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The relation between multicriteria robustness concepts and set valued optimization

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Abstract

In this paper, we discuss the connection between the concepts of robustness for multi-objective optimization problems and set order relations. We extend some of the existing concepts to general spaces and cones. We point out that uncertain multi-objective robust optimization can be interpreted as an application of set-valued optimization. Furthermore, we use algorithms developed for uncertain multi-objective optimization problems to solve a special class of set-valued optimization problems.

Keywords. Robust optimization, multi-objective optimization, scalarization, vectorization

1 Introduction

Dealing with uncertainty in multi-objective optimization problems is very important in many applications. On the one hand, most real world optimization problems are contaminated with uncertain data, especially traffic optimization problems, scheduling problems, portfolio optimization, network flow and network design problems. On the other hand, many real world optimization problems require the minimization of multiple conflicting objectives (see [26]), e.g. the maximization of the expected return versus the minimization of risk in portfolio optimization, the minimization of production time versus the minimization of the cost of manufacturing equipment, or the maximization of tumor control versus the minimization of normal tissue complication in radiotherapy treatment design.

For an optimization problem contaminated with uncertain data it is typical that at the time when it is solved these data are not completely known. It is very important to estimate the effects of this uncertainty and so it is necessary to evaluate how sensitive an optimal solution is to perturbations of the input data. One way to deal with this question is *sensitivity analysis* (for an overview see [23]). Sensitivity analysis is an a posteriori approach and provides ranges for input data within which a solution remains feasible or optimal. It does not, however, provide a course of action for changing a solution should the perturbation be outside this range. In contrast, *stochastic programming* (see e.g. Birge and Louveaux [4] for an introduction) and *robust optimization* (see e.g. [14, 2] for an overview) take the uncertainty into account during the optimization process. While stochastic programming assumes some knowledge about the probability distribution of the uncertain data and the objective usually is to find a solution that is feasible with a certain probability and that optimizes the expected value of some objective function, robust optimization hedges against the worst case. Hence robust optimization does not require any probabilistic information. Depending from the concrete application one can decide whether robust or stochastic optimization is the more appropriate way of dealing with uncertainty.

Robust optimization is usually applied to problems where a solution is required which hedges against all possible scenarios. For example, the emergency department with landing place for rescue helicopters in a ski resort should be chosen in such a way that the flight time to all ski slopes in the resort that are to be protected is minimized in the worst case, even though flight times are uncertain due to unknown weather conditions. Similarly, if an aircraft-schedule of an airline is to be determined, one would want to be able to provide service to as many passengers as possible in a cost-effective manner, even though the exact number of passengers is not known at the time the schedule is fixed.

Generally, in the concept of robustness it is not assumed that all data are known, but one allows different scenarios for the input parameters and looks for a solution that works well in every uncertain scenario.

Unfortunately, at the time the uncertain optimization problem has to be solved, it is not known which scenario is going to be realized. Therefore, a definition of a "good" (or robust against the perturbations in the uncertain parameter) solution is necessary.

Robust optimization is a growing up field of research, we refer to Ben-Tal, L. El Ghaoui, Nemirovski [2], Kouvelis and Yu [14] for an overview of results and applications for the most prominent concepts. Several

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other concepts of robustness were introduced more recently, e.g. the concept of light robustness by Fischetti and Monaci [10] or the concept of recovery-robustness in Liebchen et al. [21], for a unified approach, see [13]. A scenario-based approach is suggested in Goerigk and Schöbel [11]. In all these approaches, the uncertain optimization problem is replaced by a deterministic version, called the *robust counterpart* of the uncertain problem.

One of the most common approaches is the concept of minmax robustness, introduced by Soyster [24] and studied e.g. by Ben-Tal and Nemirovski [3]. Here, a solution is said to be robust, if it minimizes the worst case of the objective function over all scenarios. We do not go into detail here as for this paper we mostly consider concepts of robustness for multi-objective optimization problems.

Now, if we consider the objective function in the problem definition to be not a single-objective, but a multi-objective function, the concepts of robustness do not apply naturally anymore. The problem obviously is that there is no total order on $\mathbb{R}^n$ and the robustness concepts for uncertain single-objective optimization problems rely on the total order of $\mathbb{R}$. Therefore, new definitions of what is seen as a robust solution to an uncertain multi-objective optimization problem are necessary.

The first approach to handling uncertainty for multi-objective optimization problems was presented by Deb and Gupta [6] who extended the concept Branke [5] introduced for single-objective functions. Here each objective function is replaced by their mean function and an efficient solution to the resulting multi-objective optimization problem is called a robust solution. The authors also presented a second definition where the uncertainty is modeled into the constraints which restrict the variation of the original objective functions to their means. Barrico and Antunes [1] extended the concept of Deb and Gupta and introduced the degree of robustness as a measure how much a predefined neighborhood of the solution considered can be extended without containing solution whose function values are too bad. An overview of the existing concepts of robustness for multi-objective optimization problems can be found in [25] and [27].

A first approach to extending the concept of minmax robustness to multi-objective optimization was presented by Kuroiwa and Lee [19]. Here, the worst case in each component is calculated separately, and an efficient solution to the problem of minimizing the vector of worst cases is then called a robust solution to the original problem. This definition has been extended by Ehrgott, et al. [7], where the authors replace the objective function by a set-valued objective function. Furthermore, the authors present solution algorithms for calculating minmax robust efficient solutions, one of which is closely connected to the concept of robustness presented by Kuroiwa and Lee [19]. Furthermore, the authors in [19] present solution concepts for obtaining robust points of uncertain multi-objective optimization and study optimality conditions for the special case of convex objective functions in [20].

Set-valued optimization on the other hand deals with the problem of minimizing a function where the image of a point is in fact a set. Minimizing a set is not totally intuitive since on a power set there is no total order as well as on $\mathbb{R}^k$. Therefore, a definition of what can be seen as an optimal solution to minimizing a set-valued objective function is necessary. In order to compare sets, several preorders have been introduced (see e.g. [9, 15, 18, 22, 28]). With these preorders it is then possible to define a set-valued optimization problem.

Now, as Ide and Köbis [12] pointed out, the concept of minmax robust efficiency is closely connected to a certain set order relation, introduced by Kuroiwa [15, 18], namely the *upper-type set-relation*.

Replacing the set order relation implicitly used in the definition of minmax robust efficiency, Ide and Köbis [12] presented various other concepts of robustness for multi-objective optimization, derived by replacing the upper-type set-relation with another set ordering from the literature. The authors then presented algorithms to calculate the resulting robust efficient solutions.

Now, this paper is structured as follows: After fixing the notation in Section 2, in Section 3 we show that a lot of the Theorems presented in [7] and [12] can be extended to general spaces. Using this information, we extend the algorithms presented in these publications to our general setting and then use these algorithms in to solve a certain class of set-valued optimization problems. We conclude the paper with some final remarks and an outlook to future research.

2 Preliminaries

Throughout the paper, let Y be a linear topological space partially ordered by a proper closed convex pointed (i.e., $C \cap (-C) = \{0\}$) cone C. The dual cone to C is denoted by $C^* := \{y^* \in Y^* | \forall y \in C : \; y^*(y) \geq 0\}$ and the quasi-interior of C^* is defined by $C^\# := \{y^* \in C^* | \forall y \in C \setminus \{0\} : \; y^*(y) > 0\}$. Furthermore, let X be a linear space, $F : X \rightrightarrows Y$ (with the "$\rightrightarrows$"-notation we denote that F is a set-valued objective function whose function values are sets in Y), and $\mathcal{X}$ a subset of X. As usual, we denote the graph of the set-valued map F by $\operatorname{graph} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. Furthermore, we define $F(\mathcal{X}) := \cup_{x\in\mathcal{X}} F(x)$.

In set optimization, the following set-relations play an important role, see Young [28], Nishnianidze [22], Kuroiwa [15, 18] and Eichfelder, Jahn [9]. We will use these set-relations to introduce several concepts of robustness.

Definition 1 (Set less order relation ([28, 22, 9])). *Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \subset Y$ be arbitrarily chosen sets. Then the set less order relation is defined by*

$$A \preceq^s_C B :\Longleftrightarrow A \subseteq B - C \text{ and } A + C \supseteq B.$$

Remark 1. *Of course, we have*

$$A \subseteq B - C \Longleftrightarrow \forall a \in A \; \exists b \in B : a \leq_C b$$

and

$$A + C \supseteq B \Longleftrightarrow (\forall b \in B \; \exists a \in A : a \leq_C b).$$

Definition 2 (Upper-type set-relation ([15, 18])). *Let $A, B \subset Y$ be arbitrary chosen sets and $C \subset Y$ a proper closed convex and pointed cone. Then the u-type set-relation $\preceq^u_C$ is defined by*

$$A \preceq^u_C B :\Longleftrightarrow A \subseteq B - C \Longleftrightarrow \forall a \in A \; \exists b \in B : a \leq_C b.$$

Another important set order relation is the lower-type set-relation:

Definition 3 (Lower-type set-relation ([15, 18])). *Let $A, B \subset Y$ be arbitrarily chosen sets and $C \subset Y$ a proper closed convex and pointed cone. Then the l-type set-relation $\preceq^l_C$ is defined by*

$$A \preceq^l_C B :\Longleftrightarrow A + C \supseteq B \Longleftrightarrow \forall b \in B \; \exists a \in A : a \leq_C b.$$

Remark 2. *Note that*

(i) $A \subset B - \operatorname{int} C$ and

(ii) $A + N \subset B - C$ for some neighborhood N of the null vector

are not equivalent when A is not compact. Clearly (ii) implies (i). From a theoretical viewpoint, (ii) may, in some cases, be more appropriate.

Remark 3. *There is the following relationship between the l-type set-relation $\preceq^l_C$ and the u-type set-relation $\preceq^u_{-C}$:*

$$A \preceq^l_C B :\Longleftrightarrow A + C \supseteq B \Longleftrightarrow B \subseteq A - (-C) \Longleftrightarrow: B \preceq^u_{-C} A.$$

Remark 4. *If we use the set-relation $\preceq^l_C$ introduced in Definition 3 in the formulation of the solution concept, i.e., we study the set-valued optimization problem of $(SP - \preceq^l_C)$ we observe that this solution concept is based on comparisons among sets of minimal points of values of F. Furthermore, considering the u-type set-relation $\preceq^u_C$ (Definition 2), i.e., considering the problem $(SP - \preceq^u_C)$ we recognize that this solution concept is based on comparisons of maximal points of values of F. When $\overline{x} \in \mathcal{X}$ is a minimal solution of problem $(SP - \preceq^l_C)$ there does not exist $x \in S$ such that $F(x)$ is strictly smaller than $F(\overline{x})$ with respect to the set order $\preceq^l_C$.*

To conclude the notation, we repeat the definition of a set-valued optimization problem: Consider $F : X \rightrightarrows Y$, and $\mathcal{X}$ a subset of X. Furthermore, let $\preceq$ be a preorder on Y given by Definition 1, 2, 3, respectively. Then a set-valued optimization problem $(\mathcal{SP}\!-\preceq)$ is defined as

$$\preceq \text{-minimize } F(x) \text{ subject to } x \in \mathcal{X}, \qquad (\mathcal{SP}\!-\preceq)$$

where minimal solutions of $(\mathcal{SP}\!-\preceq)$ are defined in the following way:

Definition 4 (Minimal solutions of $(\mathcal{SP}\!-\preceq)$ w.r.t. the preorder $\preceq$). *Given a set-valued optimization problem $(\mathcal{SP}\!-\preceq)$, an element $\overline{x} \in \mathcal{X}$ is called a minimal solution to $(\mathcal{SP}\!-\preceq)$ if*

$$(F(x) \preceq F(\overline{x}) \quad \text{for some} \quad x \in \mathcal{X}) \Longrightarrow F(\overline{x}) \preceq F(x).$$

The definition of a minimizer of ($\mathcal{SP}-\preceq$) is very often used in the theory of set optimization and given below. However, the solution concept introduced in Definition $\mathcal{SP}-\preceq$ is more natural and useful as we can see in Example 1.

Definition 5 (Minimizer of (SP))**.** *Let $\overline{x} \in \mathcal{X}$ and $(\overline{x}, \overline{y}) \in \operatorname{graph} F$. The pair $(\overline{x}, \overline{y}) \in \operatorname{graph} F$ is called a minimizer of the problem (SP) if $\overline{y} \in \operatorname{Min}\big(F(\mathcal{X}), C\big)$, where $\operatorname{Min}\big(F(\mathcal{X}), C\big) := \{y \in Y \mid F(\mathcal{X}) \cap (\overline{y} - C\setminus\{0\}) = \emptyset\}$.*

For our approach to robustness of uncertain vector optimization problems, minimal solutions in the sense of Definition $\mathcal{SP}-\preceq$ are useful and therefore, when considering robustness concepts, we will deal with this solution concept in the following.

In order to get an insight to the issue of set-valued optimization problems, we give two examples (see Kuroiwa [17]) of set-valued optimization problems.

Example 1. *Consider the set-valued optimization problem*

$$\preceq^l_C -\textit{minimize} \quad F_1(x), \quad \textit{subject to} \quad x \in \mathcal{X}, \qquad (SP - \preceq^l_C)$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $\mathcal{X} = [0,1]$ and $F_1 : \mathcal{X} \rightrightarrows Y$ is given by

$$F_1(x) := \begin{cases} [(1,0),(0,1)] & \text{if } x = 0 \\ [(1-x, x), (1,1)] & \text{if } x \in (0,1], \end{cases}$$

where $[(a,b),(c,d)]$ is the line segment between (a,b) and (c,d). Only the element $\overline{x} = 0$ is a minimal solution of $(SP - \preceq^l_C)$. However, all elements $(\overline{x}, \overline{y}) \in \operatorname{graph} F_1$ with $\overline{x} \in [0,1]$, $\overline{y} = (1-\overline{x}, \overline{x})$ for $\overline{x} \in (0,1]$ and $\overline{y} = (1,0)$ for $\overline{x} = 0$ are minimizers of the set-valued optimization problem in the sense of Definition 5. This example shows that the solution concept with respect to the set-relation $\preceq^l_C$ (see Definitions 3 and 4) is more natural and useful than the concept of minimizers introduced in Definition 5.

Example 2. *In this example we are looking for minimal solutions of a set-valued optimization problem with respect to the set-relation $\preceq^u_C$ introduced in Definition 2.*

$$\preceq^u_C -\textit{minimize} \quad F_2(x), \quad \textit{subject to} \quad x \in \mathcal{X}, \qquad (SP - \preceq^u_C)$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $\mathcal{X} = [0,1]$ and $F_2 : \mathcal{X} \rightrightarrows Y$ is given by

$$F_2(x) := \begin{cases} [[(1,1),(2,2)]] & \text{if } x = 0 \\ [[(0,0),(3,3)]] & \text{if } x \in (0,1], \end{cases}$$

where $[[(a,b),(c,d)]] := \{(y_1, y_2) \mid a \le y_1 \le c,\ b \le y_2 \le d\}$. Then the only minimal solution of $(SP - \preceq^u_C)$ in the sense of Definition 4 is $\overline{x} = 0$.

In Section 3, we will apply the preorders introduced in Definitions 1, 2, 3 in order to define several concepts of robustness for uncertain multi-objective optimization problems.

3 Extending robust efficiency to general spaces

Talking about an uncertain optimization problem, we consider the uncertain data to be given as a parameter (also called *scenario*) $\xi \in \mathcal{U}$ where $\mathcal{U} \subseteq \mathbb{R}^m$ is the so called *uncertainty set*. For a realization of this parameter we obtain a single optimization problem

$$\begin{aligned} & f(x,\xi) \to \min \\ & \text{s.t. } x \in \mathcal{X}, \end{aligned} \qquad (\mathcal{P}(\xi))$$

where $f : X \times \mathcal{U} \mapsto Y$ is the objective function and $\mathcal{X} \subseteq X$ is the set of feasible solutions (note that we assume the feasible set to be unchanged for every realization of the uncertain parameter). We use the notation

$$f_{\mathcal{U}}(x) := \{f(x,\xi) | \xi \in \mathcal{U}\} \qquad (1)$$

for the image of the uncertainty set $\mathcal{U}$ and x under f (note that $f_{\mathcal{U}}(x)$ in general is a set and not a singleton). Now, when searching for an optimal solution, one has to overcome the problem that we do not know anything about the different scenarios, e.g., which one is most likely to occur, any kind of probability distribution and so on. Therefore, an uncertain (multi-objective) optimization problem is defined as the family of optimization problems

$$(\mathcal{P}(\xi), \xi \in \mathcal{U}). \tag{$\mathcal{P}(\mathcal{U})$}$$

Now it is not clear what solution to this problem $\mathcal{P}(\mathcal{U})$ would be seen as desirable. Throughout the paper we repeat several concepts of robustness for multi-objective optimization problems from the literature.

In this section we extend the robustness concepts presented in [7] and [12] to general spaces using the preorders introduced in Definitions 1, 2, 3. In particular, we are interested in extending the theorems which provide the foundation for the algorithms for calculating the respective robust solutions. We shortly repeat the various concepts which relate to different set orderings, extend the theorems and then formulate the algorithms. With this, we present some ideas for solving set-valued optimization problems in this section.

3.1 $\preceq^u_C$-robustness

We extend the definitions and results presented by Ehrgott, et al. [7] about minmax robust efficiency.

In [7] a feasible solution $x \in \mathcal{X}$ to $\mathcal{P}(\mathcal{U})$ is called *minmax robust efficient* if there is no other feasible solution $\overline{x} \in \mathcal{X} \setminus \{x\}$, such that

$$f_{\mathcal{U}}(\overline{x}) \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}$$

where $\mathbb{R}^k_{\geqq} := \{\lambda \in \mathbb{R}^k : \lambda_i \geq 0 \ \forall\, i = 1, \ldots, k\}$,

With the definitions of *upper-type set-relation*, see Definition 2, and minmax robust efficiency in mind we can see the close connection between minmax robust efficiency and the upper-type set-relation since a solution $x \in \mathcal{X}$ to $\mathcal{P}(\mathcal{U})$ is minmax robust efficient if there is no other feasible solution $\overline{x} \in \mathcal{X} \setminus \{x\}$, such that

$$f_{\mathcal{U}}(\overline{x}) \preceq^u_C f_{\mathcal{U}}(x),$$

where $Y = \mathbb{R}^k$ and $C = \mathbb{R}^k_{\geqq}$.

Since all the concepts considered in this paper are closely related to a set order relation $\preceq$, in order to keep the names of the concepts readable we call the respective solution $\preceq$-robust. Using this notation, the concept of minmax robust efficiency can be re-defined as a concept of robustness in the sense of set optimization in the following way:

Definition 6. *Given an uncertain multi-objective optimization problem $\mathcal{P}(\mathcal{U})$, we call a solution $x \in \mathcal{X}$* $\preceq^u_C$-robust *for $\mathcal{P}(\mathcal{U})$ if there is no solution $\overline{x} \in \mathcal{X} \setminus \{x\}$ such that*

$$f_{\mathcal{U}}(\overline{x}) \preceq^u_C f_{\mathcal{U}}(x).$$

Note that the definition of $\preceq^u_C$-robustness is valid for general spaces and general cones, while the definition of minmax robust efficiency in [7] is for $Y = \mathbb{R}^k$ and $C = \mathbb{R}^k_{\geqq}$ only.

The motivation behind this concept is the following: When comparing sets with the u-type set-relation, the upper bounds of these sets, i.e., the "worst cases", are considered. Minimizing these worst cases is closely connected to the concept of minmax robust efficiency where one wants to minimize the objective function in the worst case. This risk averse approach would reflect a decision makers strategy to hedge against a worst case and is rather pessimistic.

Remark 5. *The first scenario-based concept to uncertain multi-objective optimization, or* minmax-*robustness adapted to multi-objective optimization, has been introduced by Kuroiwa and Lee [19] and studied in [20]. In [19, 20] robust solutions of multi-objective optimization problems are introduced in the following way. The authors propose to consider the robust counterpart to $P(\mathcal{U})$*

$$\operatorname{Min}(f^{\mathcal{U}}_{RC}(\mathcal{X}), \mathbb{R}^k_{\geq}), \tag{2}$$

where the objective vector for $x \in \mathcal{X}$ is given by

$$f^{\mathcal{U}}_{RC}(x) := \begin{pmatrix} \max_{\xi \in \mathcal{U}_1} f_1(x, \xi_1) \\ \cdots \\ \max_{\xi \in \mathcal{U}_k} f_k(x, \xi_k) \end{pmatrix}, \tag{3}$$

with functionals $f_i : \mathbb{R}^n \times \mathcal{U}_i \to \mathbb{R}$ for $i = 1, \ldots, k$ and the convex and compact uncertainty sets $\mathcal{U} := (\mathcal{U}_1, \ldots, \mathcal{U}_k)$ $(\mathcal{U}_i \subseteq \mathbb{R}^m,\ i = 1, \ldots, k)$. They call solutions to (2) robust.

In difference to [19, 20], we develop new concepts of robustness based on upper-type and lower-type set-relations (see Definitions 3 and 3). Our approach is related to the robustness approach based in [7], furthermore, we derive new concepts of robustness.

Remark 6. *Robustness* in the sense of vector optimization *is introduced by Kuroiwa and Lee [19, 20] in the following way: Consider the objective function as defined by* (3) *with* $\mathcal{U} := (\mathcal{U}_1, \ldots, \mathcal{U}_k)$*, where* $\mathcal{U}_i \subseteq \mathbb{R}^m$*,* $i = 1, \ldots, k$ *are convex and compact uncertain sets. An element* $\bar{x} \in \mathcal{X}$ *is a* ***robust solution*** **in the sense of vector optimization** *if*

$$f^{\mathcal{U}}_{RC}(\bar{x}) \cap \mathrm{Min}(f^{\mathcal{U}}_{RC}(\mathcal{X}), C) \neq \emptyset. \tag{4}$$

Note that robustness in the sense of set optimization*, introduced in Definition 6, and robustness* in the sense of vector optimization*, see* (4)*, are different.*

With the extension of the definition of minmax robust efficiency we can extend an algorithms for computing minmax robust efficient solutions which is an extension of the well-known weighted sum scalarization technique for calculating efficient solutions to multi-objective optimization problems (compare e.g. Ehrgott [8]).

The general idea is to form a scalar optimization problem by multiplying each objective function with a positive weight and summing up the weighted objectives. The resulting (single-objective) problem in a more general setting is

$$\min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} \; y^* \circ f(x, \xi), \tag{$(\mathcal{P}(\mathcal{U})_{y^*})$}$$

where $f : X \times \mathcal{U} \to Y$ and $y^* \in C^*$, i.e., $y^* : Y \to \mathbb{R}$.

Now, solving this problem one can obtain $\preceq^u_C$-robust solutions as shown in Theorem 4.3 in [7]. Before extending this theorem, we need a lemma which will help during the proofs:

Lemma 1. *Consider the uncertain multi-objective optimization problem* $\mathcal{P}(\mathcal{U})$*. Then it holds for all* x'*,* $\bar{x} \in \mathcal{X}$ *and for* $Q = \mathrm{int}\, C$ *(*$Q = C \setminus \{0\}$*,* $Q = C$*, respectively),*

$$f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(\bar{x}) - Q \Longleftrightarrow \forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U} : \ f(x', \xi) \in f(\bar{x}, \eta) - Q. \tag{5}$$

Proof. "$\Longrightarrow$": Suppose the contrary. Then

$$\begin{aligned} \exists \xi \in \mathcal{U}\ \forall \eta \in \mathcal{U} : \ f(x', \xi) \notin f(\bar{x}, \eta) - Q \quad &\Longrightarrow \quad \exists \xi \in \mathcal{U} \ : \ f(x', \xi) \notin f_{\mathcal{U}}(\bar{x}) - Q \\ &\Longrightarrow \quad f_{\mathcal{U}}(x') - C \not\subseteq f_{\mathcal{U}}(\bar{x}) - Q \end{aligned}$$

"$\Longleftarrow$": Suppose the contrary. Then

$$\exists \xi \in \mathcal{U} : \ f(x', \xi) \notin f_{\mathcal{U}}(\bar{x}) - Q \quad \Longrightarrow \quad \exists \xi \in \mathcal{U}\ \forall \eta \in \mathcal{U} \ : \ f(x', \xi) \notin f(\bar{x}, \eta) - Q$$

□

With this, we can extend Theorem 4.3 from [7] in the following way:

Theorem 1. *Consider an uncertain multi-objective optimization problem* $\mathcal{P}(\mathcal{U})$*. The following statements hold:*

(a) *If* $x^0 \in \mathcal{X}$ *is a unique optimal solution of* $(\mathcal{P}(\mathcal{U})_{y^*})$ *for some* $y^* \in C^* \setminus \{0\}$*, then* x^0 *is a* $\preceq^u_C$*-robust solution for* $\mathcal{P}(\mathcal{U})$*.*

(b) *If* $x^0 \in \mathcal{X}$ *is an optimal solution of* $(\mathcal{P}(\mathcal{U})_{y^*})$ *for some* $y^* \in C^\#$ *and* $\max_{\xi \in \mathcal{U}} \; y^* \circ f(x, \xi)$ *exists for all* $x \in \mathcal{X}$*, then* x^0 *is a* $\preceq^u_{C \setminus \{0\}}$*-robust solution for* $\mathcal{P}(\mathcal{U})$*.*

(c) *If* $x^0 \in \mathcal{X}$ *is an optimal solution of* $(\mathcal{P}(\mathcal{U})_{y^*})$ *for some* $y^* \in C^* \setminus \{0\}$ *and* $\max_{\xi \in \mathcal{U}} \; y^* \circ f(x, \xi)$ *exists for all* $x \in \mathcal{X}$*, then* x^0 *is a* $\preceq^u_{\mathrm{int}\, C}$*-robust solution for* $\mathcal{P}(\mathcal{U})$*.*

Proof. Suppose that x^0 is not ($\preceq^u_C$, $\preceq^u_{C \setminus \{0\}}$, $\preceq^u_{\mathrm{int}\, C}$)-robust. Then there exists an element $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_{\mathcal{U}}(\bar{x}) \subseteq f_{\mathcal{U}}(x^0) - Q, \tag{6}$$

for $Q = C$, ($Q = (C \setminus \{0\})$, $Q = \mathrm{int}\, C$, respectively).

This implies

$$\forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U}:\ f(\overline{x}, \xi) \in f(x^0, \eta) - Q.$$

taking into account Lemma 1.

Choose now $y^* \in C^* \setminus \{0\}$ for $Q = C$ ($y^* \in C^{\#}$ for $Q = C \setminus \{0\}$, $y^* \in (C^* \setminus \{0\})$ for $Q = \operatorname{int} C$, respectively) arbitrary but fixed.

$$\begin{aligned}
\Longrightarrow &\quad \forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U}: \quad y^* \circ f(\overline{x}, \xi) \leq\ (<, <, \textit{respectively}) \quad y^* \circ f(x^0, \eta) \\
\Longleftrightarrow &\quad \forall \xi \in \mathcal{U}: \quad y^* \circ f(\overline{x}, \xi) \leq\ (<, <, \textit{respectively}) \sup_{\eta' \in \mathcal{U}} \ y^* \circ f(x^0, \eta') \\
\Longleftrightarrow &\quad \sup_{\xi' \in \mathcal{U}} \ y^* \circ f(\overline{x}, \xi') \leq\ (<, <, \textit{respectively}) \sup_{\eta' \in \mathcal{U}} \ y^* \circ f(x^0, \eta')
\end{aligned}$$

The last equation holds because for (b) and (c) $\max_{\xi' \in \mathcal{U}} \ y^* \circ f(\overline{x}, \xi')$ exists. But this means that x^0 is not the unique optimal (an optimal, an optimal, respectively) solution of $(\mathcal{P}(\mathcal{U})_{y^*})$ for $y^* \in C^* \setminus \{0\}$ ($y^* \in C^{\#}$, $y^* \in C^* \setminus \{0\}$, respectively). □

With this theorem we can now formulate a first algorithm for finding ($\preceq^u_C$, $\preceq^u_{C \setminus \{0\}}$, $\preceq^u_{\operatorname{int} C}$)-robust solutions.

Algorithm 1 for deriving ($\preceq^u_C$, $\preceq^u_{C \setminus \{0\}}$, $\preceq^u_{\operatorname{int} C}$)-robust solutions to $\mathcal{P}(\mathcal{U})$ based on weighted sum scalarization:

Input: Uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$, solution sets $\mathrm{Opt}_C = \mathrm{Opt}_{C \setminus \{0\}} = \mathrm{Opt}_{\operatorname{int}\ C} = \emptyset$.

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$.

Step 2: If $\overline{C} = \emptyset$: **STOP. Output:** Set of $\preceq^u_C$-robust solutions Opt_C, set of $\preceq^u_{C \setminus \{0\}}$-robust solutions $\mathrm{Opt}_{C \setminus \{0\}}$, set of $\preceq^u_{\operatorname{int} C}$-robust solutions $\mathrm{Opt}_{\operatorname{int}\ C}$.

Step 3: Choose $y^* \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^*\}$.

Step 4: Find an optimal solution x^0 of $(\mathcal{P}(\mathcal{U})_{y^*})$.

a) If x^0 is a unique optimal solution of $(\mathcal{P}(\mathcal{U})_{y^*})$, then x^0 is strictly robust for $\mathcal{P}(\mathcal{U})$, thus

$$\mathrm{Opt}_C := \mathrm{Opt}_C \cup \{x^0\}.$$

b) If $\max_{\xi \in \mathcal{U}} \ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$, then x^0 is weakly robust for $\mathcal{P}(\mathcal{U})$, thus

$$\mathrm{Opt}_{\operatorname{int}\ C} := \mathrm{Opt}_{\operatorname{int}\ C} \cup \{x^0\}.$$

c) If $\max_{\xi \in \mathcal{U}} \ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$ and $y^* \in C^{\#}$, then x^0 is robust for $\mathcal{P}(\mathcal{U})$, thus

$$\mathrm{Opt}_{C \setminus \{0\}} := \mathrm{Opt}_{C \setminus \{0\}} \cup \{x^0\}.$$

Step 5: Go to Step 2.

Furthermore, we present an interactive algorithm for finding a desired ($\preceq^u_C$, $\preceq^u_{C \setminus \{0\}}$, $\preceq^u_{\operatorname{int} C}$)-robust solution to the uncertain multi-objective optimization problem $\mathcal{P}(\mathcal{U})$. This algorithm uses the input of the decision maker whether she accepts the calculated solution or not:

Algorithm 2 for deriving a single accepted ($\preceq^u_C$, $\preceq^u_{C \setminus \{0\}}$, $\preceq^u_{\operatorname{int} C}$)-robust solution to $\mathcal{P}(\mathcal{U})$ based on weighted sum scalarization:

Input: Uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$.

Step 1: Choose a set $\overline{C} \subset C^* \setminus \{0\}$.

Step 2: Set j:=0, choose $y^0 \in \overline{C}$. Set $\overline{C} := \overline{C} \setminus \{y^0\}$.

Step 3: Find an optimal solution x^0 to $(\mathcal{P}(\mathcal{U})_{y^*})$.

a) If x^0 is a unique optimal solution of $(\mathcal{P}(\mathcal{U})_{\bar{y}^0})$, then x^0 is $\preceq^u_C$-robust for $\mathcal{P}(\mathcal{U})$.

b) If $\max_{\xi\in\mathcal{U}} \quad y^* \circ f(x,\xi)$ exists for all $x \in \mathcal{X}$, then x^0 is $\preceq^u_{\text{int}\, C}$-robust for $\mathcal{P}(\mathcal{U})$.

c) If $\max_{\xi\in\mathcal{U}} \quad y^* \circ f(x,\xi)$ exists for all $x \in \mathcal{X}$ and $\bar{y}^* \in C^{\#}$, then x^0 is $\preceq^u_{C\setminus\{0\}}$-robust for $\mathcal{P}(\mathcal{U})$.

If x^0 is accepted by the decision-maker: **STOP. Output:** x^0.

Step 4: Choose $\bar{y}^{j+1} \in C$, such that $\bar{y}^{j+1} \neq y^i$ for all $0 \leq i \leq j$.
Set $l := 0$, $t_0 := 0$.

Step 5: Choose t_{l+1} with $t_l < t_{l+1} \leq 1$ and compute an optimal solution x^j_{l+1} to

$$\min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} \quad \bar{y}^* \circ f(x,\xi) \qquad (\mathcal{P}(\mathcal{U})_{\bar{y}^j + t_{l+1}(\bar{y}^{j+1} - y^j)})$$

with $\bar{y}^* := \bar{y}^j + t_{l+1}(\bar{y}^{j+1} - \bar{y}^j)$. If no optimal solution to $(\mathcal{P}(\mathcal{U})_{\bar{y}^j + t_{l+1}(\bar{y}^{j+1} - y^j)})$ can be found for all $t > t^l$, go to Step 2.

Step 6: If x^j_{l+1} is accepted by the decision maker: **STOP. Output:** x^j_{l+1}.

Step 7: If $t_{l+1} = 1$, then set $j := j+1$ and go to Step 4. Otherwise, set $l := l+1$ and go to Step 5.

3.2 $\preceq^l_C$-robustness

We derive the concept of $\preceq^l_C$-robustness, defined analogously to $\preceq^u_C$-robustness (Definition 6):

Definition 7. *Given an uncertain multi-objective optimization problem $\mathcal{P}(\mathcal{U})$, a solution $x \in \mathcal{X}$ is called $\preceq^l_C$-robust if there is no $\overline{x} \in \mathcal{X} \setminus \{x\}$ such that*

$$f_{\mathcal{U}}(\overline{x}) \preceq^l_C f_{\mathcal{U}}(x)$$

The $\preceq^l_C$-robustness can be interpreted as an optimistic approach. The following example illustrates this concept.

Example 3. *Here, x is $\preceq^l_C$-robust, while it is not $\preceq^u_C$-robust.*

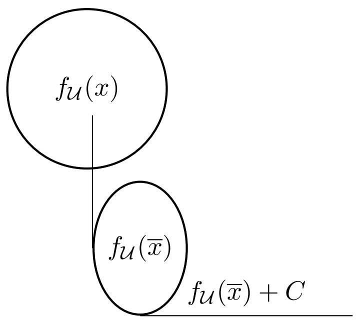

Figure 1: x is $\preceq^l_C$-robust.

Remark 7. *The $\preceq^l_C$-robustness is an alternative tool for the decision maker for obtaining solutions of another type to an uncertain multi-objective optimization problem. This rather optimistic approach focuses on the lower bound of a set $f_{\mathcal{U}}(\bar{x})$. In order to compare this set to a set $f_{\mathcal{U}}(x^0)$, the lower bound is considered. In particular, a point $x^0 \in \mathcal{X}$ is called an optimistic solution if there is no other point $\bar{x} \in \mathcal{X}$ such that $f_{\mathcal{U}}(x^0)$ is a subset of $f_{\mathcal{U}}(\bar{x}) + Q$. In that sense, this approach would reflect a decision maker's preferences if she is interested in solutions whose objective function* may be *smaller in a future scenario ξ. Contrary to the $\preceq^u_C$-robustness approach, the $\preceq^l_C$-robustness is hence not a worst-case concept, thus the decision maker is not considered to be risk averse but risk affine. This optimistic concept thus hedges against perturbations in the best-case scenarios.*

For calculating $\preceq^l_C$-robust solutions again the weighted sum scalarization is helpful, but in order to later on compute $\preceq^l_C$-robust solutions to $\mathcal{P}(\mathcal{U})$, we define a new weighted sum problem in a general setting:
Let $y^* \in C^* \setminus \{0\}$ $(y \in C^{\#},\ y \in C^* \setminus \{0\})$. Consider the weighted sum scalarization problem

$$\min_{x \in \mathcal{X}} \ \inf_{\xi \in \mathcal{U}} y^* \circ f(x, \xi). \qquad (\mathcal{P}(\mathcal{U})^{opt}_{y^*})$$

Theorem 2. *Consider an uncertain vector optimization problem $\mathcal{P}(\mathcal{U})$. The following statements hold:*

(a) If x^0 is a unique optimal solution of $(\mathcal{P}(\mathcal{U})^{opt}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$, then x^0 is a $\preceq^l_C$-robust solution to $\mathcal{P}(\mathcal{U})$.

(b) If x^0 is an optimal solution of $(\mathcal{P}(\mathcal{U})^{opt}_{y^*})$ for some $y^* \in C^{\#}$ and $\min_{\xi \in \mathcal{U}} \ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$, then x^0 is a $\preceq^l_{C \setminus \{0\}}$-robust solution to $\mathcal{P}(\mathcal{U})$.

(c) If x^0 is an optimal solution of $(\mathcal{P}(\mathcal{U})^{opt}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$ and $\min_{\xi \in \mathcal{U}} \ y^* \circ f(x, \xi)$ exists for all $x \in \mathcal{X}$, then x^0 is a $\preceq^l_{\operatorname{int} C}$-robust solution to $\mathcal{P}(\mathcal{U})$.

Proof. Suppose x^0 is not a $(\preceq^l_C / \preceq^l_{C\setminus\{0\}} / \preceq^l_{\operatorname{int} C})$-robust. Consequently, there exists an $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ s.t. $f_{\mathcal{U}}(\bar{x}) + Q \supseteq f_{\mathcal{U}}(x^0)$ for $Q = C$ ($Q = C \setminus \{0\}$, $Q = \operatorname{int} C$, respectively). That is equivalent to

$$\begin{aligned} &\forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U}: \ f(\bar{x}, \eta) + Q \ni f(x^0, \xi) \\ \iff &\forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U}: \ f(\bar{x}, \eta) \in f(x^0, \xi) - Q \end{aligned} \tag{7}$$

Now choose $y^* \in C^* \setminus \{0\}$ for $Q = C$ ($y^* \in C^{\#}$ for $Q = C \setminus \{0\}$, $y^* \in C^* \setminus \{0\}$ for $Q = \operatorname{int} C$, respectively) arbitrary, but fixed. Hence, we obtain from (7)

$$\begin{aligned} &\Longrightarrow \ \forall \xi \in \mathcal{U}\ \exists \eta \in \mathcal{U}: \ \ y^* \circ f(\bar{x}, \eta) \leq \ (<, <, \text{respectively}) \ \ y^* \circ f(x^0, \xi) \\ &\Longrightarrow \ \exists \eta \in \mathcal{U}: \ \ y^* \circ f(\bar{x}, \eta) \leq \ (<, <, \text{respectively}) \ \inf_{\xi \in \mathcal{U}} \ \ y^* \circ f(x^0, \xi) \\ &\Longrightarrow \ \inf_{\eta \in \mathcal{U}} \ \ y^* \circ f(\bar{x}, \eta) \leq \ (<, <, \text{respectively}) \ \inf_{\xi \in \mathcal{U}} \ \ y^* \circ f(x^0, \xi), \end{aligned}$$

in contradiction to the assumption. □

Based on these results, we are able to present the following algorithm that computes $(\preceq^l_C / \preceq^l_{C\setminus\{0\}} / \preceq^l_{\operatorname{int} C})$-robust solutions to $\mathcal{P}(\mathcal{U})$:

Algorithm 3 for deriving $(\preceq^l_C / \preceq^l_{C\setminus\{0\}} / \preceq^l_{\operatorname{int} C})$-robust solutions to $\mathcal{P}(\mathcal{U})$ based on weighted sum scalarization:

Input & Step 1-5: Analogous to Algorithm 1, only replacing $(\mathcal{P}(\mathcal{U})_{y^*})$ by $(\mathcal{P}(\mathcal{U})^{opt}_{y^*})$ and replacing $\max_{\xi \in \mathcal{U}} \ y^* \circ f(x^0, \xi)$ by $\min_{\xi \in \mathcal{U}} \ y^* \circ f(x^0, \xi)$.

The next algorithm computes $(\preceq^l_C / \preceq^l_{C\setminus\{0\}} / \preceq^l_{\operatorname{int} C})$-robust solutions via weighted sum scalarization by altering the weights:

Algorithm 4 for calculating a single desired $(\preceq^l_C / \preceq^l_{C\setminus\{0\}} / \preceq^l_{\operatorname{int} C})$-robust solution to $\mathcal{P}(\mathcal{U})$ based on weighted sum scalarization:

Input & Step 1-5: Analogous to Algorithm 2, only replacing $(\mathcal{P}(\mathcal{U})_{y^0})$ by $(\mathcal{P}(\mathcal{U})^{opt}_{y^*})$, $\max_{\xi \in \mathcal{U}} \ y^* \circ f(x^0, \xi)$ by $\min_{\xi \in \mathcal{U}} \ y^* \circ f(x^0, \xi)$ and $(\mathcal{P}(\mathcal{U})_{\bar{y}^j + t_{l+1}(\bar{y}^{j+1} - \bar{y}^j)})$ by $(\mathcal{P}(\mathcal{U})^{opt}_{\bar{y}^j + t_{l+1}(\bar{y}^{j+1} - \bar{y}^j)})$.

3.3 $\preceq^s_C$-robustness

Based on the definition of the *set less order relation*, we can now introduce the concept of $\preceq^s_C$-robustness:

Definition 8. *A solution x^0 of $\mathcal{P}(\mathcal{U})$ is called $(\preceq^s_C \ / \ \preceq^s_{C\setminus\{0\}} \ / \ \preceq^s_{\operatorname{int} C})$-robust if there is no $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ such that*

$$f_{\mathcal{U}}(\bar{x}) \preceq^s_Q f_{\mathcal{U}}(x^0)$$

for $Q = C$ ($Q = C \setminus \{0\}$, $Q = \operatorname{int} C$, respectively).

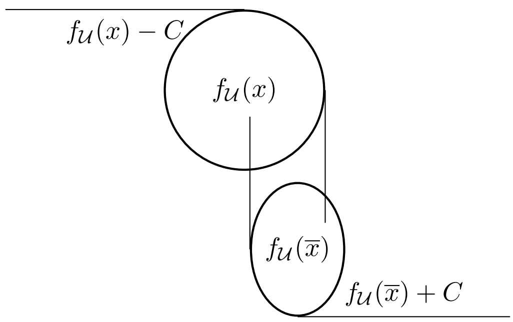

Figure 2: x is $\preceq^s_C$-robust.

Example 4. *This Example shows that x is $\preceq^s_C$-robust, while it is not $\preceq^u_{\operatorname{int} C}$-robust.*

Remark 8. *Note that a $\preceq^l_C$-robust solution is as well $\preceq^s_C$-robust by definition. The same holds for a $\preceq^u_C$-robust solution.*

The concept of $\preceq^s_C$-robustness can be interpreted in the following way: In a situation where it is not clear if one should follow a risk affine or risk averse strategy (e.g., the decision maker is not at hand or wants to get a feeling for the variety of the solutions) this concept might be helpful as it calculates solutions which reflect these different strategies. Therefore, this concept can serve as a pre-selection before deciding a definite strategy.

Computing $\preceq^s_C$-robust solutions is possible with the help of the following optimization problem:

$$\begin{pmatrix} \inf_{\xi\in\mathcal{U}} & y^* \circ f(x,\xi) \\ \sup_{\xi\in\mathcal{U}} & y^* \circ f(x,\xi) \end{pmatrix} \to v - \min_{x\in\mathcal{X}} \qquad (\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$$

with $y^* \in C^* \setminus \{0\}$ ($y^* \in C^\#$). We have the following theorem:

Theorem 3.
- *If x^0 is strictly Pareto efficient for problem $(\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$, then x^0 is $\preceq^s_C$-robust.*
- *If x^0 is weakly Pareto efficient for problem $(\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$ for some $y^* \in C^* \setminus \{0\}$ and $\min_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi)$ and $\max_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi)$ exist for all $x \in \mathcal{X}$ and the chosen weight $y^* \in C^* \setminus \{0\}$, then x^0 is $\preceq^s_{\operatorname{int} C}$-robust.*
- *If x^0 is weakly Pareto efficient for problem $(\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$ for some $y^* \in C^\#$ and $\min_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi)$ and $\max_{\xi\in\mathcal{U}} \; y^* \circ f(x,\xi)$ exist for all $x \in \mathcal{X}$ and the chosen weight $y^* \in C^\#$, then x^0 is $\preceq^s_{C\setminus\{0\}}$-robust.*

Proof. Let x^0 be strictly Pareto efficient (weakly Pareto efficient, weakly Pareto efficient) for problem $(\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$ with some some $y^* \in C^* \setminus \{0\}$ ($y^* \in C^* \setminus \{0\}$, $y^* \in C^\#$, respectively), i.e., there is no $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$\begin{aligned} \inf_{\xi\in\mathcal{U}} \; y^* \circ f(\overline{x},\xi) \leq \; (<,\; <) \inf_{\xi\in\mathcal{U}} \; y^* \circ f(x^0,\xi) \\ \text{and } \sup_{\xi\in\mathcal{U}} \; y^* \circ f(\overline{x},\xi) \leq \; (<,\; <) \sup_{\xi\in\mathcal{U}} \; y^* \circ f(x^0,\xi). \end{aligned}$$

Now suppose x^0 is not $(\preceq^s_C / \preceq^s_{\operatorname{int} C} / \preceq^s_{C\setminus\{0\}})$-robust. Then there exists an $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ such that

$$f_\mathcal{U}(\bar{x}) + Q \supseteq f_\mathcal{U}(x^0) \text{ and } f_\mathcal{U}(\bar{x}) \subseteq f_\mathcal{U}(x^0) - Q$$

for $Q = C$ ($Q = \operatorname{int} C,\; Q = C \setminus \{0\}$). That implies

$$\exists \bar{x} \in \mathcal{X} : \; \forall \xi_1, \xi_2 \in \mathcal{U} \; \exists \eta_1, \eta_2 \in \mathcal{U} : \; f(\bar{x},\eta_1) + Q \ni f(x^0,\xi_1) \text{ and } f(\bar{x},\xi_2) \in f(x^0,\eta_2) - Q \tag{8}$$

for $Q = C$ ($Q = \operatorname{int} C,\; Q = C \setminus \{0\}$). Choose now $y^* \in C^* \setminus \{0\}$ ($y^* \in C^* \setminus \{0\}$, $y^* \in C^\#$) as in problem $(\mathcal{P}(\mathcal{U})^{biobj}_{y^*})$. We obtain from (8)

$$\begin{aligned} &\exists \bar{x} \in \mathcal{X} : \; \forall \xi_1, \xi_2 \in \mathcal{U} \; \exists \eta_1, \eta_2 \in \mathcal{U} : \; y^* \circ f(\bar{x},\eta_1) \leq \; (<,\; <) \; y^* \circ f(x^0,\xi_1) \\ &\qquad\qquad \text{and } y^* \circ f(\bar{x},\xi_2) \leq \; (<,\; <) \; y^* \circ f(x^0,\eta_2) \\ &\Rightarrow \inf_{\xi\in\mathcal{U}} \; y^* \circ f(\bar{x},\xi) \leq \; (<,\; <) \; \inf_{\xi\in\mathcal{U}} \; y^* \circ f(x^0,\xi) \text{ and } \sup_{\xi\in\mathcal{U}} \; y^* \circ f(\bar{x},\xi) \leq \; (<,\; <) \sup_{\xi\in\mathcal{U}} \; y^* \circ f(x^0,\xi). \end{aligned}$$

The last two strict inequalities hold because the minimum and maximum exist. But this is a contradiction to the assumption. $\square$

Based on these observations, we can formulate the following algorithm for computing $\preceq_C^s$-robust solutions to $\mathcal{P}(\mathcal{U})$.

Algorithm 5 for computing $\preceq_C^s$-robust solutions using a family of problems $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$:

Input & Step 1-3, Step 5: Analogous to Algorithm 1.

Step 4: Find a set of (strictly, weakly, $\cdot$) Pareto efficient solutions $\{\overline{x}^1, \ldots, \overline{x}^s\}$ of $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$.

- **a)** If $\overline{x}^j$ (j=1, ...,s) is a strictly Pareto efficient solution of $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$, then $\overline{x}^j$ is $\preceq_C^s$-robust for $\mathcal{P}(\mathcal{U})$, thus
$$\text{Opt}_C := \text{Opt}_C \cup \{x^0\}.$$
- **b)** If $\overline{x}^j$ (j=1, ...,s) is a weakly Pareto efficient solution of $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$ and $\max_{\xi\in\mathcal{U}}\ y^* \circ f(x,\xi)$ and $\min_{\xi\in\mathcal{U}}\ y^* \circ f(x,\xi)$ exist for all $x \in \mathcal{X}$, then $\overline{x}^j$ is $\preceq_{\text{int}\,C}^s$-robust for $\mathcal{P}(\mathcal{U})$, thus
$$\text{Opt}_{\text{int }C} := \text{Opt}_{\text{int }C} \cup \{\overline{x}^j\}.$$
- **c)** If $\overline{x}^j$ (j=1, ...,s) is weakly Pareto efficient for problem $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$ and $y^* \in C^\#$ and $\min_{\xi\in\mathcal{U}}\ y^* \circ f(x,\xi)$ and $\max_{\xi\in\mathcal{U}}\ y^* \circ f(x,\xi)$ exist for all $x \in \mathcal{X}$ and the chosen weight $y^* \in C^\#$, then $\overline{x}^j$ is $\preceq_{\text{int}\,C}^s$-robust for $\mathcal{P}(\mathcal{U})$, thus
$$\text{Opt}_{C\setminus\{0\}} := \text{Opt}_{C\setminus\{0\}} \cup \{\overline{x}^j\}.$$

In the following we present an algorithm that computes $\preceq_C^s$-robust solutions while varying the weights in the vector of objectives of problem $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$.

Algorithm 6 for computing set less ordered robust solutions using a family of problems $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$:

Input & Step 1-3 & Step 5-8: Analogous to Algorithm 2, only replacing $(\mathcal{P}(\mathcal{U})_{y^*})$ by $(\mathcal{P}(\mathcal{U})_{y^*}^{biobj})$ and $(\mathcal{P}(\mathcal{U})_{\bar{y}^j+t_{l+1}(\bar{y}^{j+1}-\bar{y}^j)})$ by $(\mathcal{P}(\mathcal{U})_{\bar{y}^j+t_{l+1}(\bar{y}^{j+1}-\bar{y}^j)}^{biobj})$.

Step 4: Analogous to **Step 4** of Algorithm 5.

3.4 Alternative Set Less Ordered Robustness

Another way of combining the u- and l-type set-relations is the alternative set less order relation:

Definition 9 (Alternative set less order relation (compare Ide and Köbis [12])). *Let $C \subset Y$ be a proper closed convex and pointed cone. Furthermore, let $A, B \subset Y$ be arbitrarily chosen sets. Then the set less order relation is defined by*
$$A \preceq_C^s B :\Longleftrightarrow A \preceq_C^u B \text{ or } A \preceq_C^l B.$$

Based on this definition we can now define the concept of $\preceq_C^a$-robustness for general cones:

Definition 10. *A solution x^0 of $\mathcal{P}(\mathcal{U})$ is called ($\preceq_C^a$ / $\preceq_{C\setminus\{0\}}^a$ / $\preceq_{\text{int}\,C}^a$)-robust if there is no $\bar{x} \in \mathcal{X} \setminus \{x^0\}$ such that*
$$f_{\mathcal{U}}(\bar{x}) \preceq_Q^a f_{\mathcal{U}}(x^0)$$
for $Q = C$ ($Q = C \setminus \{0\}$, $Q = \text{int}\, C$, respectively).

The following example illustrates $\preceq_C^a$-robust solutions.

Example 5. *In Figure 3, both x and y are $\preceq_C^a$-robust.*

The next lemma follows directly from the definitions:

Lemma 2. *Note that a solution of $\mathcal{P}(\mathcal{U})$ is $\preceq_C^a$-robust if and only if it is $\preceq_C^l$-robust and $\preceq_C^u$-robust.*

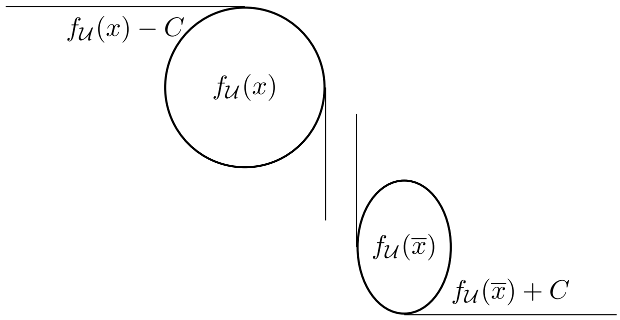

Figure 3: Both x and $\overline{x}$ are strictly $\preceq^a_C$-robust.

As this lemma shows, the concept of $\preceq^a_C$-robustness is rather restrictive as only solutions which are $\preceq^u_C$-robust and $\preceq^l_C$-robust, thus reflect both a risk averse and a risk affine strategy, are also $\preceq^a_C$-robust. Therefore, this concept is fit for a decision maker who does not want to make any mistake in terms of the best or worst cases. We can see easily that such an approach would be very restrictive against the solutions and that only very few solutions should fulfill these conditions.

Due to this Lemma 2, from Algorithms 1 and 3, we can deduce the following algorithm for calculating $\preceq^a_C$-robust solutions to $\mathcal{P}(\mathcal{U})$:

Algorithm 7 for deriving $\preceq^a_C$-robust solutions to $\mathcal{P}(\mathcal{U})$:

Input: Uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$, solution sets $\text{Opt}^a_C = \text{Opt}_{C\setminus\{0\}^a} = \text{Opt}^a_{\text{int } C} = \emptyset$.

Step 1: Compute a set of $(\preceq^l_C / \preceq^l_{\text{int}\, C} / \preceq^l_{C\setminus\{0\}})$-robust solutions $(\text{Opt}^l_C,\ \text{Opt}^l_{\text{int } C},\ \text{Opt}^l_{C\setminus\{0\}})$ using Algorithm 6.

Step 2: Compute a set of $(\preceq^u_C / \preceq^u_{\text{int}\, C} / \preceq^u_{C\setminus\{0\}})$-robust solutions $(\text{Opt}^u_C,\ \text{Opt}^u_{\text{int } C},\ \text{Opt}^u_{C\setminus\{0\}})$ using Algorithm 1 or 2.

Output: Set of $(\preceq^a_C / \preceq^a_{\text{int}\, C} / \preceq^a_{C\setminus\{0\}})$-robust solutions

$$\begin{aligned} \text{Opt}^a_C &= \text{Opt}^u_C \cap \text{Opt}^l_C, \\ \text{Opt}^a_{\text{int } C} &= \text{Opt}^u_{\text{int } C} \cap \text{Opt}^l_{\text{int } C}, \\ \text{Opt}^a_{C\setminus\{0\}} &= \text{Opt}^u_{C\setminus\{0\}} \cap \text{Opt}^l_{C\setminus\{0\}}\,. \end{aligned}$$

3.5 Further Relationships Between the Concepts

From Remark 8 we can see that every $\preceq^u_C$-robust solution and every $\preceq^l_C$-robust solution are also $\preceq^s_C$-robust solutions. The inverse direction does not hold. The following example in Figure 4 shows that a solution can be $\preceq^s_C$-robust but neither $\preceq^u_C$-robust nor $\preceq^l_C$-robust.

We summarize the relationship between the various robustness concepts in Figure 5.

4 Conclusions

In the following we will explain that our algorithms presented in Section 3 can be used for solving special classes of set-valued optimization problems.

Having a close look at all the concepts of robustness from Section 3, we can see that in fact all of these are set-valued optimization problems.

Consider a set-valued optimization problem of the form

$$\preceq -\text{minimize} \quad F(x), \quad \text{subject to} \quad x \in \mathcal{X}, \qquad (SP - \preceq)$$

with some given preorder $\preceq$ and an objective function $F : \mathcal{X} \rightrightarrows Y$ we can see the following:

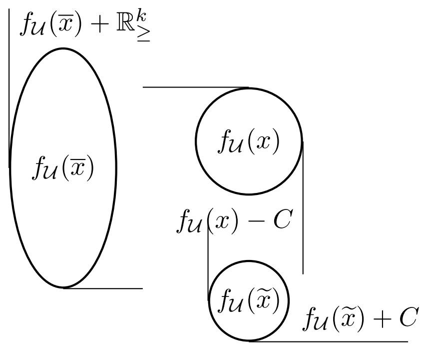

Figure 4: x is $\preceq^s_C$-robust, but neither $\preceq^u_C$-robust nor $\preceq^l_C$-robust.

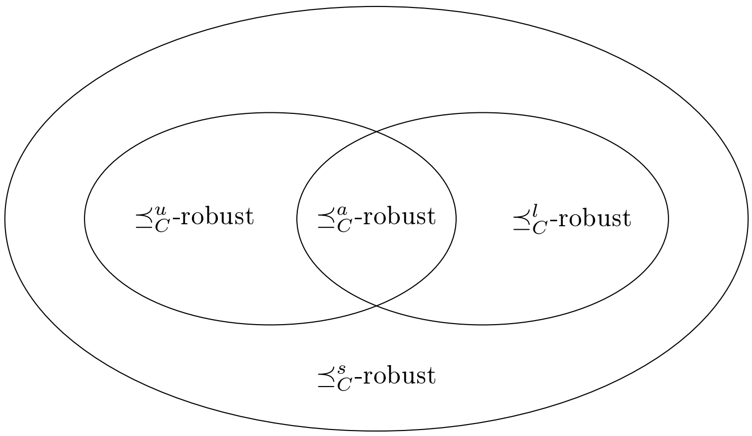

Figure 5: Scheme of solutions to an uncertain multicriteria optimization problem.

If the preorder $\preceq$ is equal to $\preceq^a_C$, $\preceq^l_C$, $\preceq^u_C$, or $\preceq^s_C$ with some cone C and $F(x)$ can be parametrized by a parameter $\xi \in \mathcal{U}$ with some set $\mathcal{U}$ in the way that

$$F(x) := f_{\mathcal{U}}(x) \text{ for all } x \in \mathcal{X},$$

where $f_{\mathcal{U}}(x) = \{f(x,\xi) | \xi \in \mathcal{U}\}$, then the set-valued optimization problem $(\mathcal{SP}- \preceq)$ is equivalent to finding $\preceq$-robust solutions to the uncertain multi-objective problem $\mathcal{P}(\mathcal{U})$ and can therefore be solved by solving one of the respective algorithms presented in Section 3.

We revealed strong connections between set-valued optimization and uncertain multi-objective optimization. Furthermore, we generalized the results achieved in [7] and [12] to more general sets Q. In particular, we provided solution algorithms for a certain class of set-valued optimization problems. This class of set-valued optimization problems is general enough to provide new algorithms for a lot of set-valued optimization problems. Furthermore, it seems possible to extend this class of problems to a more general one, but this is future work and of interest for the next steps in this area of research.

Moreover, this paper made very clear that finding robust solutions to uncertain multi-objective optimization problems can be interpreted as an application of set-valued optimization. Thus, robust solutions to uncertain multi-objective optimization problems can be obtained by using the solution techniques from set-valued optimization. Formulating concrete algorithms of this kind is another topic for future research.

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