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How fast do coercive polynomials grow?

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Abstract

In this article we analyze the relationship between the order of growth of a coercive polynomial and the stability of its coercivity. In particular, for coercive polynomials, we consider the order of coercivity which expresses how fast they grow at infinity and we link it to the so-called degree of stable coercivity. The latter describes how stable the coercivity is with respect to small perturbations of the coefficients. Our main result is an explicit relation between these numbers. This extends existing results on polynomials regarding their growth and the stability of their coercivity. Finally, we illustrate our results by constructing two families of coercive polynomials either with a bounded number of variables or with a bounded degree that possess an arbitrarily small positive order of coercivity.

Keywords: coercivity, order of growth, stability of coercivity, Łojasiewicz exponent at infinity

AMS subject classifications: Primary 26C05, 26C15. Secondary 14A99.

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1 Introduction

Motivation

For coercive multivariate polynomials, we consider the order of growth at infinity and how this relates to the stability of coercivity with respect to perturbations of the coefficients.

As a motivation, let us first consider the univariate case. A polynomial $f \in \mathbb{R}[X]$ is called coercive on $\mathbb{R}$ if $f(x) \to +\infty$ whenever $|x| \to +\infty$. This is the case if and only if the leading coefficient of f is positive and the degree $\deg(f)$ of f is positive and even. This, in turn, is equivalent to the property $f(x)/|x|^q$ being coercive for all $q \in [0, \deg(f))$. Hence, the number $\deg(f)$ expresses how fast f grows for large x, and, thus, it can be viewed as a meaningful measure for the order of growth of f at infinity. We call this number the *order of coercivity* of f.

We observe further that, in the univariate case, small perturbations of a coercive polynomial f by another univariate polynomial g preserves coercivity. In fact, if f is coercive, so is $f + g$ whenever $\deg(g) \leq \deg(f)$ and if the leading coefficient of g is sufficiently close to zero. On the other hand, $f + g$ will not necessarily be coercive if the degree of g exceeds the degree of f, and, thus, the number $\deg(f)$ can also be viewed as a measure expressing how *stable* the coercivity of f on $\mathbb{R}$ is. We call this number the *degree of stable coercivity* of f.

Consequently, for a univariate coercive polynomial f, the order of coercivity coincides with the degree of stable coercivity, and both are equal to the degree of f. Once these two numbers are properly defined in the multivariate setting, it is only natural to ask if the order of coercivity again equals the degree of stable coercivity, and if so, whether these numbers again coincide with the degree of f.

In [BS15b] the first question is answered affirmatively for a broad class of coercive polynomials whereas the authors give a dissenting answer to the second question. More precisely, using properties of the underlying Newton polytopes, a class of coercive polynomials f is identified for which the order of coercivity coincides with the degree of stable coercivity, and both are equal to a so-called *degree of convenience* of f which, in general, differs from $\deg(f)$.

In the present article we shall show that for coercive polynomials f the degree of stable coercivity of f may differ from the order of coercivity of f in general, but not "too much". More precisely, our main results show that for any coercive polynomial, its degree of stable coercivity is always equal to the integral part of the order of coercivity. We shall further characterize the case when the order of coercivity of f is maximum possible by positivity of its leading form. The latter turns out to be equivalent to the degree of stable coercivity of f also being maximum possible (see Theorems 15 and 16).

Related literature

Coercivity of multivariate polynomials itself is an interesting property for various reasons. In polynomial optimization theory it is a recurring question whether a given polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ attains its infimum over $\mathbb{R}^n$ (see, e.g. [BS15a, ED08, GSED14, GSED11, NDS06, Sch06, VP07, VP10]); a similar question is equally relevant in the integer and mixed-integer programming variant [BHS15a]. Coercivity of f is a sufficient condition for f having this property, and, thus, it is a natural task to verify or disprove whether f is coercive.

As a further consequence of coercivity, f is bounded below on $\mathbb{R}^n$ by some $v \in \mathbb{R}$, so that $f - v$ is positive semi-definite on $\mathbb{R}^n$. Also, since coercivity of f is equivalent to the boundedness of its lower level sets $\{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ for all $\alpha \in \mathbb{R}$, understanding coercivity can be useful to decide whether a basic semi-algebraic set is bounded. Furthermore, properness of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ can be characterized by coercivity of the polynomial $\|F\|_2^2$. This is useful to decide whether F is globally invertible (see, e.g. [BS16, BA07, CDTT14]).

Coercivity of polynomials is partially analyzed in [JLL14] and, in the convex setting, in [JPL14], while the coercivity of a polynomial f defined on a basic closed semi-algebraic set and its relation to the Fedoryuk and Malgrange conditions are examined in [VP10]. A connection between coercivity of multivariate polynomials and their Newton polytopes is given in [BS15a]. In [MN14], the authors study how fast – not necessarily coercive – polynomials grow on semi-algebraic sets.

Outline of the article

This article is structured as follows. In Section 2, for coercive multivariate polynomials $f \in \mathbb{R}[X_1, \dots, X_n]$, we recall the notion and also some of the properties of the so-called order of coercivity $o(f)$ and we link them to the Łojasiewicz exponent at infinity. For coercive polynomials f we further recall the definition of the degree of stable coercivity $s(f)$ and introduce the degree of strongly stable coercivity $\tilde{s}(f)$. Also, by studying the order of coercivity of rational functions, we show that $o(f)$ is always positive for any coercive polynomial f.

For a coercive polynomial f, Section 3 describes how the order of coercivity $o(f)$, the degree of stable and strongly stable coercivity, $s(f)$ and $\tilde{s}(f)$, respectively, are related. One of the two main results, Theorem 15, gives an explicit relation between these three numbers. The other main result, Theorem 16, shows that coercive polynomials f whose order of coercivity $o(f)$ is maximum possible, that is, $o(f) = \deg(f)$, are exactly the polynomials with a positive definite leading form. We also show that this is equivalent to the fact that their degree of stable coercivity $s(f)$ is maximum possible, that is, $s(f) = \deg(f)$.

In Section 4 we explicitly construct two families of coercive polynomials with the corresponding order of coercivity being positive but tending to zero. For the first family the number of variables is hold fixed but the degree varies, and, for the second family, the degree is fixed but the number of variables varies. This gives rise to the question, addressed in Section 5, of determining the minimal possible order of coercivity $\mathfrak{o}(n, d)$ for the set of coercive polynomials $f \in \mathbb{R}[X_1, \ldots, X_n]$ with degree not exceeding d.

Notation

Let $\mathbb{R}[X_1, \ldots, X_n]$ denote the ring of polynomials in n variables with real coefficients. We write $f \in \mathbb{R}[X_1, \ldots, X_n]$ in multi-index notation as $f = \sum_{\alpha \in A(f)} a_\alpha X^\alpha$ with $A(f) \subseteq \mathbb{N}_0^n$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $a_\alpha \in \mathbb{R}$ for $\alpha \in A(f)$, and $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$. We will assume that the set $A(f)$ is chosen minimally in the sense that $A(f) = \{\alpha \in \mathbb{N}_0^n \mid a_\alpha \neq 0\}$ holds. The evaluation of f at some $x \in \mathbb{R}^n$ is then expressed by $f(x) = \sum_{\alpha \in A(f)} a_\alpha x^\alpha$. The degree of f is defined as $\deg(f) := \max_{\alpha \in A(f)} |\alpha|$ with $|\alpha| = \sum_{i=1}^n \alpha_i$. Clearly, every polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ decomposes uniquely into its *homogeneous components* $f_0, \ldots, f_d \in \mathbb{R}[X_1, \ldots, X_n]$, where every f_i is homogeneous of degree i.

For the subspace of polynomials of degree bound by $d \in \mathbb{N}_0$ defined by

$$\mathbb{R}[X_1, \ldots, X_n]_d := \{f \in \mathbb{R}[X_1, \ldots, X_n] \mid \deg(f) \leq d\},$$

a basic result [Mar08, Remark 1.2.5] states

$$\dim \mathbb{R}[X_1, \ldots, X_n]_d = \binom{n+d}{d}. \tag{1.1}$$

In this article, $\|\cdot\|$ stands for an arbitrary norm on $\mathbb{R}^n$ unless specified otherwise and for $f \in \mathbb{R}[X_1, \ldots, X_n]$ we define

$$\|f\|_\infty := \max \{|a_\alpha| \mid \alpha \in A(f)\}.$$

We recall the following immediate estimate; for completeness, its proof can be found in the Appendix.

Observation 1. *For $f \in \mathbb{R}[X_1, \ldots, X_n]_d$, where $n \in \mathbb{N}$, $d \in \mathbb{N}_0$, and any $q \in [d, +\infty)$, the following estimate holds:*

$$|f(x)| \leq \binom{n+d}{d} \cdot \|f\|_\infty \cdot (\|x\|_\infty^q + 1), \quad x \in \mathbb{R}^n.$$

2 Order and stability of coercivity

The order of coercivity

A function $f : S \to \mathbb{R}$, defined on a subset $S \subseteq \mathbb{R}^n$, is coercive if for all $c \in \mathbb{R}$ there exists some $M \in \mathbb{R}$ such that for all $x \in S$ the implication

$$\|x\| \geq M \Rightarrow f(x) \geq c$$

holds. The function f is called q-coercive for some $q \geq 0$ if $f(x)/\|x\|^q$ is coercive. Note that coercivity and q-coercivity are properties that are independent of the choice of the norm on $\mathbb{R}^n$. The following characterization of q-coercivity, $q \geq 0$, turns out to be useful for our later purposes. For completeness, we give its short proof in the Appendix.

Observation 2. *Let $f : S \to \mathbb{R}$ defined on a subset $S \subseteq \mathbb{R}^n$ and $q \geq 0$ be given. Then f is q-coercive if and only if*

$$\forall c > 0 \quad \exists M \geq 0 \quad \forall x \in S,\ \|x\| \geq M \ :\ f(x) \geq c \cdot \|x\|^q \tag{A}$$

holds.

For coercive $f : S \to \mathbb{R}$, the number

$$o(f) := \sup\{q \geq 0 \mid f \text{ is } q\text{-coercive}\}$$

is called the *order of coercivity* of f. A coercive function f is q'-coercive for all q' with $0 \leq q' < o(f)$, but f need not be $o(f)$-coercive. Now, if property (A) does not hold for all but only some $c > 0$, we may not conclude q-coercivity of f. However, the following holds:

Observation 3. *Let $f : S \to \mathbb{R}$ defined on a subset $S \subseteq \mathbb{R}^n$ and $q > 0$ be given. Then the property*

$$\exists c > 0 \quad \exists M \geq 0 \quad \forall x \in S,\ \|x\| \geq M \ :\ f(x) \geq c \cdot \|x\|^q \tag{B}$$

implies $o(f) \geq q$.

For a proof we refer to the Appendix. Note that the converse statement does not necessarily hold.

The following example shows that for quadratic coercive polynomials, the equality $o(f) = \deg(f)$ is always fulfilled.

Example 4. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$, $f(x) = x^T Q x + L^t x + c$ with $Q \in \mathbb{R}^{n \times n}$ symmetric, $L \in \mathbb{R}^n$ and $c \in \mathbb{R}$ be given. If f is coercive, then $o(f) = 2$. Indeed, as f is coercive, Q must be positive definite. It is well-known that this implies the existence of a unique global minimal point $x_0 \in \mathbb{R}^n$ of f, and one finds $f(x) = (x - x_0)^T Q(x - x_0) + f(x_0)$ (see, e.g. [BHS15b]). Denoting the smallest eigenvalue of Q by λ, one obtains that $f(x) \geq \lambda(x - x_0)^T(x - x_0) + f(x_0) = \lambda\|x - x_0\|_2^2 + f(x_0)$ holds for all $x \in \mathbb{R}^n$, and thus, by Observation 3, the inequality $o(f) \geq 2$ follows. On the other hand, Observation 1 implies $o(f) \leq \deg(f)$, and, due to $\deg(f) = 2$, one obtains $o(f) \leq 2$.

Property (B) shows how the order of coercivity is related to the so-called *Łojasiewicz exponent at infinity* (see, e.g. [Kra07]). For a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^m$ it is defined as

$$\mathcal{L}_\infty(F) := \sup\left\{\nu \in \mathbb{R} \mid \exists c, M > 0\ \forall x \in \mathbb{R}^n : \|x\| \geq M \Rightarrow \|F(x)\| \geq c\,\|x\|^\nu\right\}.$$

Indeed, for coercive polynomials, the order of coercivity and Łojasiewicz exponent at infinity coincide:

Observation 5. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive. Then*

$$o(f) = \mathcal{L}_\infty(f).$$

Proof. From the definitions, $o(f) \leq \mathcal{L}_\infty(f)$. Note that the coercivity of f implies $\mathcal{L}_\infty(f) \geq 0$ and $o(f) \geq 0$. Suppose at first that $\mathcal{L}_\infty(f) = 0$, then $o(f) = 0$ and hence $o(f) = \mathcal{L}_\infty(f) = 0$ follows. Suppose next that $\mathcal{L}_\infty(f) > 0$. It is enough to show that for any $0 \leq q < \mathcal{L}_\infty(f)$, we also have $q < o(f)$. Let $\varepsilon > 0$ with $q + \varepsilon \leq \mathcal{L}_\infty(f)$. By definition of $\mathcal{L}_\infty(f)$, there is $c > 0$ and $M \geq 0$ with $f(x) \geq c \cdot \|x\|^{q+\varepsilon} = (c \cdot \|x\|^\varepsilon) \cdot \|x\|^q$ whenever $\|x\| \geq M$. As $c \cdot \|x\|^\varepsilon$ grows without bound, this yields $o(f) > q$. □

Since the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ is known to be rational (see [Gor61]), Observation 5 yields the following:

Corollary 6. *If $f \in \mathbb{R}[X_1, \ldots, X_n]$ is coercive, then $o(f) \in \mathbb{Q}$.*

The stability of coercivity

Given a coercive polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ we are interested in how stable this coercivity property is under small perturbations of f by other polynomials. This gives rise to the following definition for stability of coercivity which was already analyzed from the viewpoint of the underlying Newton polytopes in [BS15b] and is inspired by the concept of stable boundedness of polynomials [Mar03].

Definition 7 (Stable coercivity)**.** A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called q-stably coercive for $q \in \mathbb{N}_0$, if there exists an $\varepsilon > 0$ such that for all $g \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg g \leq q$ and all coefficients of g bounded in absolute value by ε it holds that $f + g$ is coercive. The degree of stable coercivity $s(f)$ of f is the largest q such that f is q-stable coercive.

We also introduce the following stronger notion for the stability of coercivity.

Definition 8 (Strong stable coercivity)**.** A polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ is called strongly q-stable coercive for $q \in \mathbb{N}_0$, if for all $g \in \mathbb{R}[x]$ with $\deg g \leq q$ it holds that $f + g$ is coercive. The degree of strongly stable coercivity $\tilde{s}(f)$ of f is the largest q such that f is strongly q-stable coercive.

Observations on the order of coercivity

In this section we collect some preliminary results on the order of coercivity. The following result is not only useful for our purposes but interesting in its own right: It states that any coercive rational function has a positive order of growth. To this end we denote the vanishing set of the polynomial $g \in \mathbb{R}[X_1, \dots, X_n]$ by $V(g) := \{x \in \mathbb{R}^n \mid g(x) = 0\}$ and its complement by $V^c(g) := \mathbb{R}^n \setminus V(g)$.

Theorem 9. *Let $f, g \in \mathbb{R}[X_1, \dots, X_n]$, $g \neq 0$, such that $f/g : V^c(g) \to \mathbb{R}$ is coercive. Then*

$$o\,(f/g) > 0.$$

As a corollary, every coercive polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ has a positive order of growth, which is a known result (see, e.g. [Gor61]). For the proof of Theorem 9, we use the following.

Theorem 10 ([Gor61, Theorem 4.1])**.** *Let $P(x, z, w)$ be a real polynomial of $n' = n_1 + n_2 + n_3$ variables $x \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$, $w \in \mathbb{R}^{n_3}$ where n_1, n_2, n_3 are non-negative integers. If the surface M given by the equation*

$$P(x, z, w) = 0$$

is not empty and lies in the domain defined by the inequality

$$\|z\|_2 \geq \varphi(\|x\|_2),$$

where $\varphi(t) \to +\infty$ as $t \to +\infty$, then there exists constants $h > 0$ and b such that this surface also lies in the domain defined by the inequality

$$\|z\|_2 \geq \|x\|_2^h - b.$$

Our choice of a suitable φ is given in the next lemma.

Lemma 11. *In the setting of Theorem 9, let $\varphi : [0, \infty) \to \mathbb{R}$ be defined as follows: Let*

$$\tilde{\varphi}(t) := \inf \left\{ \left| \frac{f(y)}{g(y)} \right| \mid y \in V^c(g),\ \|y\|_2 = t \right\}$$

and put

$$\varphi(t) = \begin{cases} \tilde{\varphi}(t), & \tilde{\varphi}(t) < \infty, \\ 0, & \textit{else}. \end{cases}$$

Then φ is coercive.

Proof. Note at first that there can at most be $d := \deg(g)$ many $t \geq 0$ with the property that $g(x) = 0$ whenever $\|x\| = t$ (resulting in $\tilde{\varphi}(t) = \infty$). Indeed, suppose there were $d+1$ points $t_0 < \ldots < t_d$ with that property. Consider the leading form g_d of g and pick $x \in \mathbb{R}^n$ with $g_d(x) \neq 0$. Then $\lambda \mapsto g_d\left(\lambda \cdot \frac{x}{\|x\|}\right)$ is a univariate polynomial of degree d with zeros at $t_0, \ldots, t_d$, which is impossible. Now suppose φ is not coercive. Thus there is $C > 0$ and an increasing sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of reals with $\tau_k \to +\infty$ and $\varphi(\tau_k) \leq C$. We may assume τ_1 is larger than any of the at most d points t_i from above. Fix $\varepsilon > 0$. Thus there is a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq V^c(g)$ with $\|x_k\| = \tau_k$ and $\left|\frac{f(x_k)}{g(x_k)}\right| - \varepsilon \leq \varphi(\tau_k) \leq C$, contradicting coercivity of f/g. □

We may now prove Theorem 9.

Proof of Theorem 9. Let $f, g \in \mathbb{R}[X_1, \ldots, X_n]$ with f/g coercive. To apply the theorem by Gorin, we let $n_1 = n$, $n_2 = n_3 = 1$ and define $P \in \mathbb{R}[X_1, \ldots, X_n, Z, W]$ via

$$P(x, z, w) := (f(x) - zg(x))^2 + (wg(x) - 1)^2, \quad x \in \mathbb{R}^n,\ z, w \in \mathbb{R}.$$

The surface $M = V(P)$ is not empty, as

$$\begin{aligned} M &= \left\{(x, z, w) \in \mathbb{R}^{n+2} \mid f(x) = zg(x) \text{ and } wg(x) = 1\right\} \\ &= \left\{(x, z, w) \in \mathbb{R}^{n+2} \mid x \in V^c(g),\ f(x)/g(x) = z \text{ and } w = 1/g(x)\right\}. \end{aligned}$$

Consider the function φ from Lemma 11. We now show that M lies in the domain defined by the inequality $\|z\|_2 \geq \varphi(\|x\|_2)$. To this end let a point $(x, z, w) \in M$ be given. Then $g(x) \neq 0$ and so we conclude

$$\|z\|_2 = |z| = \left|\frac{f(x)}{g(x)}\right| \geq \inf\left\{\left|\frac{f(y)}{g(y)}\right| \mid y \in V^c(g),\ \|y\|_2 = \|x\|_2\right\} = \varphi(\|x\|_2).$$

Hence, we may apply Gorin's theorem, so there are constants $h > 0$ and b such that M also lies in the domain defined by the inequality

$$|z| = \|z\|_2 \geq \|x\|_2^h - b.$$

This means $|f(x)/g(x)| \geq \|x\|_2^h - b$ whenever $g(x) \neq 0$. From Observation 3 we conclude $o(|f/g|) \geq h$. Since f/g is coercive, $f(x)/g(x) > 0$ for $x \in V^c(g)$ with $\|x\|_2$ large enough, which implies $o(f/g) \geq h$, too. □

We note that for a q-coercive polynomial f, the number q is strictly bound above by the order of growth of f. This is implicit in [Gor61]; we give an explicit proof in the setting of this article for completeness.

Lemma 12. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive. Then f is not $o(f)$-coercive.*

Proof. Suppose the contrary and let f be $o(f)$-coercive. By Corollary 6 and Theorem 9, the number $o(f)$ is rational and positive, so $o(f) = p/q$, with some $p, q \in \mathbb{N}$ and we may further assume that p is even. Thus by definition, $f(x)/\|x\|_2^{p/q}$ is coercive, and hence $r(x) := f(x)^q/\|x\|_2^p$ is coercive. However, as p is even, r is a coercive rational function, so by Theorem 9, there is $h > 0$ such that r is h-coercive. Hence by Observation 2 and continuity of f, there are $c_1 > 0$, $c_2 \geq 0$ with

$$\frac{f(x)^q}{\|x\|_2^p} \geq c_1\|x\|_2^h - c_2.$$

Hence, for any fixed $0 < \varepsilon < h$,

$$\frac{f(x)^q}{\|x\|_2^{p+\varepsilon}} \geq c_1\|x\|_2^{h-\varepsilon} - \frac{c_2}{\|x\|_2^{\varepsilon}},$$

which means f^q is $(p + \varepsilon)$-coercive. As f, being coercive, attains positive values for large x, this implies that f is $((p + \varepsilon)/q)$-coercive, and we conclude $o(f) \geq (p + \varepsilon)/q$, contradicting the assumption $o(f) = p/q$. □

Although, by Lemma 12, a coercive $f \in \mathbb{R}[X_1, \ldots, X_n]$ is not $o(f)$-coercive, one can still underestimate f by an $o(f)$-power of a norm for large values of $\|x\|$. That is, for coercive polynomials, we have a converse statement to Observation 3. Several variants of this result are known; one may argue by Tarski-Seidenberg [Gor61] or, in the complex setting, by curve selection at infinity [Kra07]. Our contribution is a proof by elementary methods.

Lemma 13. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive. Then there exist $c > 0$, $M \geq 0$ with*

$$f(x) \geq c \cdot \|x\|^{o(f)}, \quad \|x\| \geq M.$$

Proof. Assume to the contrary that the assertion does not hold. Then for every sequence $\{c_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ with $c_k \downarrow 0$ there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$ with $\|x_k\| \to +\infty$ such that

$$f(x_k) < c_k\|x_k\|^{o(f)}, \quad k \in \mathbb{N}.$$

Since f is coercive and $\|x_k\| \to +\infty$, we may further assume $f(x_k) \geq 0$ for all $k \in \mathbb{N}$, hence

$$0 \leq \frac{f(x_k)}{\|x_k\|^{o(f)}} < c_k, \quad k \in \mathbb{N}.$$

Using the decomposition $f = \sum_{i=0}^{d} f_i$ of f into its homogeneous components $f_i \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $i = 0, \ldots, d$, with $\xi_k := x_k/\|x_k\|$ the latter property yields

$$0 \leq \sum_{i=\lceil o(f)\rceil}^{d} \|x_k\|^{i-o(f)} f_i(\xi_k) < c_k - \sum_{i=0}^{\lceil o(f)\rceil -1} \|x_k\|^{i-o(f)} f_i(\xi_k), \quad k \in \mathbb{N}. \tag{2.1}$$

Due to $c_k \downarrow 0$ and $i - o(f) < 0$ holding for all $i = 0, \ldots, \lceil o(f)\rceil - 1$, the right hand side in (2.1) converges to zero as k approaches infinity. This implies

$$\lim_{k\to\infty} \sum_{i=\lceil o(f)\rceil}^{d} \|x_k\|^{i-o(f)} f_i(\xi_k) = 0 \tag{2.2}$$

Passing to an appropriate convergent subsequence of the sequence $\xi_k = x_k/\|x_k\|$ with a limit point ξ, due to continuity of each homogeneous component f_i of f, we may assume that $\lim_{k\to\infty} f_i(\xi_k) = f_i(\xi) \in \mathbb{R}$ for all $i = \lceil o(f)\rceil, \ldots, d$. In fact, property (2.2) yields $f_i(\xi) = 0$ for all $i = \lceil o(f)\rceil, \ldots, d$, and, hence again, using the homogeneous decomposition of f one obtains

$$f(t \cdot \xi) = \sum_{i=0}^{d} f_i(t \cdot \xi) = t^{\lceil o(f)\rceil -1} f_{\lceil o(f)\rceil -1}(\xi) + \cdots + f_0(\xi) \quad \text{for all } t \in \mathbb{R}$$

resulting in $o(f) \leq \lceil o(f)\rceil - 1$, a contradiction. □

3 Main result

In this section we show how the degree of stable and strongly stable coercivity are tied to the order of growth (Theorem 15). In case of a positive definite leading form, a stronger characterization is available (Theorem 16). We use the following estimate in the proof of both.

Proposition 14. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive. Then the following inequalities are fulfilled:*

$$\tilde{s}(f) \leq s(f) \leq o(f) \leq \tilde{s}(f) + 1.$$

Proof. The first inequality $\tilde{s}(f) \leq s(f)$ follows obviously from the Definitions 7 and 8. To see $s(f) \leq o(f)$, assume $q := s(f) > o(f)$. We introduce polynomials

$$f_{c,\sigma} := f - c \cdot \left(\sum_{j=1}^{n} \sigma_j X_j \right)^q,$$

parametrized by $c \in \mathbb{R}$ and $\sigma \in \Sigma := \{-1, 1\}^n$. As $s(f) = q$, for every $\sigma \in \Sigma$ there is $\varepsilon_\sigma > 0$ such that $f_{c,\sigma}$ is coercive whenever $c \in [-\varepsilon_\sigma, \varepsilon_\sigma]$. Let $\hat{\varepsilon} := \min_{\sigma\in\Sigma} \varepsilon_\sigma$ and fix $\hat{c} \in (0, \hat{\varepsilon})$. Hence $f_{\hat{c},\sigma}$ is coercive for all $\sigma \in \Sigma$ and thus also bounded from below. Boundedness from below means for every σ there is $k_\sigma \geq 0$ with

$$f(x) \geq \hat{c} \left(\sum_{j=1}^n \sigma_j x_j \right)^q - k_\sigma, \quad x \in \mathbb{R}^n, \ \sigma \in \Sigma.$$

Put $\hat{k} := \max_{\sigma \in \Sigma} k_\sigma$. Then for $x \in \mathbb{R}^n$

$$f(x) \geq \hat{c} \cdot \max_{\sigma \in \Sigma} \left(\sum_{j=1}^n \sigma_j x_j \right)^q - \hat{k} = \hat{c} \cdot \left(\sum_{j=1}^n |x_j| \right)^q - \hat{k} = \hat{c} \cdot \|x\|_1^q - \hat{k},$$

so Observation 3 implies $o(f) \geq q = s(f)$, a contradiction.

Now we proceed to prove the third inequality $o(f) \leq \tilde{s}(f) + 1$. Assume the contrary: Let $q := \tilde{s}(f)$ and suppose $o(f) > q + 1$. We have arrived at a contradiction if we may show that for any $g \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $q+1$, $f+g$ is coercive, as in this case $\tilde{s}(f) \geq q + 1 = \tilde{s}(f) + 1$. To this end fix an arbitrary $g \in \mathbb{R}[X_1, \ldots, X_n]_{q+1}$. Now choose $c_1 > \binom{n+d}{d} \cdot \|g\|_\infty$. As $o(f) > q+1$, f is $q+1$-coercive, therefore, by Observation 2 and continuity of f, there is $c_2 \geq 0$ such that $f(x) \geq c_1 \|x\|_\infty^{q+1} - c_2$ holds for $x \in \mathbb{R}^n$, and hence, by Observation 1,

$$\begin{aligned} f(x) + g(x) \geq f(x) - |g(x)| &\geq c_1 \|x\|_\infty^{q+1} - c_2 - \binom{n+d}{d} \cdot \|g\|_\infty \left(\|x\|_\infty^{q+1} + 1 \right) \\ &= c_1' \cdot \|x\|_\infty^{q+1} - c_2', \quad x \in \mathbb{R}^n, \end{aligned}$$

for some appropriately chosen $c_1' > 0$, $c_2' \in \mathbb{R}$. Thus $f + g$ is coercive.

□

We show now how the integer part of the order of growth and our notions of stability are related to each other.

Theorem 15. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive.*

1. *If $o(f)$ is integer, then*
$$\tilde{s}(f) + 1 = s(f) = o(f).$$
2. *If $o(f)$ is fractional, then*
$$\tilde{s}(f) = s(f) = \lfloor o(f) \rfloor.$$

Proof. In order to prove 1., we show $\tilde{s}(f) + 1 = o(f)$ first. By integrality of $\tilde{s}(f)$, $o(f)$ and by the property $o(f) \in [\tilde{s}(f), \tilde{s}(f) + 1]$ holding due to Proposition 14, it is enough to show that $\tilde{s}(f) < o(f)$. Suppose the contrary, that is $\tilde{s}(f) = o(f) =: q$. Now for $c > 0$ and $\sigma \in \Sigma := \{-1, 1\}^n$, define

$$f_{c,\sigma} := f - c \cdot \left(\sum_{j=1}^n \sigma_j X_j \right)^q \in \mathbb{R}[X_1, \ldots, X_n].$$

By definition of $\tilde{s}(f)$, the polynomial $f_{c,\sigma}$ is coercive and hence bounded from below for all $c > 0$ and $\sigma \in \Sigma$. That is, for every $c > 0$ and $\sigma \in \Sigma$, there exists $k_{c,\sigma} \geq 0$ such that

$$f(x) \geq c \cdot \left(\sum_{j=1}^{n} \sigma_j x_j \right)^q - k_{c,\sigma}, \quad x \in \mathbb{R}^n,\ c > 0,\ \sigma \in \Sigma,$$

and hence with $k_c := \max_{\sigma \in \Sigma} k_{c,\sigma}$, we have for all $x \in \mathbb{R}^n$ and $c > 0$ the property

$$f(x) \geq c \cdot \max_{\sigma \in \Sigma} \left(\sum_{j=1}^{n} \sigma_j x_j \right)^q - k_c = c \cdot \left(\sum_{j=1}^{n} |x_j| \right)^q - k_c = c \cdot \|x\|_1^q - k_c.$$

In view of Observation 2, the polynomial f is q-coercive. Since $q = \tilde{s}(f) = o(f)$ is holding by assumption, f is $o(f)$-coercive. This is impossible by Lemma 12, and we may conclude that $\tilde{s}(f) + 1 = o(f)$.

For the second equality $s(f) = o(f)$, put $q := o(f)$. By Lemma 13 and continuity of f, there are constants $c_1, c_2 > 0$ such that

$$f(x) \geq c_1 \|x\|_\infty^q - c_2 \text{ holds for all } x \in \mathbb{R}^n.$$

Define $\varepsilon := \frac{c_1}{2} \cdot \binom{n+q}{q}^{-1}$. Now for any $g \in \mathbb{R}[X_1, \ldots, X_n]_q$ with $\|g\|_\infty \leq \varepsilon$ and all $x \in \mathbb{R}^n$, we have from Observation 1

$$\begin{aligned} f(x) + g(x) &\geq f(x) - |g(x)| \\ &\geq c_1 \|x\|_\infty^q - c_2 - \varepsilon \cdot \binom{n+q}{q} \left(\|x\|_\infty^q + 1 \right) \\ &= \frac{c_1}{2} \|x\|_\infty^q - c_2 - \frac{c_1}{2}. \end{aligned}$$

To summarize, $f + g$ is coercive whenever $\deg g \leq q$ and $\|g\|_\infty \leq \varepsilon$, that is, f is q-stably coercive, or $s(f) = q = o(f)$.

Statement 2. follows at once from Proposition 14. □

Our next result shows that more characterizations are available for a maximal order of coercivity.

Theorem 16. *Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $d \geq 2$. Then, the following assertions are equivalent:*

1. *$f_d(x) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*
2. *There exists $\delta > 0$ such that $f_d(x) \geq \delta \|x\|^d$ for all $x \in \mathbb{R}^n$.*
3. *$o(f) = d$.*
4. *$o(f) > d - 2$.*
5. *$s(f) = d$.*

6. $s(f) \geq d-1$.
7. $\tilde{s}(f) = d-1$.
8. $\tilde{s}(f) \geq d-2$.

Proof. "1 ⇒ 2" For $x = 0$ the assertion is trivial. For nonzero $x \in \mathbb{R}^n$ one obtains

$$f_d(x) = \|x\|^d f_d\left(\frac{x}{\|x\|}\right) \geq \|x\|^d \inf_{y \in S^{n-1}} f_d(y).$$

The infimum is positive by compactness of the sphere. Now for "2 ⇒ 3", let $c_j = \inf_{y \in S^{n-1}} f_j(y)$ for $j = 0, \dots, n-1$ and put $c_d = \delta$. Then by homogeneity of the f_j,

$$f(x) = \sum_{j=0}^{d} f_j(x) \geq \sum_{j=0}^{d} c_j \|x\|^j,$$

hence $o(f) = d$. The implication "3 ⇒ 4" is trivial. The implication "4 ⇒ 1" holds as follows: Suppose $o(f) > d-2$ but $f_d(\tilde{x}) = 0$ for some $\tilde{x} \in \mathbb{R}^n$ with $\tilde{x} \neq 0$. By assumption $o(f)$ is positive, hence f is coercive. Let us show that this implies $f_{d-1}(\tilde{x}) = 0$. Indeed, we find that for all $\lambda \in \mathbb{R}$ it holds

$$f(\lambda \tilde{x}) = \sum_{j=0}^{d} f_j(\lambda \tilde{x}) = \sum_{j=0}^{d-1} \lambda^j f_j(\tilde{x}),$$

which, as a function of λ is unbounded from below unless $f_{d-1}(\tilde{x}) = 0$. In fact, this holds since as $d-1$ is odd. Hence

$$|f(\lambda \tilde{x})| \leq \sum_{j=0}^{d-2} |f_j(\lambda \tilde{x})| = \sum_{j=0}^{d-2} |\lambda|^j |f_j(\tilde{x})|,$$

implying $o(f) \leq d-2$, a contradiction, so 1 through 4 are equivalent.

To see "2 ⇒ 5", let $g \in \mathbb{R}[X_1, \dots, X_n]$ of degree d be given, and let $c' = \max_{x \in S^{n-1}} g_d(x)$. Then $|g_d(x)| \leq c' \|x\|^d$ by homogeneity, so for $\varepsilon \in [-\frac{\delta}{2c'}, \frac{\delta}{2c'}]$,

$$f_d(x) + \varepsilon g_d(x) \geq f_d(x) - |\varepsilon g_d(x)| \geq \delta \|x\|^d - \frac{\delta}{2}\|x\|^d = \frac{\delta}{2}\|x\|^d,$$

hence $f + \varepsilon g$ is still coercive, and we conclude $s(f) = d$.

We show now that 5. implies 6 and 7. The first implication is trivial. To see "5 ⇒ 7", note that Proposition 14 implies $\tilde{s}(f) \geq d-1$. As $\tilde{s}(f) \geq d$ is not possible for a degree d polynomial, $\tilde{s}(f) = d-1$. Since both 6 and 7 imply 8 trivially, all equivalences are shown once "8 ⇒ 4" holds.

So suppose $\tilde{s}(f) \geq d-2$. From the definition of strong stable coercivity, this implies coercivity of f, and d must be even. The function $g(x) = \|x\|_2^{d-2}$ is a polynomial of degree $d-2$. The assumption $\tilde{s}(f) \geq d-2$ implies that $f - c_1 g$ is coercive for all $c_1 > 0$. Hence there is M, depending on c_1, such that

$$f(x) - c_1 \|x\|_2^{d-2} \geq 0$$

holds for $\|x\| \geq M$. As $d \geq 2$, we may use Observation 2 to find that f is $d-2$-coercive. Now Lemma 12 states that $o(f) > d-2$, which finishes the proof. □

4 Example families

Introductory remarks

In this section, we give two explicit example families of coercive polynomials with arbitrarily small but positive order of growth. The first family has a bounded number of variables (two) but varying degree and the second family has a bounded degree (four) but a varying number of variables.

There are some examples families of $\{f_i\}_{i \in I} \subseteq \mathbb{R}[X_1, \dots, X_n]$, where I is some index set, in the literature where the Łojasiewicz exponents at infinity $\mathcal{L}_\infty(f_i)$ of the f_i – and hence the order of coercivity $o(f_i)$ of f_i, if f_i is coercive – are explicitly computed, e.g., [Gor61], [Kra07]. These example families are extensive in the following sense: For every $q \in \mathbb{Q}$ there is $i \in I$ with $\mathcal{L}_\infty(f_i) = q$. Hence, example polynomials with arbitrarily small order of growth are easily given.

However, these example families from the literature were not created with the objective in mind to keep the number of variables and the degree of the resulting polynomials low. The examples we present are, in this sense, not only some further polynomials with known Łojasiewicz exponents at infinity.

In the literature, the computations are rather terse. We take a different route and carefully prove all assertions. These proofs are simplified by partitioning the domain of definition. Specifically, given $S' \subseteq S$, we write $o(f|S')$ for the order of coercivity of f restricted to S'. Then, in view of the immediate Observation 17, we may compute the order of coercivity on more suitable subsets of $\mathbb{R}^n$ instead of on all of $\mathbb{R}^n$.

Observation 17. *Let $S_1, \dots, S_k \subseteq \mathbb{R}^n$, $S := \cup_{i=1}^k S_i$ and $f : S \to \mathbb{R}$ coercive. Then*

$$o(f) = \min_{1 \leq i \leq k} o(f|S_i).$$

The following handy observation is immediate.

Observation 18. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \sum_{j=1}^n c_j |x_j|^{\alpha_j}$ for some $c_j > 0$, $\alpha_j > 0$. Then $o(f) = \min_j \alpha_j$.*

Fixed number of variables

We give now an example of a family of coercive polynomials of arbitrarily small (but, of course, positive) order of growth in two variables. The key observation is that the function $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2$, is $\frac{1}{k}$-coercive on (the image of) the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, t^{2k})$.

Proposition 19. *Consider the polynomial $f_k \in \mathbb{R}[X, Y]$, $k \in \mathbb{N}$, given by*

$$f_k = X^2 + \left(Y - X^{2k}\right)^2. \tag{4.1}$$

Then $o(f_k) = \frac{1}{k}$.

Note that by Theorem 15, $s(f_k) = \tilde{s}(f_k) = 0$ holds for all $k \geq 2$, thus even small linear perturbations of f_k may lead to the loss of coercivity.

Corollary 20. *For any given $\rho > 0$ there is a polynomial that is coercive but not ρ-coercive; this even holds if the number of variables is fixed to 2.*

We split the proof of Proposition 19 into two Lemmata.

Lemma 21. *For f_k as in (4.1), $o(f_k) \geq \frac{1}{k}$.*

Proof. The proof is by case distinction on a given point $(x, y) \in \mathbb{R}^n$. Put

$$\begin{aligned} S^{\downarrow} &:= \{(x, y) \in \mathbb{R}^2 \mid y < 0\}, \\ S^{\leftrightarrow} &:= \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 2x^{2k}\}, \\ S^{\uparrow} &:= \{(x, y) \in \mathbb{R}^2 \mid 2x^{2k} \leq y\}, \end{aligned}$$

and observe that these sets are a partition of $\mathbb{R}^n$.

1. $(x, y) \in S^{\downarrow}$. Then $y < 0$ and hence

$$f_k(x, y) = x^2 + (-|y| - x^{2k})^2 \geq x^2 + y^2 + x^{4k},$$

   thus $o(f_k|S^{\downarrow}) \geq 2$ by Observation 18.

2. $(x, y) \in S^{\leftrightarrow}$. Thus $x^{2k} > \frac{1}{2}|y|$, or $x^2 > \frac{1}{\sqrt[k]{2}}|y|^{1/k}$ and we find

$$f_k(x, y) \geq \frac{1}{2}x^2 + \frac{1}{2}x^2 \geq \frac{x^2}{2} + \frac{1}{\sqrt[k]{2}}|y|^{1/k},$$

   hence $o(f_k|S^{\leftrightarrow}) \geq \frac{1}{k}$.

3. $(x, y) \in S^{\uparrow}$. Then $y \geq 2x^{2k}$, equivalently, $y - x^{2k} \geq \frac{1}{2}y$. As y is non-negative,

$$f_k(x, y) \geq x^2 + \left(\frac{y}{2}\right)^2 = x^2 + \frac{y^2}{4},$$

   which yields $o(f_k|S^{\uparrow}) \geq 2$.

The claim follows now from Observation 17.

□

Lemma 22. *For f_k as in (4.1), $o(f_k) \leq \frac{1}{k}$.*

Proof. Assume $o(f_k) > \frac{1}{k}$. By Observation 2 and continuity of f_k, there are $c_1 > 0$, $c_2 \geq 0$ and $\rho > \frac{1}{k}$ with

$$f_k(x, y) \geq c_1 \|(x, y)\|_1^\rho - c_2, \quad (x, y) \in \mathbb{R}^2.$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of reals with $\lim_{n\to\infty} x_n = +\infty$. We define another sequence $y_n := x_n^{2k}$. Thus $x_n^2 = \sqrt[k]{y_n}$ and

$$f_k(x_n, y_n) = x_n^2 = y_n^{1/k} \geq c_1\|(x_n, y_n)\|_1^\rho - c_2 \geq c_1\|(0, y_n)\|_1^\rho - c_2 = c_1 y_n^\rho - c_2.$$

We shorten the last estimate to the inequality

$$y_n^{1/k} \geq c_1 y_n^\rho - c_2, \quad n \in \mathbb{N},$$

which yields a contradiction: Since $\rho > \frac{1}{k}$, $c_1 > 0$ and $\lim_{n\to\infty} y_n = +\infty$, so this inequality will eventually be violated. □

Fixed degree

Our second example is a family of coercive polynomials of arbitrarily small order of growth with a degree fixed to four. The geometric idea behind this family is similar to the one before: The function $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto x_1^2$ is 2^{2-n} coercive on (the image of) the curve

$$\gamma : \mathbb{R} \to \mathbb{R}^n,\ t \mapsto (t, t^2, t^4, t^8, \ldots, t^{2^{n-2}}, t^{2^{n-1}},)$$

To model this curve as the zero set of a single polynomial, we use the fact that for real polynomials $h_1, \ldots, h_s \in \mathbb{R}[X_1, \ldots, X_n]$ and $x \in \mathbb{R}^n$, the following holds:

$$h_1(x) = \ldots = h_s(x) = 0 \iff \sum_{i=1}^{s} h_i(x)^2 = 0.$$

More specifically, the term $\sum_{i=1}^{n-1} \left(X_{i+1} - X_i^2\right)^2$ vanishes at x if and only if $x_{i+1} - x_i^2 = 0$ for all $i \in \{2, \ldots, n\}$ if and only if x lies on the curve γ, i.e., if and only if x satisfies $x_n = x_{n-1}^2 = x_{n-2}^4 = x_{n-3}^8 = \cdots = x_1^{2^{n-1}}$.

Proposition 23. *Consider the polynomial $g_n \in \mathbb{R}[X_1, \ldots, X_n]$, $n \in \mathbb{N}$, given by*

$$g_n = X_1^2 + \sum_{i=2}^{n} \left(X_i - X_{i-1}^2\right)^2. \tag{4.2}$$

Then $o(g_n) = 2^{2-n}$.

Note that by Theorem 15, $s(g_n) = \tilde{s}(g_n) = 0$ holds for all $n \geq 3$, thus even small linear perturbations of g_n may lead to the loss of coercivity.

Corollary 24. *For any given $\rho > 0$ there is a polynomial that is coercive but not ρ-coercive; this even holds if the degree is fixed to 4.*

The proof of Proposition 23 is divided into three lemmata.

Lemma 25. *Let*

$$C := \{x \in \mathbb{R}^n \mid |x_i| \geq 1 \text{ for all } i \in [n]\}.$$

Then $o(g_n|C) \geq 2^{2-n}$ for g_n as in (4.2).

Proof. We introduce the functions on C

$$T_1(x) := x_1^2, \quad T_i(x) := (x_i - x_{i-1}^2)^2, \quad i = 2, \ldots, n,$$
$$Q_i(x) := \frac{1}{8^i}|x_i|^{2^{2-i}}, \quad i = 1, \ldots, n,$$

so $g_n(x) = \sum_{i=1}^n T_i(x)$ on C. The claim follows from Observation 18 if we can prove by induction

$$\sum_{i=1}^{j} T_i(x) \geq \sum_{i=1}^{j-1} Q_i(x) + 2Q_j(x), \quad x \in C, \ j = 1, \ldots, n. \tag{4.3}$$

The claim in 4.3 trivially holds for $j = 1$. Assume it holds for some $j < n$. For the inductive step it suffices to show that for an arbitrary $x \in \mathbb{R}^n$ one of

$$T_{j+1}(x) \geq 2Q_{j+1}(x) \tag{4.4}$$

or

$$Q_j(x) \geq 2Q_{j+1}(x) \tag{4.5}$$

holds. Indeed, in case (4.4) holds at x, then adding this inequality to (4.3) yields

$$\sum_{i=1}^{j+1} T_i(x) \geq \sum_{i=1}^{j-1} Q_i(x) + 2Q_j(x) + 2Q_{j+1}(x) \geq \sum_{i=1}^{j} Q_i(x) + 2Q_{j+1}(x)$$

In the other case, (4.5) holds at x. Then

$$\sum_{i=1}^{j+1} T_i(x) \geq \sum_{i=1}^{j} T_i(x) \geq \sum_{i=1}^{j} Q_i(x) + Q_j(x) \geq \sum_{i=1}^{j} Q_i(x) + 2Q_{j+1}(x).$$

Now let us show by case distinction on $x \in \mathbb{R}^n$ why (4.4) or (4.5) holds. Again, we introduce a partition

$$S_i^{\downarrow} := \{x \in C \mid x_i < 0\},$$
$$S_i^{\leftrightarrow} := \{x \in C \mid 0 \leq x_i < 2x_{i-1}^2\},$$
$$S_i^{\uparrow} := \{x \in C \mid 2x_{i-1}^2 \leq x_i\},$$

for $i = 1, \ldots, n-1$. Now fix $x \in C$ and consider the cases

1. $x \in S^{\downarrow}_{j+1}$. Thus $x_{j+1} < 0$, and as $x \in C$, we may use monotonicity of exponentials to find

$$T_{j+1}(x) = \left(-|x_{j+1}| - x_j^2\right)^2 \geq x_{j+1}^2 \geq |x_{j+1}|^{2^{2-(j+1)}} \geq 2Q_{j+1}(x),$$

so 4.4 holds.

2. $x \in S^{\leftrightarrow}_{j+1}$. Thus $|x_j|^2 > \frac{1}{2}|x_{j+1}|$ and raising both sides to the $2^{2-(j+1)}$-th power,

$$|x_j|^{2^{2-j}} \geq \frac{1}{2^{2^{2-(j+1)}}}|x_{j+1}|^{2^{2-(j+1)}} \tag{4.6}$$

As $2-(j+1) \leq 0$, we have $\frac{1}{2^{2^{2-(j+1)}}} \geq \frac{1}{4}$. Hence, dividing both sides of 4.6 by 8^j, we see that (4.5) holds.

3. $x \in S^{\uparrow}_{j+1}$. Equivalently, $x_{j+1} - x_j^2 \geq \frac{1}{2}x_{j+1}$, thus by monotonicity again,

$$T_{j+1}(x) \geq \frac{1}{4}x_{j+1}^2 \geq \frac{1}{4}|x_{j+1}|^{2^{2-(j+1)}} \geq 2Q_{j+1}(x),$$

that is, (4.4) holds.

Hence, (4.3) holds for $j+1$ and all $x \in C$, so the induction step is proven. □

Lemma 26. *Let*

$$D := \{x \in \mathbb{R}^n \mid |x_i| < 1 \text{ for some } i \in [n]\}.$$

Then $o(g_n|D) \geq 2^{2-n}$ *for* g_n *as in* (4.2).

Proof. Suppose not. Thus there is a sequence $\{x_m\}_{m\in\mathbb{N}} \subseteq D$ with $\|x_m\|_\infty \to +\infty$ for $m \to \infty$, and

$$g_n(x_m) = x_{m,1}^2 + \sum_{i=2}^{n}\left(x_{m,i} - x_{m,i-1}^2\right)^2 \leq c\|x_m\|_\infty^{2^{2-n}}, \quad m \in \mathbb{N}.$$

Especially,

$$x_{m,1}^2 \leq c\|x_m\|_\infty^{2^{2-n}}, \quad \left(x_{m,i} - x_{m,i-1}^2\right)^2 \leq c\|x_m\|_\infty^{2^{2-n}}, \quad i = 2, \ldots, n. \tag{4.7}$$

We arrive at a contradiction if we can show by induction that there are $N_{n-1} \geq \ldots \geq N_0$ with

$$|x_{m,n-j}| \geq \frac{1}{2^j}\|x_m\|_\infty^{2^{-j}}, \quad m \geq N_j, \quad j = 0, \ldots, n-1. \tag{4.8}$$

Indeed, once the induction is complete, inequality (4.8) holds for all j and all $m \geq N_{n-1}$, and as $\|x_m\|_\infty$ grows without bound, (4.8) forces x_m to leave the set D, contradicting the assumption $x_m \in D$ for all m.

For the basis of the induction, we use the second inequality in (4.7) to find $|x_{m,i-1}^2 - x_{m,i}| \leq c^{1/2}\|x_m\|_\infty^{2^{1-n}}$ and thus

$$x_{m,i-1}^2 \leq |x_{m,i}| + c^{1/2}\|x_m\|_\infty^{2^{1-n}}$$

by the reverse triangle inequality. Using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b \ge 0$, the last inequality yields

$$|x_{m,i-1}| \le |x_{m,i}|^{1/2} + c^{1/4} \|x_m\|_\infty^{2^{-n}} \le \|x_m\|_\infty^{1/2} + c^{1/4} \|x_m\|_\infty^{2^{-n}}$$

By assumption on x_m, $\|x_m\|$ grows without bound, so there is $N_{-1} \in \mathbb{N}$ with $\|x_m\| \ge 1$ for $m \ge N_{-1}$, and then $\|x_m\|_\infty^{2^{-n}} \le \|x_m\|_\infty^{1/2}$. Thus

$$|x_{m,i-1}| \le \left(1 + c^{1/4}\right) \|x_m\|_\infty^{1/2}. \tag{4.9}$$

Also, there is $N_0 \ge N_{-1}$ with $\left(1 + c^{1/4}\right)^2 < \|x_m\|_\infty$ for $m \ge N_0$, which together with (4.9) implies $|x_{m,i-1}| < \|x_m\|_\infty$ for $m \ge N_0$ and $i = 2, \ldots, n$, that is,

$$|x_{m,n}| = \|x_m\|_\infty \quad \text{for } m \ge N_0,$$

a rewording of the basis of the induction.

For the inductive hypothesis, suppose (4.8) holds for some $j < n - 1$. We now prove the inductive step. Using the reverse triangle inequality on (4.7) the other way, we find

$$x_{m,j-1}^2 \ge |x_{m,j}| - c^{1/2} \|x_m\|_\infty^{2^{1-n}}, \quad j = 2, \ldots, n. \tag{4.10}$$

With (4.10) and the inductive hypothesis,

$$x_{m,n-(j+1)}^2 \ge |x_{m,n-j}| - c^{1/2} \|x_m\|_\infty^{2^{1-n}} \ge \frac{1}{2^j} \|x_m\|_\infty^{2^{-j}} - c^{1/2} \|x_m\|_\infty^{2^{1-n}} \tag{4.11}$$

On the other hand, as $\|x_m\|_\infty$ grows without bound, there is $N_{j+1} \ge N_j$ with

$$\begin{aligned}
& \|x_m\|_\infty \ge \left(2^{j+1} c^{1/2}\right)^{1/(2^{-j} - 2^{1-n})} \\
\iff & \|x_m\|_\infty^{2^{-j} - 2^{1-n}} \ge 2^{j+1} c^{1/2} \\
\iff & \|x_m\|_\infty^{2^{-j}} \ge 2^{j+1} c^{1/2} \cdot \|x_m\|_\infty^{2^{1-n}} \\
\iff & \frac{1}{2^{j+1}} \|x_m\|_\infty^{2^{-j}} - c^{1/2} \cdot \|x_m\|_\infty^{2^{1-n}} \ge 0 \\
\iff & \frac{1}{2^j} \|x_m\|_\infty^{2^{-j}} - c^{1/2} \cdot \|x_m\|_\infty^{2^{1-n}} \ge \frac{1}{2^{j+1}} \|x_m\|_\infty^{2^{-j}}
\end{aligned}$$

for $m \ge N_{j+1}$. With (4.11) we deduce

$$x_{m,n-(j+1)}^2 \ge \frac{1}{2^j} \|x_m\|_\infty^{2^{-j}} - c^{1/2} \|x_m\|_\infty^{2^{1-n}} \ge \frac{1}{2^{2(j+1)}} \|x_m\|_\infty^{2^{-j}}$$

and hence

$$|x_{m,n-(j+1)}| \ge \frac{1}{2^{j+1}} \|x_m\|_\infty^{2^{-(j+1)}},$$

proving the induction step. □

Lemma 27. *For g_n as in* (4.2), $o(g_n) \le 2^{2-n}$.

Proof. Assume $o(g_n) > 2^{2-n}$. Using Observation 2 and continuity of g_n, there are $c_1 > 0$, $c_2 \ge 0$ and $\rho > 2^{2-n}$ with

$$g_n(x) \ge c_1 \|x\|_1^\rho - c_2 \quad \forall x \in \mathbb{R}^n. \tag{4.12}$$

Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of reals with $\lim_{m\to\infty} a_m = +\infty$, and define $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^n$ with components $(x_m)_1 := a_m$ and

$$(x_m)_2 := \left((x_m)_1\right)^2,\ (x_m)_3 := \left((x_m)_2\right)^2,\ \ldots,\ (x_m)_n := \left((x_m)_{n-1}\right)^2.$$

Observe that $(x_m)_1 = \left((x_m)_n\right)^{2^{2-n}}$ and $(x_m)_n \to +\infty$ for $n \to \infty$. Then

$$g_n(x_m) = (x_m)_1^2 + \left(\sum_{i=2}^{n-1} 0^2\right) = (x_m)_1^2 = (x_m)_n^{2^{2-n}} \ge c_1 \|x_m\|_1^\rho - c_2$$

by definition of x_m and by (4.12), and we may estimate further

$$\ge c_1 \|\,(0,\ldots,0,(x_m)_n)\,\|_1^\rho - c_2 = c_1 |(x_m)_n|^\rho - c_2 = c_1 (x_m)_n^\rho - c_2$$

which contains the contradictory inequality

$$(x_m)_n^{2^{2-n}} \ge c_1 (x_m)_n^\rho - c_2, \quad m \in \mathbb{N}. \tag{4.13}$$

Indeed, as $(x_m)_n \to +\infty$ for $m \to \infty$ and $c_1 > 0$, $\rho > 2^{2-n}$, inequality (4.13) will eventually be violated.

□

5 Final remarks

In Section 4 we have seen explicit examples of slowly growing coercive polynomials where either the number of variables n or the degree d are fixed. It is thus only a natural question to ask how small the order of growth can get when both the number of variables and the degree of the polynomial are fixed. In other words, we consider for $n \in \mathbb{N}$ and $d \in 2\mathbb{N}$ the number

$$\mathfrak{o}(n,d) = \inf \left\{ o(f) \mid f \in \mathbb{R}[X_1,\ldots,X_n]_d \text{ is coercive} \right\}.$$

We call $\mathfrak{o}(n,d)$ the *minimum possible order of coercivity* of a coercive polynomial in n variables of degree d. It is not known to us whether there is a closed formula for $\mathfrak{o}(n,d)$ or if at least $\mathfrak{o}(n,d) > 0$ for all $n \in \mathbb{N}$ and $d \in 2\mathbb{N}$. Also, we do not know if our example families f_k and g_n from Section 4 are minimal examples in the sense that $o(f_k) = \mathfrak{o}(2,4k)$ or $o(g_n) = \mathfrak{o}(n,4)$.

n	d	Upper bound on $\mathfrak{o}(n,d)$	Attainment	Reference
$*$	2	2	yes	Example 4
1	$*$	d	yes	cf. Section 1
2	$4k$	$1/k$	?	Proposition 19
≥ 2	4	2^{2-n}	?	Proposition 23

Table 1: Upper bounds on the minimum possible order of coercivity $\mathfrak{o}(n,d)$.

Table 1 summarizes the special cases and examples discussed in this article. A star ($*$) indicates arbitrary values; that is, $n \in \mathbb{N}$ or $d \in 2\mathbb{N}$.

In [BS15a] a class of coercive polynomials is identified where coercivity can be verified by analyzing properties of the underlying Newton polytopes at infinity. Then, in [BS15b], it is shown that for each polynomial from the aforementioned class one always has $o(f) = s(f) = c(f) \in 2\mathbb{N}$ with $c(f)$ denoting the so-called degree of convenience of f – which is the length of the shortest intercept of the Newton polytope at infinity with the n coordinate axes. So, for coercive polynomials with a fractional order of growth, for example such as those from Section 4, it would be an interesting question whether their order of growth is encoded in their Newton polytopes as well.

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A Appendix

Proof of Observation 1. Fix $n \in \mathbb{N}$, $d \in \mathbb{N}_0$, $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most d and $q \geq d$. In multi-index notation, $f = \sum_{\alpha \in A(f)} a_\alpha X^\alpha$, and in view of (1.1), we get $|A(f)| \leq \binom{n+d}{d}$. Also, for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq q$, we have

$$|x^\alpha| \leq \|x\|_\infty^{|\alpha|} \leq \max\left(\|x\|_\infty^q, 1\right) \leq \|x\|_\infty^q + 1.$$

The estimates combine to

$$\begin{aligned} |f(x)| = |\sum_{\alpha \in A(f)} a_\alpha x^\alpha| &\leq \|f\|_\infty \sum_{\alpha \in A(f)} |x^\alpha| \leq \|f\|_\infty \sum_{\alpha \in A(f)} \left(\|x\|_\infty^q + 1\right) \\ &= |A(f)| \cdot \|f\|_\infty \cdot \left(\|x\|_\infty^q + 1\right) \leq \binom{n+d}{d} \cdot \|f\|_\infty \cdot \left(\|x\|_\infty^q + 1\right). \end{aligned}$$

□

Proof of Observation 2. Let f be q-coercive and $c > 0$. By definition of q-coercivity, there is $M \geq 0$ such that $f(x)/\|x\|^q \geq c$ whenever $x \in S$ and $\|x\| \geq M$. Multiplication by $\|x\|^q$ gives the claim. Now suppose (A) holds and fix $c > 0$ with the corresponding $M \geq 0$. Division by $\|x\|^q$ yields for all nonzero $x \in S$ with $\|x\| \geq M$ the inequality

$$\frac{f(x)}{\|x\|^q} \geq c,$$

and, thus, $\liminf_{\|x\|\to\infty} f(x)/\|x\|^q \geq c$. Since $c > 0$ was arbitrary, f is q-coercive. □

Proof of Observation 3. For any $0 < \varepsilon < q$ and all nonzero $x \in S$ with $\|x\| \geq M$ one has

$$\frac{f(x)}{\|x\|^{q-\varepsilon}} \geq c\|x\|^{\varepsilon},$$

and, as the right hand side is coercive, f is $(q - \varepsilon)-$coercive. Thus the inequality $o(f) \geq q - \varepsilon$ holds and since ε was arbitrary, $o(f) \geq q$ follows. □

References

[BA07] Carles Bivià-Ausina. Injectivity of real polynomial maps and Łojasiewicz exponents at infinity. *Mathematische Zeitschrift*, 257(4):745–767, 2007.

[BHS15a] Sönke Behrends, Ruth Hübner, and Anita Schöbel. Norm bounds and underestimators for unconstrained polynomial integer minimization. *arXiv preprint arXiv:1502.05107*, 2015.

[BHS15b] Christoph Buchheim, Ruth Hübner, and Anita Schöbel. Ellipsoid bounds for convex quadratic integer programming. *SIAM Journal on Optimization*, 25(2):741–769, 2015.

[BS15a] Tomáš Bajbar and Oliver Stein. Coercive polynomials and their Newton polytopes. *SIAM Journal on Optimization*, 25(3):1542–1570, 2015.

[BS15b] Tomáš Bajbar and Oliver Stein. Coercive polynomials: Stability, order of growth, and Newton polytopes. *Optimization Online, Preprint ID 2015-10-5158, 2015*, 2015.

[BS16] Tomáš Bajbar and Oliver Stein. On globally diffeomorphic polynomial maps via Newton polytopes and circuit numbers. *arXiv preprint arXiv:1602.01977*, 2016.

[CDTT14] Ying Chen, Luis Renato Gonçalves Dias, Kiyoshi Takeuchi, and Mihai Tibar. Invertible polynomial mappings via Newton non-degeneracy, to appear in ann. *Annales de l'Institut Fourier*, 54(5):1807–1822, 2014.

[ED08] Mohab Safey El Din. Computing the global optimum of a multivariate polynomial over the reals. In *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 71–78. ACM, 2008.

[Gor61] Evgenii Alekseevich Gorin. Asymptotic properties of polynomials and algebraic functions of several variables. *Russian mathematical surveys*, 16(1):93–119, 1961.

[GSED11] Aurélien Greuet and Mohab Safey El Din. Deciding reachability of the infimum of a multivariate polynomial. In *Proceedings of the 36th international symposium on Symbolic and algebraic computation*, pages 131–138. ACM, 2011.

[GSED14] Aurélien Greuet and Mohab Safey El Din. Probabilistic algorithm for polynomial optimization over a real algebraic set. *SIAM Journal on Optimization*, 24(3):1313–1343, 2014.

[JLL14] Vaithilingam Jeyakumar, Jean Bernard Lasserre, and Guoyin Li. On polynomial optimization over non-compact semi-algebraic sets. *Journal of Optimization Theory and Applications*, 163(3):707–718, 2014.

[JPL14] Vaithilingam Jeyakumar, Tiên So'n Phạm, and Guoyin Li. Convergence of the Lasserre hierarchy of SDP relaxations for convex polynomial programs without compactness. *Operations Research Letters*, 42(1):34–40, 2014.

[Kra07] Tadeusz Krasiński. On the Łojasiewicz exponent at infinity of polynomial mappings. *Acta Math. Vietnam*, 32:2–3, 2007.

[Mar03] Murray Marshall. Optimization of polynomial functions. *Canadian Mathematical Bulletin*, 46:575–587, 2003.

[Mar08] Murray Marshall. *Positive polynomials and sums of squares*. Number 146. American Mathematical Soc., 2008.

[MN14] Pinaki Mondal and Tim Netzer. How fast do polynomials grow on semialgebraic sets? *Journal of Algebra*, 413:320–344, 2014.

[NDS06] Jiawang Nie, James Demmel, and Bernd Sturmfels. Minimizing polynomials via sum of squares over the gradient ideal. *Mathematical programming*, 106(3):587–606, 2006.

[Sch06] Markus Schweighofer. Global optimization of polynomials using gradient tentacles and sums of squares. *SIAM Journal on Optimization*, 17(3):920–942, 2006.

[VP07] Hà Huy Vui and Tiên So'n Phạm. Minimizing polynomial functions. *Acta Mathematica Vietnamica*, 32(1):71–82, 2007.

[VP10] Hà Huy Vui and Tiên So'n Phạm. Representations of positive polynomials and optimization on noncompact semialgebraic sets. *SIAM Journal on Optimization*, 20(6):3082–3103, 2010.

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