50</td>
<td>779  2,700  1,688.6</td>
<td>2   472  126.7</td>
</tr>
<tr>
<td>200</td>
<td>2</td>
<td>1.06  2.77  1.79</td>
<td>1,929  4,890  3,204.0</td>
<td>44  1,451  258.4</td>
</tr>
<tr>
<td>500</td>
<td>2</td>
<td>2.81  7.50  4.52</td>
<td>2,152  5,185  3,385.4</td>
<td>13  303  139.0</td>
</tr>
<tr>
<td>1,000</td>
<td>2</td>
<td>3.92  20.59  12.60</td>
<td>1,401  7,611  4,747.1</td>
<td>2   395  128.8</td>
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<tr>
<td>2,000</td>
<td>2</td>
<td>13.30  59.58  34.20</td>
<td>2,568  11,015  6,392.6</td>
<td>2   979  247.2</td>
</tr>
<tr>
<td>5,000</td>
<td>2</td>
<td>49.35  277.35  128.60</td>
<td>3,700  20,085  9,450.5</td>
<td>2   2,711  423.1</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.46  1.66  0.96</td>
<td>3,409  11,707  6,734.0</td>
<td>286  2,182  947.9</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>1.88  26.98  6.61</td>
<td>7,271  79,666  22,602.1</td>
<td>567  7,663  2,879.6</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>9.38  79.60  33.42</td>
<td>11,605  95,584  45,969.4</td>
<td>152  7,333  2,660.9</td>
</tr>
<tr>
<td>100</td>
<td>3</td>
<td>41.56  217.94  128.50</td>
<td>32,710  133,468  83,211.5</td>
<td>112  10,028  3,079.1</td>
</tr>
</tbody>
</table>

Table 2: Numerical results for the p-median problem.

As can be seen, for the $p$-median problem the number of solved pure location problems increases with the number $m$ of demand points. However, even with $m = 5,000$ the 2-median problem could be solved by considering only a few hundred single facility problems.

9 Conclusions

In this paper, we suggested an extension of well-known geometric branch-and-bound methods for mixed continuous and combinatorial optimization problems. The main contribution of our work were some general bounding operations and we proved their theoretical rate of convergence. Hence, we provided a general algorithm to solve mixed continuous and combinatorial optimization problems. Using the rectilinear norm, the method was demonstrated on some location problems and we succeeded in finding exact optimal solutions. Note that the example problems can also be solved in the same manner using the Euclidean norm and the Weiszfeld algorithm for the solution of the pure location problems, see Weiszfeld (1937).

However, as in the classical pure continuous algorithms, the method is only suitable
if the number $n$ of continuous variables is relatively small. Moreover, for all bounding operations we assumed that for any fixed $x$ the pure combinatorial problem can be solved easily.

In general, it might be possible to find some branch-and-bound algorithms for the pure combinatorial part depending on the specific problem. In particular, if $\Omega(Y) \leq M$ for some boxes $Y$, we can find the optimal solution for all $x \in Y$ and all $\pi \in \Omega(Y)$ branching the integer variables $\pi$.

Furthermore, the presented techniques and theoretical results can be directly applied for example to simultaneous scheduling and location problems (ScheLoc), see Elvikis et al. (2009) and Kalsch and Drezner (2010). In particular, a further point of interest is to solve some ScheLoc problems up to an exact optimal solution using the results given in Section 6.

References


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