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A Local Projection Stabilization FEM for the linearized stationary MHD problem

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Abstract. We present a local projection stabilization (LPS) type finite element (FE) method for the linearized stationary magnetohydrodynamics (MHD) problem which is essentially based on the ideas of the residual-based stabilization given in [1], [2], [3]. In contrast to the residual-based stabilization, we investigate a symmetric LPS comparable to the term-by-term stabilization in [3].

1 Introduction

Following the time discretization and linearization approach in [1]-[2], we want to investigate the stationary MHD model which reads

$$-\nu\Delta\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \nabla p - (\nabla\times\mathbf{b})\times\mathbf{d} = \mathbf{f_u}\ ,\ \nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\lambda\nabla\times(\nabla\times\mathbf{b}) + \nabla r - \nabla\times(\mathbf{u}\times\mathbf{d}) = \mathbf{f_b}\ ,\ \nabla\cdot\mathbf{b} = 0, \tag{2}$$

in a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$ with $\nabla\cdot\mathbf{a}=0$. $\mathbf{a}$ and $\mathbf{d}$ are the vector-fields for the velocity and magnetic field at linearization. The velocity field $\mathbf{u}$, the magnetic field $\mathbf{b}$, the pressure p and the magnetic pseudo-pressure r are sought. For the weak formulation, we introduce the function spaces

$$V = \left\{\mathbf{v}\in\left[H^1(\Omega)\right]^d \,:\, \mathbf{v}=0\ \text{on}\,\partial\Omega\right\} \quad , \quad Q = L^2_0(\Omega),$$

$$C = \{\mathbf{c}\in H(curl\,;\Omega) \,:\, \mathbf{n}\times\mathbf{c}=0\ \text{on}\,\partial\Omega\} \quad , \quad S = H^1_0(\Omega)$$

where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are appropriate inner and dual products. The variational problem is to find $\mathbf{U} := (\mathbf{u},\mathbf{b},p,r)\in V\times C\times Q\times S$ such that for all $\mathbf{V} := (\mathbf{v},\mathbf{c},q,s)\in V\times C\times Q\times S$

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \langle\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v}\rangle - (p,\nabla\cdot\mathbf{v}) - \langle(\nabla\times\mathbf{b})\times\mathbf{d},\mathbf{v}\rangle = \langle\mathbf{f_u},\mathbf{v}\rangle, \tag{3}$$

$$(\nabla\cdot\mathbf{u},q) = 0, \tag{4}$$

$$\lambda(\nabla\times\mathbf{b},\nabla\times\mathbf{c}) + (\nabla r,\mathbf{c}) - \langle\nabla\times(\mathbf{u}\times\mathbf{d}),\mathbf{c}\rangle = \langle\mathbf{f_b},\mathbf{c}\rangle, \tag{5}$$

$$-(\mathbf{b},\nabla s) = 0 \tag{6}$$

holds. Using the bilinear and linear forms

$$\begin{aligned}\mathcal{A}_G(\mathbf{U},\mathbf{V}) &= \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \langle\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v}\rangle - (p,\nabla\cdot\mathbf{v}) - \langle(\nabla\times\mathbf{b})\times\mathbf{d},\mathbf{v}\rangle\\ &\quad + (\nabla\cdot\mathbf{u},q) - (\mathbf{b},\nabla s)\\ &\quad +\lambda(\nabla\times\mathbf{b},\nabla\times\mathbf{c}) + (\nabla r,\mathbf{c}) - \langle\nabla\times(\mathbf{u}\times\mathbf{d}),\mathbf{c}\rangle,\\ \mathcal{F}_G(\mathbf{V}) &= \langle\mathbf{f_u},\mathbf{v}\rangle + \langle\mathbf{f_b},\mathbf{c}\rangle,\end{aligned}$$

we now want to find $\mathbf{U}\in V\times C\times Q\times S$ such that

$$\mathcal{A}_G(\mathbf{U},\mathbf{V}) = \mathcal{F}_G(\mathbf{V}) \tag{7}$$

is fulfilled for all $\mathbf{V}\in V\times C\times Q\times S$.

Let $\mathcal{T}_h$ be the primal grid with FE spaces of Taylor-Hood type

$$V_h\times Q_h/C_h\times S_h = \mathbb{P}^r_{\mathcal{T}_h}\times\mathbb{P}^{r-1}_{\mathcal{T}_h} \text{ or } \mathbb{Q}^r_{\mathcal{T}_h}\times\mathbb{Q}^{r-1}_{\mathcal{T}_h}. \tag{8}$$

These spaces $V_h\times Q_h$ are discretely-divergence-free, thus

$$V_h^{div} := \{\mathbf{v_h}\in V_h:\ (\nabla\cdot v_h,q_h)=0\ \forall q_h\in Q_h\}\neq\{0\}.$$

Let $\mathcal{M}_h=\mathcal{T}_h$ or $\mathcal{M}_h=\mathcal{T}_{2h}$ be the macro grid with discontinuous FE spaces

$$\begin{aligned}D_h^{u/b} &:= \{\mathbf{v}\in[L^2(\Omega)]^d:\ \mathbf{v}|_M\in\mathbb{P}^{r-1}_{\mathcal{M}_h}\ \text{ or }\ \mathbb{Q}^{r-1}_{\mathcal{M}_h},\ \forall M\in\mathcal{M}_h\ \},\\ D_h^{p/r} &:= \{v\in L^2(\Omega):\ v|_M\in\mathbb{P}^{k-1}_{\mathcal{M}_h}\ \text{ or }\ \mathbb{Q}^{r-1}_{\mathcal{M}_h},\ \forall M\in\mathcal{M}_h\ \}\end{aligned}$$

with $k\in\{0,\dots,r-2\}$. The local L^2-projections are defined through $\pi_M^{u/b}: [L^2(M)]^d\to D_h^{u/b}|_M$, $\pi_M^{p/r}: L^2(M)\to D_h^{p/r}|_M$ whereas the global projections $\pi_h^{u/b}: [L^2(\Omega)]^d\to D_h^{u/b}(\pi_h^{u/b}\mathbf{w})|_M := \pi_M^{u/b}(\mathbf{w}|_M)$ are defined through $(\pi_h^{u/b}\mathbf{w})|_M := \pi_M^{u/b}(\mathbf{w}|_M)$ The scalar-valued fluctuation operator which is given by $\kappa_h^{p/r}: [L^2(\Omega)]\to[L^2(\Omega)]$ is then determined through $\kappa_h^{p/r} := id-\pi_h^{p/r}$, whereas the vector-valued fluctuation operator has to be understood in a component-wise manner. From now, the superscript latters will be skipped.

Let $\mathbf{U_h}=(\mathbf{u_h},\mathbf{b_h},p_h,r_h),\mathbf{V_h}=(\mathbf{v_h},\mathbf{c_h},q_h,s_h)\in V_h\times C_h\times Q_h\times S_h\subset V\times C\times Q\times S$. Then the LPS term reads

$$\begin{aligned}\mathcal{S}_{lps}(\mathbf{U_h},\mathbf{V_h}) = \sum_M \{&\tau_1(\kappa((\mathbf{a_M}\cdot\nabla)\mathbf{u_h}),\kappa((\mathbf{a_M}\cdot\nabla)\mathbf{v_h}))_M + \tau_2(\nabla\cdot\mathbf{u_h},\nabla\cdot\mathbf{v_h})_M\\ &+\tau_3(\kappa((\nabla\times\mathbf{b_h})\times\mathbf{d_M}),\kappa((\nabla\times\mathbf{c_h})\times\mathbf{d_M}))_M\\ &+\tau_4(\kappa(\nabla\times(\mathbf{u_h}\times\mathbf{d_M})),\kappa(\nabla\times(\mathbf{v_h}\times\mathbf{d_M})))_M\\ &+\tau_5(\nabla r_h,\nabla s_h)_M + \tau_6(\nabla\cdot\mathbf{b_h},\nabla\cdot\mathbf{c_h})_M\}\end{aligned}$$

where $(\cdot,\cdot)_M$ describes the L^2 scalar products on cells M. Here, $\mathbf{a_M}$ and $\mathbf{d_M}$ are elementwise approximations of $\mathbf{a}|_\mathbf{M}$ and $\mathbf{d}|_\mathbf{M}$. The LPS problem consists of finding $\mathbf{U_h}\in V_h\times C_h\times Q_h\times S_h$ such that

$$\mathcal{A}_{stab}(\mathbf{U_h},\mathbf{V_h}) = \mathcal{A}_G(\mathbf{U_h},\mathbf{V_h}) + \mathcal{S}_{lps}(\mathbf{U_h},\mathbf{V_h}) = \mathcal{F}_G(\mathbf{V_h}) \tag{9}$$

holds for all $\mathbf{V_h}\in V_h\times C_h\times Q_h\times S_h$.

2 Stability of the proposed method

We want to examine some properties of the LPS term $\mathcal{S}_{lps}$.

Lemma 1. *For* $\mathbf{U} = (\mathbf{u}, \mathbf{b}, p, r)$ *and* $\mathbf{V} = (\mathbf{v}, \mathbf{c}, q, s)$*, it holds*

$$\mathcal{S}_{lps}(\mathbf{U}, \mathbf{U}) \geq 0 \text{ and } |\mathcal{S}_{lps}(\mathbf{U}, \mathbf{V})| \leq (\mathcal{S}_{lps}(\mathbf{U}, \mathbf{U}))^{\frac{1}{2}} (\mathcal{S}_{lps}(\mathbf{V}, \mathbf{V}))^{\frac{1}{2}}$$

with $\mathbf{U}, \mathbf{V} \in V \times C \times Q \times S$.

Proof. The first property is a consequence of $\mathcal{S}_{lps}$ whereas the second inequality follows by the symmetry of $\mathcal{S}_{lps}$ and the Cauchy-Schwarz inequality. □

Moreover, we receive the approximate Galerkin orthogonality.

Lemma 2. *Let* $\mathbf{U}$ *and* $\mathbf{U_h}$ *be the given solutions of (7) and (9) and let* $\mathbf{V_h} = (\mathbf{v_h}, \mathbf{c_h}, q_h, s_h) \in V_h \times C_h \times Q_h \times S_h \subset V \times C \times Q \times S$*. Then*

$$(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U} - \mathbf{U_h}, \mathbf{V_h}) = \mathcal{S}_{lps}(\mathbf{U}, \mathbf{V_h})$$

holds for all $\mathbf{V_h} \in V_h \times C_h \times Q_h \times S_h$.

Proof. By subtracting (7) and (9), we get the result. □

Let $\mathbf{V} = (\mathbf{v}, \mathbf{c}, q, s) \in V \times C \times Q \times S$. Integration by parts yields

$$\langle \mathbf{a} \cdot \nabla \mathbf{v}, \mathbf{v} \rangle = -\frac{1}{2} \langle (\nabla \cdot \mathbf{a}) \mathbf{v}, \mathbf{v} \rangle = 0 \,, \; \langle (\nabla \times \mathbf{c}) \times \mathbf{d}, \mathbf{v} \rangle = -\langle \nabla \times (\mathbf{v} \times \mathbf{d}), \mathbf{c} \rangle.$$

This implies

$$\mathcal{A}_G(\mathbf{V}, \mathbf{V}) = \nu \|\nabla \mathbf{v}\|_0^2 + \lambda \|\nabla \times \mathbf{c}\|_0^2$$

where $\|\cdot,\cdot\|_0$ is the induced L^2 norm on Ω. We therefore define the norms

$$\|\mathbf{V}\|_{\mathrm{G}}^2 = \nu \|\nabla \mathbf{v}\|_0^2 + \lambda \|\nabla \times \mathbf{c}\|_0^2, \qquad \|\mathbf{V}\|_{\mathrm{Stab}}^2 = \mathcal{S}_{lps}(\mathbf{V}, \mathbf{V}). \tag{10}$$

For $k \in \mathbf{N_0}$ and $D \subset \Omega$, we use the notation $|\cdot|_{k,D} := |\cdot|_{H^k(D)}$ and $\|\cdot\|_{\infty,D} := \|\cdot\|_{L^\infty(D)}$. In case of $D = \Omega$, we omit index D.

We show stability of (9). Let $\mathbf{U_h} \in V_h \times C_h \times Q_h \times S_h$ be the solution of (9). Let $\mathbf{V_h} \in V_h \times C_h \times Q_h \times S_h$. By symmetric testing $\mathbf{V_h} = \mathbf{U_h}$, we get

$$(\mathcal{A}_{\mathcal{G}} + \mathcal{S}_{lps})(\mathbf{V_h}, \mathbf{V_h}) = \|\mathbf{V_h}\|_{\mathrm{G}}^2 + \|\mathbf{V_h}\|_{\mathrm{Stab}}^2. \tag{11}$$

Using Young's inequality and the definition

$$\|(\mathbf{f_u}, \mathbf{f_b})\|_{G,\star} = \sup_{(v_h, c_h) \in V_h \times C_h} \frac{\mathcal{F}_{\mathcal{G}}(\mathbf{V_h})}{\|\mathbf{V_h}\|_G},$$

we get the unique existence of the velocity field via

$$\|\mathbf{U_h}\|_{\mathrm{G}}^2 + 2\|\mathbf{U_h}\|_{\mathrm{Stab}}^2 \leq \|(\mathbf{f_u}, \mathbf{f_b})\|_{G,\star}^2. \tag{12}$$

We want to estimate the fluid pressure. By the discrete Babuska-Brezzi-condition, we have for all $p_h \in Q_h$ the unique existence of $\mathbf{v_h} \in V_h$ with

$$\nabla \cdot \mathbf{v_h} = -p_h \ , \ |\mathbf{v_h}|_1 \leq \beta_u^{-1} \|p_h\|_0. \tag{13}$$

Examining the term $(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U_h}, (\mathbf{v_h}, \mathbf{0}, 0, 0))$, we end up with

$$\begin{aligned}\|p_h\|_0^2 \leq \|\mathbf{f_u}\|_{-1} |\mathbf{v_h}|_1 + \nu \|\nabla \mathbf{u_h}\|_0 |\mathbf{v_h}|_1 + |\mathcal{S}_{lps}((\mathbf{u_h}, \mathbf{0}, 0, 0), (\mathbf{v_h}, \mathbf{0}, 0, 0))| \\ - (\mathbf{a} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) + ((\nabla \times \mathbf{b_h}) \times \mathbf{d}, \mathbf{v_h})\end{aligned}$$

by using that $\mathbf{u_h} \in V_h^{div}$. Based on the inequalities

$$-(\mathbf{a} \cdot \nabla \mathbf{u_h}, \mathbf{v_h}) = (\mathbf{a} \cdot \nabla \mathbf{v_h}, \mathbf{u_h}) \leq C_p \|\mathbf{a}\|_\infty |\mathbf{u_h}|_1 |\mathbf{v_h}|_1 \, , \tag{14}$$

$$((\nabla \times \mathbf{b_h}) \times \mathbf{d}, \mathbf{v_h}) \leq C_p \|\mathbf{d}\|_\infty \|\nabla \times \mathbf{b_h}\|_0 |\mathbf{v_h}|_1 \, , \tag{15}$$

$$\|(\mathbf{v_h}, \mathbf{0}, 0, 0)\|_{\text{Stab}}^2 \leq \left(\max_M \left(\tau_1 |\mathbf{a_M}|^2 \right) \right) |\mathbf{v_h}|_1 + \left(\max_M \tau_2 d \right) |\mathbf{v_h}|_1 \tag{16}$$

together with (13), we get that

$$\begin{aligned}\|p_h\|_0^2 &\leq \Big(\|\mathbf{f_u}\|_{-1} + \nu \|\nabla \mathbf{u_h}\|_0 + C_p \|\mathbf{a}\|_\infty \|\nabla \mathbf{u_h}\|_0 + C_p \|\mathbf{d}\|_\infty \|\nabla \times \mathbf{b_h}\|_0 \Big) \|p_h\|_0 \\ &\quad + \|(\mathbf{u_h}, \mathbf{0}, 0, 0)\|_{\text{Stab}} \|(\mathbf{v_h}, \mathbf{0}, 0, 0)\|_{\text{Stab}} \\ &\leq \Big(\|\mathbf{f_u}\|_{-1} + \big(\sqrt{\nu} + \frac{C_p \|\mathbf{a}\|_\infty}{\sqrt{\nu}} + \frac{C_p \|\mathbf{d}\|_\infty}{\sqrt{\lambda}} \big) \|\mathbf{U_h}\|_G \Big) \|p_h\|_0 \\ &\quad + \frac{1}{\beta_u} \Big(\max_M \big(\sqrt{\tau_1} |\mathbf{a_M}| \big) + \max_M \sqrt{\tau_2 d} \Big) \|\mathbf{U_h}\|_{\text{Stab}} \|p_h\|_0\end{aligned}$$

holds. This leads to

$$\begin{aligned}\|p_h\|_0 \leq \|\mathbf{f_u}\|_{-1} + \left(\sqrt{\nu} + \frac{C_p \|\mathbf{a}\|_\infty}{\sqrt{\nu}} + \frac{C_p \|\mathbf{d}\|_\infty}{\sqrt{\lambda}} \right) \|\mathbf{U_h}\|_G \\ + \frac{1}{\beta_u} \left(\max_M (\sqrt{\tau_1} |\mathbf{a_M}|) + \max_M \sqrt{\tau_2 d} \right) \|\mathbf{U_h}\|_{\text{Stab}}\end{aligned}$$

which shows the unique existence of the fluid pressure. We define the norms

$$\|\mathbf{c}\|_C = \sqrt{\lambda} \left(L_0^{-1} \|\mathbf{c}\|_0 + \|\nabla \times \mathbf{c}\|_0 \right) \, , \ \|s\|_S = \left(\sqrt{\lambda} \right)^{-1} (\|s\|_0 + L_0 \|\nabla s\|_0) \tag{17}$$

on the spaces C and Q with a user-chosen length-scale L_0. Using integration by parts, we define the bilinear form of the Maxwell problem

$$\mathcal{C}_{Maxwell}((\mathbf{b}, r), (\mathbf{c}, s)) = \mathcal{A}_G((\mathbf{0}, \mathbf{b}, 0, r), (\mathbf{0}, \mathbf{c}, 0, s)) \, .$$

For this problem, the continuous Babuska-Brezzi-condition

$$\inf_{(\mathbf{b}, r) \in C \times S} \sup_{(\mathbf{c}, s) \in C \times S} \frac{\mathcal{C}_{Maxwell}((\mathbf{b}, r), (\mathbf{c}, s))}{(\|\mathbf{b}\|_C + \|r\|_S)(\|\mathbf{c}\|_C + \|s\|_S)} \geq \beta_m \tag{18}$$

holds. Let $(\mathbf{b_h}, 0) \in C_h \times S_h \subset C \times S$. By (18) with a projector Π there exists a unique $(\overline{\mathbf{c}}, \overline{s}) \in C \times S$ with $\|\overline{\mathbf{c}}\|_C + \|\overline{s}\|_S = 1$ such that

$$\beta_m \|\mathbf{b_h}\|_C \leq \mathcal{C}_{Maxwell}\left((\mathbf{b_h}, 0), (\overline{\mathbf{c}}, \overline{s})\right) \tag{19}$$
$$= \underbrace{\mathcal{C}_{Maxwell}\left((\mathbf{b_h}, 0), (0, \overline{s} - \Pi(\overline{s}))\right)}_{=I} + \underbrace{\mathcal{C}_{Maxwell}\left((\mathbf{b_h}, 0), (\overline{\mathbf{c}}, \Pi(\overline{s}))\right)}_{=II}$$

holds. We get with $\|\overline{s}\|_S \leq 1$ and $\|\overline{\mathbf{c}}\|_C \leq 1$ the estimates

$$\begin{aligned} I &= -\left(\nabla \times \mathbf{b_h}, \overline{s} - \Pi(\overline{s})\right) \qquad (20)\\ &\leq C \|\mathbf{U_h}\|_{\text{Stab}} \max_M \left(\frac{h_M}{\sqrt{\tau_6}}\right) |\overline{s}|_1 \leq C \frac{\sqrt{\lambda}}{L_0} \max_M \left(\frac{h_M}{\sqrt{\tau_6}}\right) \|\mathbf{U_h}\|_{\text{Stab}}, \\ II &= \lambda \left(\nabla \times \mathbf{b_h}, \nabla \times \overline{\mathbf{c}}\right) + \left(\nabla \times \mathbf{b_h}, \Pi(\overline{s})\right) \\ &\leq \sqrt{\lambda} \left(\|\mathbf{U_h}\|_G + \left(\min_M \sqrt{\tau_6}\right)^{-1} \|\mathbf{U_h}\|_{\text{Stab}} \right) \qquad (21) \end{aligned}$$

by the Cauchy-Schwarz inequality. Putting (20) and (21) into (19) leads to

$$\|\mathbf{b_h}\|_C \leq \frac{\sqrt{\lambda}}{\beta_m} \Big(\|\mathbf{U_h}\|_G + \big(\frac{1}{\min_M \sqrt{\tau_6}} + \frac{C}{L_0} \max_M \big(\frac{h_M}{\sqrt{\tau_6}} \big) \big) \|\mathbf{U_h}\|_{\text{Stab}} \Big) \tag{22}$$

and this gives the unique existence of the discrete magnetic field.

Finally, we want to estimate the magnetic pseudo-pressure. From the equation $(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U_h}, (\mathbf{0}, \mathbf{0}, 0, r_h)) = -(\mathbf{b_h}, \nabla r_h) + \sum_M \tau_5 \|\nabla r_h\|_{0,M}^2 = 0$, we conclude by using (17) that

$$\|\nabla r_h\|_0 \leq \left(\min_M \sqrt{\tau_5}\right)^{-1} \|\mathbf{b_h}\|_0 \leq L_0 \left(\sqrt{\lambda}\right)^{-\frac{1}{2}} \left(\min_M \sqrt{\tau_5}\right)^{-1} \|\mathbf{b_h}\|_C \tag{23}$$

which shows the unique existence of the discrete magnetic pseudo-pressure.

3 Error Analysis for smooth solutions

Let $\mathbf{U} \in V \times C \times Q \times S$ and $\mathbf{U_h} \in V_h \times C_h \times Q_h \times S_h$ be the solutions of (7) and (9), respectively. Lemma 2 implies for the consistency error

$$(\mathcal{A}_G + \mathcal{S}_{lps})(\mathbf{U} - \mathbf{U_h}, \mathbf{V_h}) = \mathcal{S}_{lps}(\mathbf{U}, \mathbf{V_h}) \tag{24}$$

for all $\mathbf{V_h} \in V_h \times C_h \times Q_h \times S_h$. Let $\mathbf{J} = \left(\mathbf{j^u}, \mathbf{j^b}, j^p, j^r\right)$ be appropriate interpolation operators. We therefore can decompose the error and receive $\mathbf{U} - \mathbf{U_h} = (\mathbf{U} - \mathbf{JU}) + (\mathbf{JU} - \mathbf{U_h}) = \varepsilon - \mathbf{E_h}$ with errors $\varepsilon = (\varepsilon_{\mathbf{u}}, \varepsilon_{\mathbf{b}}, \varepsilon_p, \varepsilon_r)$ and $\mathbf{E_h} = (\mathbf{e_u}, \mathbf{e_b}, e_p, e_r)$. By setting $\mathbf{V_h} = \mathbf{E_h}$ in (24), we get that

$$\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{Stab}^2 = \underbrace{\mathcal{S}_{lps}(\mathbf{U}, \mathbf{E_h})}_{=I} \underbrace{-\mathcal{A}_G(\varepsilon, \mathbf{E_h})}_{=II} - \underbrace{\mathcal{S}_{lps}(\varepsilon, \mathbf{E_h})}_{=-III}. \tag{25}$$

We obtain

$$I \leq \left(\mathcal{S}_{lps}\left(\mathbf{U},\mathbf{U}\right)\right)^{\frac{1}{2}}\left(\mathcal{S}_{lps}\left(\mathbf{E_h},\mathbf{E_h}\right)\right)^{\frac{1}{2}} \leq \|\mathbf{U}\|_{Stab}\|\mathbf{E_h}\|_{Stab}, \tag{26}$$

$$|III| = \mathcal{S}_{lps}\left(\varepsilon,\mathbf{E_h}\right) \leq \|\varepsilon\|_{Stab}\|\mathbf{E_h}\|_{Stab}, \tag{27}$$

$$-II \leq \|\varepsilon\|_{G}\|\mathbf{E_h}\|_{G} + IV \tag{28}$$

with

$$\begin{aligned} IV = {} & \left(\mathbf{a}\cdot\nabla\varepsilon_{\mathbf{u}},\mathbf{e_u}\right) - \left(\left(\nabla\times\varepsilon_{\mathbf{b}}\right)\times\mathbf{d},\mathbf{e_u}\right) - \left(\nabla\times\left(\varepsilon_{\mathbf{u}}\times\mathbf{d}\right),\mathbf{e_b}\right) \\ & - \left(\varepsilon_p,\nabla\cdot\mathbf{e_u}\right) + \left(\nabla\cdot\varepsilon_{\mathbf{u}},e_p\right) + \left(\nabla\varepsilon_r,e_p\right) - \left(\varepsilon_{\mathbf{b}},\nabla e_r\right). \end{aligned} \tag{29}$$

Then we can summarize estimates (25)-(29) as

$$\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{Stab}^2 \leq \left(\|\varepsilon\|_{Stab} + \|\mathbf{U}\|_{Stab}\right)\|\mathbf{E_h}\|_{Stab} + \|\varepsilon\|_G\|\mathbf{E_h}\|_G + |IV|. \tag{30}$$

Now we have to estimate the remaining seven terms in IV. Integration by parts and Cauchy-Schwarz inequality yield the estimations

$$\begin{aligned} \left(\mathbf{a}\cdot\nabla\varepsilon_{\mathbf{u}},\mathbf{e_u}\right) = -\left(\mathbf{a}\cdot\nabla\mathbf{e_u},\varepsilon_{\mathbf{u}}\right) &\leq \left(\sum_M \frac{\|\mathbf{a}\|_{\infty,M}^2}{\nu}\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\right)^{\frac{1}{2}}\|\mathbf{E_h}\|_G, \\ -\left(\varepsilon_p,\nabla\cdot\mathbf{e_u}\right) &\leq \left(\sum_M \frac{1}{\tau_2}\|\varepsilon_p\|_{0,M}^2\right)^{\frac{1}{2}}\|\mathbf{E_h}\|_{Stab}, \\ -\left(\varepsilon_{\mathbf{b}},\nabla e_r\right) &\leq \left(\sum_M \frac{1}{\tau_5}\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\right)^{\frac{1}{2}}\|\mathbf{E_h}\|_{Stab}, \\ \left(\nabla\varepsilon_r,\mathbf{e_b}\right) = -\left(\varepsilon_r,\nabla\cdot\mathbf{e_b}\right) &\leq \left(\sum_M \frac{1}{\tau_6}\|\varepsilon_r\|_{0,M}^2\right)^{\frac{1}{2}}\|\mathbf{E_h}\|_{Stab}, \\ -\left(\nabla\times\left(\varepsilon_{\mathbf{u}}\times\mathbf{d}\right),\mathbf{e_b}\right) &\leq \left(\sum_M \frac{1}{\lambda}\|\mathbf{d}\|_{\infty,M}^2\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\right)^{\frac{1}{2}}\|\mathbf{E_h}\|_G. \end{aligned} \tag{31}$$

The term $-(e_p,\nabla\cdot\varepsilon_{\mathbf{u}})$ vanishes using $\nabla\cdot\mathbf{u}=0$ and provided that $\mathbf{j^u u}\in V_h^{div}$ holds. Let $\mathbf{d}\in\left[\mathrm{W}^{1,\infty}\left(\Omega\right)\right]^d$. For sufficiently smooth $\mathbf{e}$ and $\mathbf{f}$, by the vector formula $\nabla\times\left(\mathbf{e}\times\mathbf{f}\right) = \mathbf{f}\cdot\nabla\mathbf{e} - \mathbf{f}\left(\nabla\cdot\mathbf{e}\right) - \mathbf{e}\cdot\nabla\mathbf{f} + \mathbf{e}\left(\nabla\cdot\mathbf{f}\right)$, the inequalities of Cauchy, Schwarz and Poincare and an estimation for the divergence by the Jacobian, it follows

$$\begin{aligned} -\left(\left(\nabla\cdot\varepsilon_{\mathbf{b}}\right)\times\mathbf{d},\mathbf{e_u}\right) = \left(\varepsilon_{\mathbf{b}},\nabla\times\left(\mathbf{e_u}\times\mathbf{d}\right)\right) = \sum_M\left(\varepsilon_{\mathbf{b}},\nabla\times\left(\mathbf{e_u}\times\mathbf{d}\right)\right)_M \\ \leq \Big(\sum_M \frac{\left(1+\sqrt{d}\right)^2}{\nu}\left(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\right)^2\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}}\|\mathbf{E_h}\|_G. \end{aligned} \tag{32}$$

We then summarize equations (30)-(32). Using Young's inequality, we obtain the inequality $\|\mathbf{E_h}\|_G^2 + \|\mathbf{E_h}\|_{Stab}^2 \leq S_1^2 + S_2^2$ with

$$S_1 := \|\varepsilon\|_G + \left(\sum_M \frac{1}{\nu}\|\mathbf{a}\|_{\infty,M}^2\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\right)^{\frac{1}{2}} + \left(\sum_M \frac{1}{\lambda}\|\mathbf{d}\|_{\infty,M}^2\|\varepsilon_{\mathbf{u}}\|_{0,M}^2\right)^{\frac{1}{2}}$$
$$+\Big(\sum_M \frac{\left(1+\sqrt{d}\right)^2}{\nu}\left(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\right)^2\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}},$$
$$S_2 := \|\varepsilon\|_{Stab} + \|\mathbf{U}\|_{Stab} + \left(\sum_M \frac{1}{\tau_2}\|\varepsilon_p\|_{0,M}^2\right)^{\frac{1}{2}} + \Big(\sum_M \frac{1}{\tau_5}\|\varepsilon_{\mathbf{b}}\|_{0,M}^2\Big)^{\frac{1}{2}}$$
$$+\left(\sum_M \frac{1}{\tau_6}\|\varepsilon_r\|_{0,M}^2\right)^{\frac{1}{2}}.$$

Now, the approximation properties of the FE spaces yield for a smooth solution $\mathbf{U} = (\mathbf{u}, \mathbf{b}, p, r) \in [H^{k+1}(\Omega)]^d \times [H^{k+1}(\Omega)]^d \times H^k(\Omega) \times H^k(\Omega)$ that

$$S_1^2 \leq C\sum_M h_k^{2k}\Big(\nu\big(1 + \frac{\|\mathbf{a}\|_{\infty,M}^2 h_M^2}{\nu^2}\big) + \lambda\frac{\|\mathbf{d}\|_{\infty,M}^2 h_M^2}{\lambda^2}\big)\Big)|\mathbf{u}|_{k+1,M}^2$$
$$+\Big(\lambda + \frac{h_M^2}{\nu}\left(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}\right)^2\Big)|\mathbf{b}|_{k+1,M}^2, \tag{33}$$
$$S_2^2 \leq C\sum_M h_k^{2k}\Big(\big(\tau_2 + \tau_1|\mathbf{a_M}|^2 + \tau_4|\mathbf{d_M}|^2\big)|\mathbf{u}|_{k+1,M}^2$$
$$+\big(\tau_3|\mathbf{d_M}|^2 + \tau_6 + \frac{h_M^2}{\tau_5}\big)|\mathbf{b}|_{k+1,M}^2 + \frac{1}{\tau_2}|p|_{k,M}^2 + \big(\tau_5 + \frac{1}{\tau_6}\big)|r|_{k,M}^2\Big). \tag{34}$$

We denote the local fluid and magnetic Reynolds numbers by

$$Re_{f,M} := \|\mathbf{a}\|_{\infty,M} h_M/\nu, \qquad Re_{m,M} := \|\mathbf{d}\|_{\infty,M} h_M/\lambda.$$

respectively. Moreover, we will call an error estimate to be of order k if the coefficients multiplying corresponding Sobolev norms of the solutions are of order h^k uniformly w.r.t. the problem data. In this case, sufficient conditions can be found by the following restrictions on the local mesh width h_M

$$\sqrt{\nu}Re_{f,M} \leq C, \quad \sqrt{\lambda}Re_{m,M} \leq C, \quad h_M(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M}) \leq C\sqrt{\nu} \tag{35}$$

and on the stabilization parameters

$$\tau_1|\mathbf{a_M}|^2, \tau_4|\mathbf{d_M}| \leq C\nu\max\{1; Re_{f,M}^2\} + \lambda Re_{m,M}^2, \tag{36}$$
$$\tau_3|\mathbf{d_M}|^2, \tau_6, \frac{h_M^2}{\tau_5} \leq C\max\{\lambda; \frac{h_M^2(\|\mathbf{d}\|_{\infty,M} + \|\nabla\mathbf{d}\|_{\infty,M})^2}{\nu}\}, \tag{37}$$
$$\tau_2 \sim 1, \quad \text{provided that} \quad |\mathbf{u}|_{k+1,M} \sim |p|_{k,M}. \tag{38}$$

Theorem 3. *Assume that the solution* $\mathbf{U} = (\mathbf{u}, \mathbf{b}, p, r)$ *of (7) belongs to* $[H^{r+1}(\Omega)]^d \times [H^{r+1}(\Omega)]^d \times H^r(\Omega) \times H^r(\Omega)$ *and that* $\mathbf{j^u u} \in V_h^{div}$*. Further, let the LPS parameters are chosen according to conditions (36)-(38) and that the local mesh width* h_M *is chosen such that (35) is valid. Then we obtain*

$$\|\mathbf{U_h}-\mathbf{JU}\|_{\mathrm{G}}^2+\|\mathbf{U_h}-\mathbf{JU}\|_{\mathrm{Stab}}^2 \leq C \sum_M h_k^{2k} \Big(|\mathbf{u}|_{k+1,M}^2+|\mathbf{b}|_{k+1,M}^2+|p|_{k,M}^2+|r|_{k,M}^2 \Big).$$

4 Outlook

Restriction (35) on the mesh is not convincing. Let us assume

$$(\mathbf{v} - \mathbf{j^u v}, \zeta_\mathbf{h}) = 0 \quad \forall \mathbf{v} \in V \text{ and } \forall \zeta_\mathbf{h} \in [D_h^\mathbf{u}(M)], \tag{39}$$

$$\big(\mathbf{c} - \mathbf{j^b c}, \eta_\mathbf{h}\big) = 0 \quad \forall \mathbf{c} \in C \text{ and } \forall \eta_\mathbf{h} \in \big[D_h^\mathbf{b}(M)\big]. \tag{40}$$

Sufficient conditions on $\mathcal{T}_h, \mathcal{M}_h$ and the FE spaces for (39)-(40) can be found in [4]. This allows modified estimates of the skew-symmetric terms, e.g.

$$(\mathbf{a} \cdot \nabla \varepsilon_\mathbf{u}, \mathbf{e_u}) = -\Big(\kappa\left(\mathbf{a} \cdot \nabla \mathbf{e_u}\right), \varepsilon_\mathbf{u}\Big) \leq \Big(\sum_M \frac{1}{\tau_1} \|\varepsilon_\mathbf{u}\|_{0,M}^2 \Big)^{\frac{1}{2}} \|\mathbf{E_h}\|_{Stab}$$

and similarly for the Lorentz terms. Then, a modification of (33) leads to

$$S_1^2 \leq C \sum_M h_k^{2k} \Big(\nu + \frac{h_M^2}{\tau_1} + \frac{h_M^2}{\tau_3} \Big) |\mathbf{u}|_{k+1,M}^2 + \Big(\lambda + \frac{h_M^2}{\tau_4} \Big) |\mathbf{b}|_{k+1,M}^2. \tag{41}$$

A calibration of the parameters in (41) and (34) gives

$$\tau_1 \sim \min \Big(\frac{h_M}{|\mathbf{a_M}|}; \frac{h_M^2}{\nu} \Big), \quad \tau_3 \sim \min \Big(\frac{h_M}{|\mathbf{d_M}|}; \frac{h_M^2}{\nu} \Big), \quad \tau_4 \sim \min \Big(\frac{h_M}{|\mathbf{d_M}|}; \frac{h_M^2}{\lambda} \Big)$$

and allows to omit the restrictions (35). A careful estimation has to consider the approximation of $\mathbf{a_M} \sim \mathbf{a}$ and $\mathbf{d_M} \sim \mathbf{d}$ which can be accomplished via a Gronwall argument for the time-dependent (linearized) MHD-problem.

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