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The line planning routing game*

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Abstract

In this paper, we take a novel perspective on line planning in public transportation: We interpret line planning as a game where the passengers are players who aim at minimizing individual objective functions composed of travel time, transfer penalties, and a share of the overall cost of the solution. We discuss the relation among equilibria of this game and line planning solutions found by optimization approaches. Furthermore, we investigate the algorithmic viability of our approach as a solution method for line planning problems, using a best-response algorithm to find equilibria. We investigate under which conditions a passenger's best-response can be calculated efficiently and which properties are needed to guarantee convergence of the best-response algorithm.

Keywords: Transportation, Game Theory, Routing, Line Planning, Routing Game

1 Introduction

Due to the high complexity of public transportation planning, the planning process is normally subdivided in subsequent steps, such as network design, line planning, timetabling, vehicle scheduling, etc. The *line planning problem* aims at determining the routes, called lines, which are served regularly by a vehicle and the frequencies of these services. When evaluating such a set of lines both the emerging *costs* and the *quality* from the passengers' perspective are taken into account. Various variants of line planning have been formulated and solved as optimization problems. We take a new perspective on line planning: we propose to model line planning as a routing game where passengers choose routes based on travel quality and a cost share, which depends on the amount of passengers who share (parts of) the route. In this paper we address the question on how to find equilibria of this so-defined *line planning routing game (LPRG)* and compare them to line planning solutions found by optimization approaches.
The remainder of this paper is structured as follows. In Section 1.1 we review literature on line planning before we detail our contribution in Section 1.2. We then briefly introduce some concepts from game theory in Section 2. In Section 3 we introduce the line planning problem we study, both in its centralized version (Section 3.1) and as line planning routing game (Section 3.2) and discuss the relations between the two problems (Section 3.3).
In Section 4 we investigate properties of the line planning routing game. We sketch the best-response algorithm used to find equilibria to LPRG, and in Section 4.1 we investigate under which

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conditions on the line planning model a passenger's best-response can be calculated efficiently. The existence of equilibria and the convergence of the best-response algorithm are investigated in Section 4.2. Section 4.3 evaluates the solutions found by the best-response algorithm with respect to solutions found with a centralized approach. Finally, in Section 5 we illustrate and compare the different models on some small line planning instances.

1.1 Related literature

Line planning is an important step in the public transportation planning process. There are many line planning models which differ with respect to the decisions covered by the term *line planning*, the level of detail with which real-world constraints are included in the model, and the way of measuring the travel quality of a line plan. In this paper, we give a brief overview on the line planning models and solution methods which are most relevant for this paper. See, e.g., [Sch11, Sch14] for more extensive overviews on line planning.

Line planning aims at finding a *line concept* (that means: line routes and frequencies) which is good from an operational point of view and offers good travel quality for the passengers. Cost-oriented line planning models focus on minimizing the operational costs subject to the constraint that passenger demand has to be satisfied (see, e.g., [CvDZ98, Bus98, GvHK06, BHK$^+$13]).

Possible ways to measure the quality of a line concept from the point of view of a passenger are the (generalized) travel time and the number of transfers on the route that a passenger would choose.

A few passenger-oriented line planning models aim at minimizing the overall travel time while keeping the costs below a predefined threshold [SS06a, Sch14]. There are also passenger-oriented models which measure quality by the number of direct travelers [Die78, Bus98, BKZ97]. Several models combine quality and cost into one objective [BGP08, GYW06, PB06].

Line planning problems are often modeled and solved as integer programs. Solution approaches for cost-oriented models often assign the demand to the network edges in a preprocessing step and formulate covering or packing models. Solution techniques include branch-and-bound [Bus98, CvDZ98], branch-and-cut [GvHK04], and variable fixing heuristics [BLL04].

Passenger-oriented line planning assumes that passengers choose the "best" route with respect to the chosen line concept (where "best" is often understood as travel-time minimal). For this purpose, passengers' routes cannot be determined in a preprocessing step but have to be determined together with the line concept. [SS06a] model passengers as flows in a *change-and-go network*, which allows to include transfer times in the travel time, and solve the LP-relaxation using Dantzig-Wolfe decomposition. However, this leads to very large IP models and relatively long solution times. [BGP07, BN10, BK12] use column generation to generate passengers' routes. In [BN10] it is shown that this can lead to a significant speed-up with respect to flow formulations in change-and-go networks. However, in order to achieve problem formulations which can be solved for practical instances, these models use several simplifications. Often, transfer times are assumed to be independent of line frequencies (see, e.g., [SS06a, BN10, BK12, Sch14]) or not taken into account at all [BGP07]. [GvHK06, GvHK04] use a model that allows to adjust transfer times to frequencies, but make a different restriction: for each passenger, the path in the network on which he travels is fixed beforehand (even if the exact connection, i.e., the sequence of lines used on this path, is not).

A further drawback of the described passenger-oriented models is that they determine a system-optimum with respect to the cumulated objective functions of all passengers. In order to achieve a system-optimal solution, single passengers may be assigned to routes which are significantly worse than their individually optimal route. [Sch14, GS17] introduce a model where only line concepts which allow *all* passengers to travel on shortest paths (with respect to the line concept) are considered feasible and propose an IP formulation as well as a genetic algorithm.

Solution approaches to line planning which are not IP-based, often concentrate on the line routes only and postpone frequency setting to a later step. They use greedy strategies [CW91, PRR95, Qua03] to construct lines or successively remove lines from a big line pool [Pat25, Son79]. Furthermore, metaheuristics like genetic algorithms [FGP02, FM06a, SW11, GS17], neighborhood search [SW11, CDLSLM17], and simulated annealing [FM06b] are used. [Man80, JBT10, Sch14] describe iterative approaches, where line planning/frequency setting and route assignment steps are iterated.

Furthermore, the trend in research goes towards the integration of different planning steps in public transportation, like line planning and rolling stock planning [CDLSLM16], line planning and timetabling [BBVL17] or even all three problems [Sch17, PSSS17].

There are also game-theoretic approaches to line planning which model line operators as players who compete for a good utilization of the lines they offer [SS06b, Sch09, BKZ09, BKZ11, SS13, Neu14]. In [LMP10], the problem of finding a line concept which is robust against link failures is modeled as a game between the network provider and an adversary. However, to the extent of our knowledge, so far no attempt has been made to model line planning as a game with *passengers* as players.

In the field of transit assignment, models from game theory are used to model passenger flows on networks (see, e.g., [SF89, She85, DCF93, NP88, SFS+11, SSJ11, CF95, FHSS17b]). These models take into account different modeling requirements from practice, like e.g., limited seat capacity or uncertain information about the next arriving vehicles. Equilibria are often found by mathematical programming.

Routing games on networks are also studied from a more theoretical perspective in the area of algorithmic game theory. A good overview of this line of research, both for atomic and non-atomic flow, is given, e.g., in [Rou07]. Questions of interest cover the existence and quality of equilibria and algorithmic approaches to identify equilibria (see, e.g., [Ros73, AAE05, Rou05, ADK+04, Rou07, TW07]

1.2 Contribution of this paper

In this paper, we propose a new perspective on line planning problems with cost and travel quality objective, which motivates a novel algorithmic approach to solve line planning problems. Instead of integrating planning and routing steps or iterating between both as done in the approaches described above, we regard only the routing step and include all planning decisions in this step. To this end, we define an individual objective function for each passenger which is composed of travel time, transfer penalties, and a share of the overall cost of the solution. This way, the line planning problem can be interpreted as a game in which the passengers are the players who aim at minimizing their objective functions.

To find equilibria we propose a best-response algorithm. We investigate the algorithmic viability of this approach, that is, under which conditions on the line planning model a passenger's best-response can be calculated efficiently and which properties are needed to guarantee convergence of the best-response algorithm. For cases where we do not have these properties, we propose heuristics which simplify the routing step.

We compare the solutions found by our algorithm to solutions found by centralized approaches, both theoretically, by investigating the price of anarchy, and experimentally. Furthermore, we show that the solutions found by our approach are more balanced in the sense that passengers with the same origin and destination are assigned to paths with the same generalized costs.

2 Basics from game theory

In this section we describe some basic concepts from game theory which are used in the remainder of this paper. See, e.g., [NRTV07] for a more comprehensive introduction to game theory.

Game theory studies the dynamics of situations where players try to minimize individual, conflicting objective functions. In a game $(\mathcal{Q}, \mathrm{Strat}, h)$, each player $q \in \mathcal{Q}$ has a set of *strategies* Strat_q among which he can choose. The individual objective function $h_q(\mathcal{S}) = h_q(S_q, \mathcal{S}^{-q})$ of player q depends on his chosen strategy S_q, but also on the strategies $\mathcal{S}^{-q} = (S_1, S_2, \ldots, S_{q-1}, S_{q+1}, \ldots, S_{|\mathcal{Q}|})$ chosen by the other players.

A central concept of game theory is the concept of equilibria. A set of strategies $(S_1, \dots, S_{|\mathcal{Q}|})$ is called *(Nash) equilibrium* if none of the players can improve his individual objective function by changing his strategy given that all other players do *not* change their strategies. I.e., $\hat{\mathcal{S}} = (\hat{S}_1, \dots, \hat{S}_{|\mathcal{Q}|})$ is an equilibrium if for all $q \in \mathcal{Q}$ it holds that

$$h_q(\hat{S}_q, \hat{\mathcal{S}}^{-q}) \leq h_q(S_q, \hat{\mathcal{S}}^{-q}) \quad \forall S_q \in \text{Strat}_q.$$

Not all games have equilibria, and even if equilibria exist, they can be hard to find and they do not need to be unique.

A special class of games with good properties is the class of potential games. We call a function $\Phi : \text{Strat} = \text{Strat}_1 \times \text{Strat}_2 \times \ldots \times \text{Strat}_{|\mathcal{Q}|} \to \mathbb{R}$ *potential function*, if it satisfies the relation

$$\Phi(\mathcal{S}) - \Phi(\mathcal{S}') = h_q(S_q, \mathcal{S}^{-q}) - h_q(S'_q, \mathcal{S}^{-q}) \tag{1}$$

for all solutions $\mathcal{S} = (S_1, \dots, S_{|\mathcal{Q}|}) \in \text{Strat}$, all players $q \in \mathcal{Q}$ and all solutions $\mathcal{S}' = (S_1, \dots, S_{q-1}, S'_q, S_{q+1}, \dots, S_{|\mathcal{Q}|}) \in \text{Strat}$ which can be obtained from $\mathcal{S}$ by exchanging the strategy of player q. A game with potential function is called *potential game*. The existence of a potential function allows us to interpret the problem of finding an equilibrium to $(\mathcal{Q}, \text{Strat}, h)$ as an optimization problem. As we can easily verify in (1), an optimal solution to Φ is an equilibrium for the considered game (although there may be equilibria which are not optimal for Φ).

Furthermore, the relation (1) implies that every time a player changes his strategy to improve his personal objective (while the other players' strategies remain unchanged), the solution becomes better with respect to Φ and, in this sense, closer to an equilibrium. This motivates the approach of using *best-response algorithms* to find equilibria: in every step, one of the players changes his strategy to the *best response* with respect to the other players' strategies, i.e., he picks a solution of the optimization problem $\min_{S_q \in \text{Strat}_q} h_q(S_q, \mathcal{S}^{-q})$ as a new strategy. If there is only a finite number of strategies, this procedure converges to an optimum of Φ, and hence to an equilibrium of the game in a finite number of steps.

A centralized way to evaluate a solution $\mathcal{S} = (S_1, \dots, S_{|\mathcal{Q}|})$ is to sum up the individual objective functions to a centralized objective function $H(\mathcal{S}) = \sum_{q \in \mathcal{Q}} h_q(S_q, \mathcal{S}^{-q})$. We call $\mathcal{S} \in \text{Strat}$ *system-optimal* if it minimizes H.

There exist different concepts to measure the inefficiency of equilibria with respect to the centralized objective. The *price of anarchy* is defined as

$$\max_{\mathcal{S}^* \text{ is an equilibrium}} \frac{H(\mathcal{S}^*)}{\min_{\mathcal{S} \in \text{Strat}} H(\mathcal{S})}.$$

Assuming that over time, selfish behavior will converge to equilibrium solutions, the price of anarchy gives a worst-case bound on the quality of such a convergence process.

The *price of stability*,

$$\min_{\mathcal{S}^* \text{ is an equilibrium}} \frac{H(\mathcal{S}^*)}{\min_{\mathcal{S} \in \text{Strat}} H(\mathcal{S})},$$

in contrast, quantifies how far the best equilibrium (i.e., the best solution that would be accepted by the players) is away from system optimality.

3 Line planning with travel quality and cost objective

3.1 The centralized approach

Line planning aims at determining routes and frequencies of vehicles like trains, metros, or buses. As a basis, we consider the underlying *public transportation network (PTN)* $G = (V, E)$. The nodes V of this network represent stations. Two stations are connected by an edge $e \in E$ if there is a direct track connection between the corresponding stations. In this paper, we consider a line pool $\mathcal{L}$ of possible lines, which are simple paths in the network, as input to the problem. The

main task of line planning is to find a *line concept*, i.e., to assign a frequency $f_l \in \mathbb{N}_0$ to every line l in the line pool $\mathcal{L}$. In many line planning models from the literature, constraints on the number of lines which can pass an edge e are imposed. While this is certainly important in practice, in order to keep our line planning model as simple as possible, we do not consider this constraint in this paper.
We denote the costs of a line, depending on its frequency, as $cost_l(f)$. We model $cost_l(f)$ as composed of a frequency-independent cost k_l^1, which represents, e.g., administration costs, and a frequency-based cost k_l^2, e.g., fuel or labor costs. We obtain $cost_l(f) = \gamma_1 k_l^1 + \gamma_2 k_l^2 f_l$ if $f_l > 0$ and 0 otherwise, where γ_1 and γ_2 are non-negative constants. The cost of a line concept represented by frequencies f is thus given as $cost(f) := \sum_{l:f_l>0}(\gamma_1 k_l^1 + \gamma_2 k_l^2 f_l)$.
We consider passenger demand per period given in form of *origin-destination (OD)*-pairs (u_q, v_q), specifying origin u_q and destination v_q of passenger q from the set of passengers Q. To be able to evaluate the quality of the line plan from the passengers' perspective, together with the line concept we determine a set of *passenger routes* $\mathcal{R} := \{R_q : q \in \mathcal{Q}\}$. A route R_q for passenger q specifies a path $P'_q = (e_1, \ldots, e_n)$ from u_q to v_q and for every edge $e_i \in P'_q$ a line l_i which is used while traveling on e_i. I.e., R_q can be written as a sequence $R_q = ((e_1, l_1), (e_2, l_2), \ldots, (e_n, l_n))$. For a given set of routes $\mathcal{R}$ we denote the number of passengers who use line $l \in \mathcal{L}$ on edge $e \in l$ by $x_{(e,l)}(\mathcal{R}) := |\{q \in \mathcal{Q} : (e, l) \in R_q\}|$.
We call a pair of frequencies f and passenger route set $\mathcal{R}$ *feasible*, if the number of passengers does not exceed the vehicle capacity in any run of any line on any edge, under the assumption that passengers spread evenly over all vehicles runs of one line. That is, if for every l and every $e \in l$ it holds that $x_{(e,l)}(\mathcal{R}) \leq f_l \cdot B$, where B denotes the capacity of a single vehicle.

To *evaluate* a line concept, we use a weighted sum of costs, travel time, and transfers. Here, travel time consists of in-vehicle time and transfer time, that is, we do not take waiting times at the origin station into account. The in-vehicle time on route R_q depends only on the chosen route in the PTN. It is given as $c_q(R_q) := \sum_{(e,l)\in R_q} c(e)$, where $c(e)$ is the in-vehicle time for an edge $e \in G$. The transfer time $\tau_q(R_q, f)$ is estimated based on the frequencies of the lines involved in the transfers on the route. In this paper, for a transfer from line l to line l' we assume a transfer time of $\frac{T}{f_l + f_{l'}}$, where T is the period length (often one hour). This models the expected transfer time under the assumption that passengers choose their route based on a periodic timetable. The overall transfer time of passenger q on route R_q is $\tau_q(R_q, f) := \sum_{i=1}^{n-1} \frac{T}{f_{l_i} + f_{l_{i+1}}}$, where $(l_1, l_2, \ldots, l_n)$ is the sequence of lines used on R_q. Furthermore, we include the number of transfers $\text{transfer}_q(R_q) = (n-1)$ into the evaluation of each route. This models the inconvenience arising for the passenger from ea ch transfer.

Definition 3.1. *Given a PTN G, a line pool $\mathcal{L}$, a capacity bound B, a set of passengers $\mathcal{Q}$, a parameter set $(\alpha_1/\alpha_2, \beta, \gamma_1/\gamma_2)$, and a period length T, the* line planning with travel quality and cost objective (LPQC) *is defined as follows: find a pair of frequencies f and routes $\mathcal{R}$ which fulfills $x_{(e,l)}(\mathcal{R}) \leq f_l \cdot B$ and minimizes the objective function*

$$H(\mathcal{R}, f) := \underbrace{\sum_{q\in\mathcal{Q}} \left(\alpha_1 \cdot c_q(R_q) + \alpha_2 \cdot \tau_q(R_q, f) + \beta \cdot \mathit{transfer}_q(R_q)\right)}_{=:\mathit{travel}(\mathcal{R},f)} + \underbrace{\gamma_1 \cdot \sum_{l:f_l>0} k_l^1 + \gamma_2 \cdot \sum_{l:f_l>0} k_l^2 f_l}_{=:\mathrm{cost}(f)} . \tag{2}$$

(LPQC) takes a *centralized* perspective on line planning: we aim to minimize the sum of costs and total travel time (summed up over all passengers). This does not necessarily mean that the travel time for each individual passenger is short. In fact, particular passengers may be forced to take detours for the 'greater good' of allowing short routes for others. See Section 3.3 for an example.

The following observation from [Sch14] will be useful in the remainder of this paper:

Observation 3.2. *Given a route set $\mathcal{R}$ we can easily determine a* corresponding line concept *$f(\mathcal{R}) = (f_l(\mathcal{R}))_{l\in\mathcal{L}}$ by setting $f_l(\mathcal{R}) := \max_{e\in l} \left\lceil \frac{x_{(e,l)}(\mathcal{R})}{B} \right\rceil$.*

Observation 3.2 allows us to omit the line concept as argument in the function H, thus in the following we use the notation $H(\mathcal{R}) := H(\mathcal{R}, f)$ when convenient. The same holds for the functions τ_q, where we write $\tau_q(\mathcal{R})$ or $\tau_q(R_q, \mathcal{R}^{-q})$ instead of $\tau_q(\mathcal{R}, f(\mathcal{R}))$.

3.2 The line planning routing game

In this paper, we interpret line planning as a routing game. The passengers $\mathcal{Q}$ are the players. The strategies of a passenger q are the routes $\mathcal{R}_q$ from his origin u_q to his destination v_q. Based on a set of routes chosen by the passengers $\mathcal{R}$, we determine the line concept as $f(\mathcal{R})$ as described in Observation 3.2. Each passenger has an individual objective function $h_q(R_q, \mathcal{R}^{-q})$ on which he bases the route choice. It depends on his chosen route R_q and the routes chosen by the other passengers $\mathcal{R}^{-q}$. We call this game *line planning routing game (LPRG)* and interpret equilibria $\mathcal{R}^*$ of this game as solutions $(\mathcal{R}^*, f(\mathcal{R}^*))$ of the line planning problem. The choice of the individual objective functions h_q is of course crucial for the quality of the obtained solutions. We want the individual objective functions to

- account for individual travel quality as well as costs in order to find a solution which is balanced between the two partly contradicting objectives of minimizing costs while maximizing quality, and
- model passengers' behavior as realistically as possible.

We propose the following general model. The passengers' individual objective functions are composed of the travel quality of the solution $\text{travel}_q := \alpha_1 \cdot c_q(R_q) + \alpha_2 \cdot \tau_q(R_q) + \beta \cdot \text{transfer}_q(R_q)$ and a share of the overall costs, $\text{cost}_q(R_q, \mathcal{R}^{-q})$, that is, we have

$$h_q(R_q, \mathcal{R}^{-q}) := \text{travel}_q(R_q, \mathcal{R}^{-q}) + \text{cost}_q(R_q, \mathcal{R}^{-q}).$$

To share the costs among the passengers, we propose two models:

1. equally divide the cost of all lines among all passengers that are choosing this line as part of their route
$$\text{cost}_q(\mathcal{R}) := \sum_{l \in R_q} \frac{\text{cost}_l(f(\mathcal{R}))}{|\{q' \in \mathcal{Q} : l \in R_{q'}\}|}$$
(called *line-based* cost model in the following), or
2. split the line costs of line l among the edges $e \in l$ as *edge costs* $\text{cost}_{(e,l)}$ (referred to as *edge-based* cost model in the following) and compute the cost for passenger q as
$$\text{cost}_q(\mathcal{R}) := \sum_{(e,l) \in R_q} \frac{\text{cost}_{(e,l)}(f(\mathcal{R}))}{x_{(e,l)}(\mathcal{R})}.$$
In this paper, we assume that the edge costs are proportional to the edge lengths $c(e)$, i.e., $\text{cost}_{(e,l)}(f(\mathcal{R})) := (\gamma_1 k_l^1 + \gamma_2 k_l^2 f_l) \frac{c(e)}{\sum_{e \in l} c(e)}$.

In Definition 3.3 we summarize the definition of the LPRG:

Definition 3.3. *In the* line planning routing game (LPRG)*, the passengers $q \in \mathcal{Q}$ act as players. Every passenger (player) chooses among the routes from his origin u_q to his destination v_q (strategies) to minimize his individual objective function $h_q(R_q, \mathcal{R}^{-q})$ which depends both on the route R_q chosen by q and the routes chosen by the other passengers $\mathcal{R}^{-q}$.*

Note that in the definition of the quality functions in Section 3.1 and the individual objective functions in the section, we implicitly assumed that all passengers have the same perception of quality of a travel route since we assume the weighting factors α_1, α_2, β, γ_1, and γ_2 to be the same for each passenger. It would be possible to replace these common weighting factors by a set of individual weighting factors for each passenger. However, for the sake of simplicity, in this paper we only consider the case of common weighting factors for all passengers.

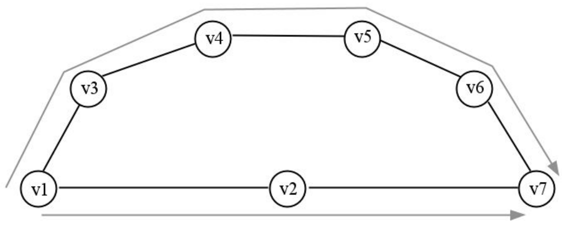


Figure 1: Example instance where LPQC finds undesirable solution.

3.3 Relation between LPQC and LPRG

In this section we discuss the relation between the objective function H of the line planning problem with travel quality and cost objective (LPQC) and the individual objective functions h_q of the line planning routing game (LPRG).

By definition $\sum_{q\in\mathcal{Q}} \text{travel}_q(R_q, \mathcal{R}^{-q}) = \text{travel}(\mathcal{R}, f(\mathcal{R}))$. Furthermore, in the line-based cost model, we have $\sum_{q\in\mathcal{Q}} \text{cost}_q(R_q, \mathcal{R}^{-q}) = \text{cost}(f(\mathcal{R}))$. This is also true in the edge-based cost model, as long as it is ensured that a line does not contain an edge which no passenger is using on this particular line, which we will assume in the following. We conclude that

$$\sum_{q\in\mathcal{Q}} h_q(R_q, \mathcal{R}^{-q}) = H(\mathcal{R}, f(\mathcal{R})).$$

That is, a system-optimal route set for LPRG corresponds to an optimal solution of LPQC. Hence, if the price of anarchy in the LPRG is small, an equilibrium $\mathcal{R}^*$ of the game provides us with a good approximation $(\mathcal{R}^*, f(\mathcal{R}^*))$ for LPQC.

Lemma 3.4. *Denote by I an instance of the LPQC. Assume that the price of anarchy for the corresponding instance I_{RG} of LPRG is bounded by ξ. Then any equilibrium $\mathcal{R}^*$ of I_{RG} is a ξ-approximation $(\mathcal{R}^*, f(\mathcal{R}^*))$ for I.*

So, on the one hand, finding an equilibrium to LPRG may be regarded as a new, decentralized, way of solving LPQC. On the other hand, one may argue that in some cases, optimal solutions to LPQC are not desirable in practice. Indeed, it may happen that the route set $\mathcal{R}$ in a solution $(\mathcal{R}, f)$ to LPQC allots very long routes to some passengers for the 'greater good' of a solution which is optimal with respect to the centralized objective function H.

We discuss an example for the latter in the remainder of this section. Consider the situation shown in Figure 1: There are seven (railway) stations and two lines (depicted by gray arrows) from station v_1 to station v_7. One is a fast line which stops only at one intermediate station, the other one is a regional line which serves a geographically different route and visits many small stations in between. Assume that the transportation capacity of each line is $B = 100$. The demand situation is as follows: 100 passengers want to travel from v_1 to v_7, 50 want to travel from v_2 to v_7, and some smaller amounts of passengers are traveling to and from the regional stations. Hence, both lines have to be established. Now, if the cost parameters γ_1 and γ_2 in the centralized objective function H are comparatively large, both lines will be established with frequency 1 in an optimal solution $(\hat{\mathcal{R}}, \hat{f})$ to LPQC. This means that 50 of the 100 passengers from v_1 to v_2 will be sent via the regional train route in an optimal solution.

However, if this solution was implemented in real life, at station v_1, when the passengers from v_1 and v_7 have to make a decision which train to board, the fast train is still empty. To implement the solution $(\hat{\mathcal{R}}, \hat{f})$ into practice, somebody would have to convince these 50 passengers to use a slower connection to reserve the seats in the fast train for the passengers from v_2 to v_7 boarding later. It is not hard to imagine, that the passengers from v_1 to v_7 would board the train anyway so that the ones starting in v_2 could not board or the train would be overcrowded.

This would not happen in the solution $(\mathcal{R}^*, f(\mathcal{R}^*))$ provided by an equilibrium $\mathcal{R}^*$ of the corresponding routing game LPRG. In this solution, all passengers from v_1 to v_7 would choose the fast train and the planner would be *forced* to provide enough frequency here to avoid overcrowding - unless taking the slow line would be cheap enough to be a favorable option for the passengers. Hence, if we assume that $\text{cost}_q(R_q)$ is an estimate of the real costs that a passenger pays on a route R_q, in this example the solution $(\mathcal{R}^*, f(\mathcal{R}^*))$ defined by an equilibrium $\mathcal{R}^*$ of LPRG models passenger behavior in a better way, provides better estimates of actual solution quality and helps to avoid overcrowding and is therefore, from this perspective, preferable to the solution $(\hat{R}, \hat{f})$ found by the centralized perspective taken in LPQC.

4 Finding equilibria to LPRG

To find equilibria to the LPRG, we use a *best-response algorithm* which is outlined below.

Algorithm 1 Best response algorithm

Require: PTN, line pool, set of passengers $\mathcal{Q}$, individual objective functions h_q, maximal number of iterations $m \in \mathbb{N} \cup \infty$

Ensure: A route set $\mathcal{R}$

```
Start with an empty route set (or with an arbitrary non-empty route set).
while improvements for the passengers possible and m not reached do
    for Passenger q ∈ Q do
        Calculate optimal passenger route R_q according to h_q.
    end for
end while
```

In the remainder of this paper we discuss under which assumptions we can find routes for passengers in the routing step of Algorithm 1 in polynomial time (Section 4.1), for which instances of the LPRG Algorithm 1 converges to an equilibrium (Section 4.2), and the quality of the equilibria (Section 4.3). We conclude the section in 4.4 with the description of heuristic modifications of the individual objective functions which guarantee polynomial solvability of the routing step and convergence.

4.1 The routing problem

In every step of Algorithm 1 we have to solve the following *routing problem* for passenger q:

Definition 4.1. *Given PTN G, line pool $\mathcal{L}$, origin u_q, destination v_q and individual objective function h_q for passenger q (defined by parameter set $(\alpha_1/\alpha_2, \beta, \gamma_1/\gamma_2)$ and period length T), and routes $R_{q'}$ for all passengers $q' \in \mathcal{Q} \setminus \{q\}$, the* routing problem for passenger q *(RP_q) consists of finding a route R_q from u_q to v_q such that $h_q(R_q, \mathcal{R}^{-q})$ is minimized.*

Unfortunately, the routing problem which has to be solved in each iteration of Algorithm 1 is NP-hard in general. We see in Section 4.1.1 that there are two components which make the problem hard: 1.) line-based costs (Theorem 4.2), and 2.) frequency-based transfer times (Theorem 4.3). However, if costs are assumed to be edge-based with $\gamma_2 = 0$ and transfer times are neglected, the problem becomes much better tractable, as we are going to discuss in Section 4.1.2. Heuristics to incorporate frequency-based transfer times are discussed in Section 4.4.

4.1.1 NP-hardness of the routing problem

For determining the complexity of our problems we use reductions from the set cover problem (SCP). An instance of SCP is given by a set of elements $\mathcal{M} = \{m_1, \ldots, m_n\}$, a set of subsets $\mathcal{C}$ with $C \subseteq \mathcal{M}$ for every $C \in \mathcal{C}$ and an integer $K \in \mathbb{N}$. The problem is to determine whether there exists a subset $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcup_{C \in \mathcal{C}'} C \supseteq \mathcal{M}$ and $|\mathcal{C}'| \leq K$.
We first show that the assumption of line-based costs leads to an NP-hard routing problem.

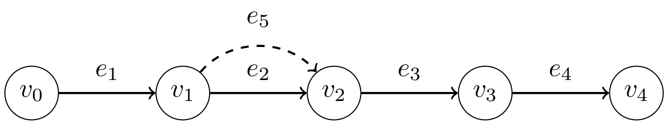


Figure 2: PTN used in the proof of Theorem 4.2.

Theorem 4.2. *The routing problem (as in Definition 4.1) with* line-based costs *is NP-hard, even if there is only one passenger and neither transfer times nor transfer penalties nor frequency-based costs are taken into account, i.e. if* $\alpha_2 = \beta = \gamma_2 = 0$.

Proof. We show that SCP given by $(\mathcal{M}, \mathcal{C}, K)$ can be reduced to the decision version of the routing problem with line-based costs. Given an instance $(\mathcal{M}, \mathcal{C}, K)$ of SCP we construct an instance of the decision version of the routing problem as follows.
We create a station v_0 and for each $m_i \in \mathcal{M}$, $i = 1, \ldots, n$ a station v_i and an edge $e_i = (v_{i-1}, v_i)$. For all $C \in \mathcal{C}$ we create a line $l_C \in \mathcal{L}$ containing all edges $\{e_i : m_i \in C\}$ and additional edges to ensure that the lines are connected paths in the PTN. We set edge lengths to $c(e) := 0$ for all edges related to $m \in \mathcal{M}$ and to $c(e) := K + 1$ for all additional edges. We consider a passenger q who wants to travel from v_0 to v_n. Furthermore we assume line costs of $cost_l = 1$ for all lines l. The parameters of the objective function are $\alpha_1 = \gamma_1 = 1$ and $\alpha_2 = \beta = \gamma_2 = 0$. T can be set to an arbitrary value since $\alpha_2 = 0$. An example for the construction is given below.
Now there is a solution to the routing problem with objective value less or equal to K if and only if there is a solution to SCP with objective value less or equal to K:
Let $\mathcal{C}'$ be a solution to SCP. Then the set of lines $\mathcal{L}' := \{l_C : C \in \mathcal{C}'\}$ has costs less or equal to K and allows q to travel from origin to destination with zero travel time. On the other hand, in every solution to the constructed instance of the routing problem with travel time less or equal to$K$, q uses the edge sequence $(e_1, \ldots, e_n)$, because otherwise his travel time would be greater than K. Hence, $\mathcal{C}' = \{C \in \mathcal{C} : q \text{ uses } l_C\}$ is a solution to SCP. $\square$

The following example illustrates the construction of an instance of the routing problem from an instance of SCP. Consider the instance of SCP given by $\mathcal{M} = \{1, 2, 3, 4\}$, $\mathcal{C} = \{C_1 = \{1, 2\}, C_2 = \{1, 3\}, C_3 = \{3, 4\}\}$, and $K = 2$. This leads to the PTN shown in Figure 2 where $e_1, \ldots, e_4$ correspond to $\mathcal{M}$ and have length $c(e_i) = 0$ for $i = 1, \ldots, 4$ and e_5 is an auxiliary edge for C_2 with $c(e_5) = K + 1 = 3$.
The line pool is $\mathcal{L} = \{l_1 = (e_1, e_2), l_2 = (e_1, e_5, e_3), l_3 = (e_3, e_4)\}$. It is easy to see that any path from v_0 to v_4 with zero travel time must contain all edges $e_i, i = 1, \ldots, 4$, and hence for each of these edges a line needs to be included.

Note that analogously, we can show that the routing problem is NP-hard even for one passenger for frequency-independent costs $\gamma_1 = 0$ (and $\alpha_2 = \beta = 0$), by interchanging the roles of frequency-based cost and frequency-independent costs in the construction made in the proof of Theorem 4.2.

Due to the result of Theorem 4.2, in the remainder of this paper we restrict ourselves to edge-based cost functions. However, even without considering costs, the routing problem with frequency-based transfer times is NP-hard.

Theorem 4.3. *The routing problem as in Definition 4.1 is NP-hard, even if transfer penalties and operational costs are not taken into account, i.e.,* $\beta = 0$ *and* $\gamma_1 = \gamma_2 = 0$.

See the appendix for a proof of this result.

4.1.2 Cases with polynomially solvable routing problem

A convenient way to represent route choice in line planning problems is the change-and-go network (CGN) $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, which was first introduced in [SS06a]. The set of nodes of the CGN consists of *station nodes* $\mathcal{V}_{stat} := \{(v, \text{board}) : v \in V\} \cup \{(v, \text{alight}) : v \in V\}$ and *travel nodes* $\mathcal{V}_{trav} := \{(v, l) : l \in \mathcal{L}, v \in l\}$. The set of arcs is $\mathcal{A} := \mathcal{A}_{OD} \cup \mathcal{A}_{trans} \cup \mathcal{A}_{\text{line}}$ with

- line arcs $\mathcal{A}_{\text{line}} := \{(e, l) : l \in \mathcal{L}, e \in l\}$ for each edge e covered by a line l,
- transfer arcs $\mathcal{A}_{trans} := \{((v, l_1), (v, l_2)) : v \in V, l_1 \ni v, l_2 \ni v\}$,
- and arcs for boarding and alighting

$$\mathcal{A}_{OD} := \{((v, \text{board}), (v, l)) : l \in \mathcal{L}, v \in l\} \cup \{((v, l), (v, \text{alight})) : l \in \mathcal{L}, v \in l\}.$$

For an example of a CGN, see Figure 4.
Now every route R_q for a passenger q can be uniquely represented in $\mathcal{G}$ as a path P_q from (u_q, board) to (v_q, alight) in $\mathcal{G}$.
For $a \in \mathcal{A}$ we denote by $x_a(\mathcal{R})$ the number of passengers, using arc a of the CGN, i.e., $x_a(\mathcal{R}) := |\{q \in \mathcal{Q} : P_q \ni a\}|$ where P_q is the path in the CGN corresponding to R_q. To abbreviate, we sometimes omit the route set and use the notation $x_a := x_a(\mathcal{R})$.
Let us now assume that, given $\mathcal{R}^{-q}$, we can express the objective value of a route R_q as the sum of edge weights over all edges contained in the corresponding path P_q, i.e., that there are arc weights $w_a^q(\mathcal{R}^{-q}) \geq 0 \forall a \in \mathcal{A}$ such that

$$h_q(R_q, \mathcal{R}^{-q}) = \sum_{a \in P_q} w_a^q(\mathcal{R}^{-q}). \tag{3}$$

This is the case if costs are edge-based with $\gamma_2 = 0$ and $\alpha_2 = 0$. Indeed, since in this case the edge cost function $cost_{(e,l)} := cost_{(e,l)}(f(\mathcal{R})) = \gamma_1 k_l^1 \frac{c(e)}{x_{(e.l)}(\mathcal{R})}$ is independent of the routing of the current passenger, it is easy to check that the weights

$$w_a^q(\mathcal{R}^{-q}) := \begin{cases} \alpha_1 c(e) + \frac{\text{cost}_{(e,l)}}{x_{(e,l)}(\mathcal{R}^{-q})+1} & \text{if } a = (e, l) \in \mathcal{A}_{\text{line}} \\ \beta & \text{if } a \in \mathcal{A}_{trans} \end{cases}$$

satisfy (3). In Section 4.4, different approaches to define arc weights are studied.
If edge weights of the form (3) can be found, we obtain the following lemma:

Lemma 4.4. *Consider an instance I of the routing problem (Definition 4.1). If there are arc weights $w_a^q(\mathcal{R}^{-q})$ as defined in* (3), *(RP$_q$) can be solved in polynomial time.*

Proof. In this case, any shortest path from (u_q, board) to (v_q, alight) with respect to the edge weights $w_a^q(\mathcal{R}^{-q})$ is an optimal solution to I. Hence, we can find a solution using, e.g., Dijkstra's algorithm. $\square$

Hence, in this case, we can use Algorithm 1 with, e.g., Dijkstra's algorithm in the routing step to search for an equilibrium of the LPRG.

4.2 Existence of equilibria and convergence of the best-response algorithm

In this section we study under which assumptions equilibria to the LPRG exist and can be found by Algorithm 1 . We start with an example which shows that in the general case the existence of an equilibrium is not guaranteed.

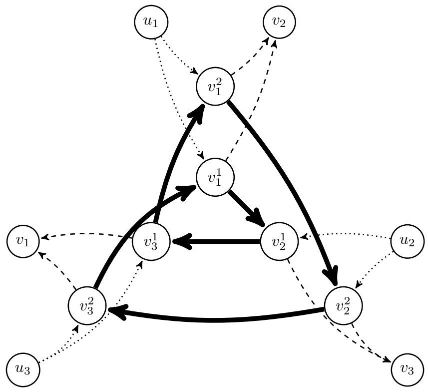


Figure 3: PTN for an example instance where there are no equilibria for LPRG and Algorithm 1 does not converge.

4.2.1 Non-existence of equilibria

In this section, we give an intuition for why some instances of LPRG do not have equilibria. A more detailed description of the example and proof of non-existence of equilibria for this example can be found in the appendix.

We regard the PTN from Figure 3 and assume that every edge is served by one directed line (which contains only this edge). Because of this one-to-one correspondence of lines and edges, in this example we use 'edges' as a synonym for 'lines'. We set the vehicle capacity to $B = 1$, so that the frequency of an edge is given by the number of passengers on it. We consider three main passengers q_1 from u_1 to v_1, q_2 from u_2 to v_2, and q_3 from u_3 to v_3. For each of these passengers, there exist two routes from origin to destination, we denote the route starting with edge (u_i, v_i^1) as R_i^1 and the route starting with edge (u_i, v_i^2) as R_i^2. Note that each of this routes consists of a sequence of dotted edge, two thick edges, and a dashed edge.

For the sake of simplicity, in our objective function we take only the transfer time into account, i.e., $(\alpha_1/\alpha_2, \beta, \gamma_1/\gamma_2) = (0/1, 0, 0/0)$. We assume that the line frequency on the dashed edges in the PTN is already very high (which we ensure by adding auxiliary OD-pairs which have to use these edges). The dotted edges, which originate in the nodes u_i, will have a frequency of 1 if the passenger q_i travels on them, or 0 otherwise. Consequently, the transfer time of a passenger only depends on whether he shares the thick edges with other passengers or not. Furthermore, transfer time towards the dashed edges is small anyway, due to their high frequency. Hence, the first two transfers on a passengers' route make up for most part of the objective function.

Now we show that in this example there is no equilibrium in which passenger q_1 travels on route R_1^1 by contradiction. Assume that $\mathcal{R}$ is an equilibrium of the described line planning routing game where q_1 travels on R_1^1. We can conclude that q_2 travels on route R_2^1, because no matter which route q_3 chooses, the transfer time on R_2^1 will be lower than on R_2^2 (see the appendix for details). Given the routes R_1^1 and R_2^1 for q_1 and q_2, it is easy to see that for q_3 the transfer times are lowest on R_3^1.

However, if q_2 travels on R_2^1 and q_3 travels on R_3^1, for q_1 transfer times would be lower on R_2^1, which contradicts the assumption that $\mathcal{R}$ is an equilibrium.

Analogously, we can show that there is no equilibrium in which q_1 travels on R_1^2. Hence, there is no equilibrium in this example.

4.2.2 Line planning routing games with potential functions

In contrast to the example from Section 4.2.1 we show in Lemma 4.5 that existence of equilibria and convergence can be guaranteed if for every $a \in \mathcal{A}$ there is an *arc weight function* $\bar{w}_a : \mathbb{N} \to \mathbb{R}$

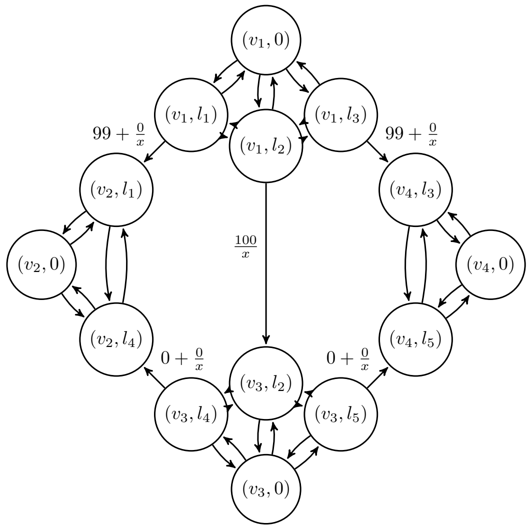


Figure 4: CGN of two equilibria with different objective values

such that

$$h_q(R_q, \mathcal{R}^{-q}) = \sum_{a \in P_q} \bar{w}_a(x_a) \tag{4}$$

for every route R_q from u_q to v_q and its corresponding path P_q in the CGN.
In case of edge-based costs with $\gamma_2 = 0$ (in this case, again, we can write $\text{cost}_{(e,l)}$ instead of $\text{cost}_{(e,l)}(f(\mathcal{R}))$) and $\alpha_2 = 0$, such arc weight functions are given by

$$\bar{w}_a(x) := \begin{cases} \alpha_1 c(e) + \frac{\text{cost}_{(e,l)}}{x} & \text{if } a = (e,l) \in \mathcal{A}_{\text{line}} \\ \beta & \text{if } a \in \mathcal{A}_{trans} \end{cases}. \tag{5}$$

Lemma 4.5. *Let $I := (G, \mathcal{L}, \mathcal{Q}, \{h_q : q \in \mathcal{Q}\})$ be an instance of the LPRG such that arc weight functions as specified in* (4) *exist. Then*

1. *$\Phi(\mathcal{R}) := \sum_{a \in \mathcal{A}} \sum_{i=1}^{x_a(\mathcal{R})} \bar{w}_a(i)$ is a potential function for I,*
2. *there exists an equilibrium to I,*
3. *Algorithm 1 converges to an equilibrium in a finite number of steps,*
4. *each of the steps can be executed in polynomial time.*

The proof follows standard arguments for convergence of atomic routing games, and can be found in the appendix.
We conclude that in particular for all line planning routing games with $\gamma_2 = 0$ and $\alpha_2 = 0$ and edge-based costs, Algorithm 1 finds an equilibrium after a finite number of steps.

4.3 Quality of equilibria

4.3.1 Two examples for 'bad' equilibria

We start with an example which illustrates that the LPRG can have different equilibria and that Algorithm 1 does not necessarily find a good one, even when convergence to some equilibrium is guaranteed because the conditions of Lemma 4.5 are fulfilled.

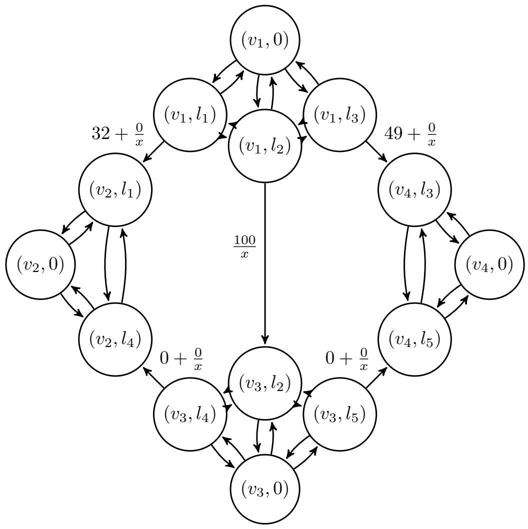


Figure 5: CGN where the system optimum is not necessarily an equilibrium.

We consider a PTN consisting of four nodes v_1, v_2, v_3, v_4, edges $\{v_1, v_2\}$ and $\{v_1, v_3\}$ with length 99 and edges $\{v_1, v_4\}$, $\{v_2, v_3\}$, and $\{v_3, v_4\}$ with length 0. Our line pool consists of five lines, the corresponding CGN is shown in Figure 4. Note that for the sake of a more compact representation, we contracted boarding and alighting node for each station v_i to a node $(v_i, 0)$.
We consider two passengers: q_1 wants to travel from v_1 to v_2 and q_2 wants to travel from v_1 to v_4. The parameters of the individual objective functions are $\alpha_1 = \gamma_1 = 1$ and $\alpha_2 = \beta = \gamma_2 = 0$, that is, we only take in-vehicle time and frequency-independent costs into account.
Line l_3 has costs 100, while all other line costs are 0.
For the reader's convenience, we specify the arc-weight functions as a sum of in-vehicle travel time and costs for the line arcs next to the corresponding arcs in Figure 4, all other arc weight functions are 0 in this example. There are two equilibria:

1. $\mathcal{R}'$: q_1 uses line 1 and q_2 uses line 3. For both passengers, the individual objective values are $h_{q_i} = 99$.
2. $\mathcal{R}^*$: q_1 uses line 2 and 4, q_2 uses line 2 and 5. For both passengers, the individual objective values are $h_{q_i} = 50$.

Clearly, the second equilibrium is preferable to the first one, since for both passengers the individual objective functions are almost twice as high in the first one. However, e.g., when starting with an empty solution, Algorithm 1 will find the first equilibrium.
It can be easily seen that in this example, the second and 'better' equilibrium is also a system-optimum, that is, it optimizes $H = h_{q_1} + h_{q_2}$, the objective function of LPQC. Hence, in this example the price of anarchy is $\frac{198}{100}$, but the price of stability is 1.

However, system-optima to LPRG (that is: optimal solutions to LPQC) are not necessarily equilibria. To illustrate this, we use a slightly modified version of the previous example:
We consider a PTN consisting of four nodes v_1, v_2, v_3, v_4, edges $\{v_1, v_2\}$ with length 32, $\{v_1, v_3\}$ with length 49, and edges $\{v_1, v_4\}$, $\{v_2, v_3\}$, and $\{v_3, v_4\}$ with length 0.
Again, our line pool consists of five lines, of which line l_2 has frequency-independent costs 100 and the other lines have costs 0. The corresponding CGN is shown in Figure 5, where, again, we contract boarding and alighting node for each station v_i to a node $(v_i, 0)$. This time, we consider three passengers: q_1 wants to travel from v_1 to v_2, q_2 wants to travel from v_1 to v_4, q_3 wants to travel from v_1 to v_3. As in the previous example, for the (individual) objective function(s) we use

the parameters $\alpha_1 = \gamma = 1$, $\alpha_2 = \beta = \gamma_2 = 0$. Again, the conditions of Lemma 4.5 are met and we specify the arc-weight functions for the line arcs next to the corresponding arcs in Figure 4, all other arc weight functions are 0.
In this case, there is only one equilibrium $\mathcal{R}^*$: q_1 uses line 1, q_2 uses line 3, q_3 uses line 2; with $H(\mathcal{R}^*) = 32 + 49 + 100 = 181$. The system-optimal solution (and optimal solution to LPQC) is defined by the route set $\hat{R}$: q_1 uses line 2 and 4, q_2 uses line 2 and 5, q_3 uses line 2, with overall objective value $H(\hat{R}) = \frac{100}{3} + \frac{100}{3} + \frac{100}{3} = 100$. So for this example, both price of anarchy and price of stability equal $\frac{181}{100}$.
By extending the example given in Figure 5 in a straight-forward way, we see that for instances with an unbounded number of passengers, the price of stability is not bounded for the considered games: for n passengers we can construct an instance with price of stability (and price of anarchy) close to $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

4.3.2 Bounding the price of anarchy

However, we can bound the price of anarchy *by the number of passengers* if the arc weight functions (4) fulfill the property described in Lemma 4.6.

Lemma 4.6. *If there exist non-increasing arc weight functions $\bar{w}_a$ with $\bar{w}_a(1) \leq x \cdot \bar{w}_a(x)$ for all $x \in \mathbb{N}$, the price of anarchy in the LPRG is at most the number of passengers.*

Proof. Let the route set $\mathcal{X} := \{X_1, \ldots, X_n\}$ represent a social optimum (in the way described in Observation 3.2) and let the route set $\mathcal{R} := \{R_1, \ldots, R_n\}$ represent an equilibrium. Assume that $H(\mathcal{R}) > |\mathcal{Q}| H(\mathcal{X})$. Then there is at least one passenger q with $h_q(\mathcal{R}) > |\mathcal{Q}| h_q(\mathcal{X})$. For this passenger q it follows that

$$h_q(X_q, \mathcal{R}^{-q}) = \sum_{a \in X_q} \bar{w}_a(x'_a) \leq \sum_{a \in X_q} \bar{w}_a(1) \leq \sum_{a \in X_q} \hat{x}_a \bar{w}_a(\hat{x}_a) \leq \sum_{a \in X_q} |\mathcal{Q}| \bar{w}_a(\hat{x}_a) < h_q(R_q, \mathcal{R}^{-q}),$$

where $\hat{x}_a := x_a(X_q, \mathcal{R}^{-q})$ denotes the number of passengers on arc a when passengers follow routing $(X_q, \mathcal{R}^{-q})$ and $x'_a := x_a(\mathcal{R})$ the number of passengers on arc a when passengers follow routing $\mathcal{R}$. This is a contradiction to the assumption that $\mathcal{R}$ is a equilibrium. □

Corollary 4.7. *If edge-based cost functions with $\gamma_2 = 0$ are considered and $\alpha_2 = 0$, the price of anarchy is bounded by the number of passengers.*

Proof. The functions given in (5) are non-increasing. Furthermore, for $x \geq 1$, we have for $a \in \mathcal{A}_{\text{line}}$ $x\bar{w}_a(x) = x\alpha_1 c(a) + \gamma_1 \text{cost}(a) \geq \alpha_1 c(a) + \gamma_1 \text{cost}(a) = \bar{w}_a(1)$ and for $a \in \mathcal{A}_{trans}$

$$x \cdot \bar{w}_a(x) = x\beta \geq \beta = \bar{w}_a(1).$$

□

To see that there are indeed instances I with a price of anarchy that equals $|\mathcal{Q}|$ consider the example given in Figure 4. If we set the travel time on (v_1, v_2) and (v_1, v_4) to 100, $\mathcal{R}'$ and $\mathcal{R}^*$ are still both equilibria and the price of anarchy is 2. We can easily extend this construction to an arbitrary number of passengers.

4.3.3 Algorithm 1 as a heuristic for LPQC

Corollary 4.7 implies that if we use Algorithm 1 for instances of LPQC with $\alpha_2 = \gamma_2 = 0$, we have an approximation ratio $|\mathcal{Q}|$ where $|\mathcal{Q}|$ is the number of passengers (as long as we ensure that each edge of each established line is used by at least one passenger).

However, convergence to an equilibrium may be slow. In the next lemma we show that we can achieve the same quality bound after computing the best response for each passenger once.

Lemma 4.8. *If there exist non-increasing arc weight functions $\bar{w}_a$ with $\bar{w}_a(1) \leq x \cdot \bar{w}_a(x)$ for all $x \in \mathbb{N}$, given an empty state of the game, calculating the best response once for every passenger in Algorithm 1 leads to a route set $\mathcal{R}$ with $\frac{H(\mathcal{R})}{H(\mathcal{X})} \leq |\mathcal{Q}|$, where $\mathcal{X}$ is a system-optimal solution.*

Proof. Let $\mathcal{Q} = \{1, \ldots, n\}$ be the set of passengers and $\mathcal{S}^q$ for $q = 1, \ldots, n$ the route combination after choosing the best response R_q for passenger q, i.e., $\mathcal{S}^q = (R_1, R_2, \ldots, R_q, \emptyset, \emptyset, \ldots, \emptyset)$. Furthermore, let $\mathcal{X}$ be the system-optimal solution, where the passengers choose the route X_q with corresponding paths Y_q in the CGN.
Since arc weight functions are non-increasing, it holds that $h_q(S^n) \leq h_q(S^q)$. Since R_q is a best response to $(R_1, R_2, \ldots, R_{q-1}, \emptyset, \emptyset, \ldots, \emptyset)$ we have

$$h_q(S_q) = \sum_{a \in P_q} \bar{w}_a(x_a(\mathcal{S}^q)) = \sum_{a \in P_q} \bar{w}_a(x_a(\mathcal{S}^{q-1}) + 1) \leq \sum_{a \in Y_q} \bar{w}_a(x_a(\mathcal{S}^{q-1}) + 1) \tag{6}$$

where P_q denotes the path in the CGN corresponding to R_q. With this, the following holds:

$$\begin{aligned}
H(\mathcal{S}^n) = \sum_{q \in \mathcal{Q}} h_q(\mathcal{S}^n) &\leq \sum_{q \in \mathcal{Q}} h_q(\mathcal{S}^q) \\
&\leq \sum_{q \in \mathcal{Q}} \sum_{a \in Y_q} \bar{w}_a(x_a(\mathcal{S}^{q-1}) + 1) && \text{due to (6)} \\
&\leq \sum_{q \in \mathcal{Q}} \sum_{a \in Y_q} \bar{w}_a(1) && \text{since } \bar{w}_a \text{ non-increasing} \\
&\leq \sum_{q \in \mathcal{Q}} \sum_{a \in Y_q} x_a(\mathcal{X}) \cdot \bar{w}_a(x_a(\mathcal{X})) && \text{since } \bar{w}_a(1) \leq x\bar{w}_a(x) \\
&\leq |\mathcal{Q}| \sum_{q \in \mathcal{Q}} \sum_{a \in Y_q} \bar{w}_a(x_a(\mathcal{X})) \\
&= |\mathcal{Q}| \sum_{q \in \mathcal{Q}} h_q(\mathcal{X}^n) = |\mathcal{Q}| \cdot H(\mathcal{X}^n).
\end{aligned}$$

□

That means that for instances of the LPQC/LPRG for which there exist non-increasing arc weight functions $\bar{w}_a$ with $\bar{w}_a(1) \leq x \cdot \bar{w}_a(x)$ for all $x \in \mathbb{N}$, that is, in particular if $\alpha_2 = \gamma_2 = 0$, a solution $(\mathcal{R}, f)$ to the line planning problem with approximation ratio $|\mathcal{Q}|$ can be found in polynomial time. As described in the previous section, we can show that this bound is tight, i.e., there are instances where Algorithm 1 can get stuck in an equilibrium whose objective value is $|\mathcal{Q}|$-times the optimal solution value.

4.4 Heuristic approaches to the routing problem

In the preceding Sections 4.1-4.3 we have seen that in order to achieve polynomial running time of Algorithm 1, to be able to prove convergence to an equilibrium, and give bounds on the quality of an equilibrium, strong restrictions on the parameters of the objective function have to be imposed.
In this section we investigate heuristic approaches to the routing problem with general individual objective functions $h_q(R_q, \mathcal{R}^{-q}) = \text{travel}_q(R_q, \mathcal{R}^{-q}) + \text{cost}_q(R_q, \mathcal{R}^{-q})$ using edge-based costs $\text{cost}_q(R_q, \mathcal{R}^{-q}) = (\gamma_1 k_l^1 + \gamma_2 k_l^2 f_l) \cdot \sum_{(e,l) \in R_q} \frac{\text{cost}_{(e,l)}(f(\mathcal{R}))}{x_{(e,l)}(R_q, \mathcal{R}^{-q})}$.
In this general case, the routing problem is NP-hard (Theorem 4.3) and Algorithm 1 does not necessarily converge (see Section 4.2.1). To overcome these difficulties in a heuristic way, we simplify the transfer time function τ_q and the edge-based cost function cost_q in this section.

4.4.1 Auxiliary frequencies

In our first approach, we replace the frequencies $f(\mathcal{R})$ by *auxiliary frequencies* $\tilde{f}(\mathcal{R}^{-q})$ when determining a route for passenger q. This small trick allows us to define arc weights in accordance to Lemma 4.4 and hence, to solve the routing problem using Dijkstra's algorithm in the CGN. Let $\mathcal{Q}$ be a set of passengers and let $\mathcal{R} = \{R_q : q \in \mathcal{Q}\}$ be a set of strategies represented by paths in the CGN. We call an *edge* $(e,l) \in \mathcal{A}$ *critical for* $\mathcal{R}$ if one additional passenger on the edge would increase the frequency, i.e., if $x_{(e,l)}(\mathcal{R}) \equiv 0 \mod B$. A *line* $l \in \mathcal{L}$ is *critical for* $\mathcal{R}$ if it contains an edge which is critical for $\mathcal{R}$. In order to find a route, given the routes for all other passengers $\mathcal{R}^{-q}$, we define the auxiliary frequencies

$$\tilde{f}_l(\mathcal{R}^{-q}) := \begin{cases} f_l(\mathcal{R}^{-q}) + 1 & \text{if } l \text{ is critical for } \mathcal{R}^{-q} \\ f_l(\mathcal{R}^{-q}) & \text{otherwise.} \end{cases}$$

We observe that for every line l and every passenger $q \in \mathcal{Q}$, $\tilde{f}_l(\mathcal{R}^{-q}) \geq f_l(\mathcal{R}) \geq f_l(\mathcal{R}^{-q})$. For all non-critical lines we even have equality. Plugging in the auxiliary frequencies into τ_q we obtain an *auxiliary* transfer time function

$$\tilde{\tau}_q^{\text{lb}}(\mathcal{R}) := \sum_{i=1}^{n-1} \frac{T}{\tilde{f}_{l_i}(\mathcal{R}^{-q}) + \tilde{f}_{l_{i+1}}(\mathcal{R}^{-q})}$$

(where $l_1, \ldots, l_n$ are the lines used in R_q) which *underestimates* the transfer times $\tau(\mathcal{R})$ in a route set $\mathcal{R}$. To find an *overestimating* heuristic measure for transfer times, we can consider

$$\tilde{\tau}_q^{\text{ub}}(\mathcal{R}) := \tau_q(\mathcal{R}^{-q}) = \sum_{i=1}^{n-1} \frac{T}{f_{l_i}(\mathcal{R}^{-q}) + f_{l_{i+1}}(\mathcal{R}^{-q})}.$$

Using the same approach, we can define overestimating auxiliary edge-based cost functions as

$$\tilde{\text{cost}}_q^{\text{ub}}(\mathcal{R}) := \tilde{\text{cost}}_q(\mathcal{R}^{-q}) := \sum_{(e,l) \in R_q} \frac{\text{cost}_{(e,l)}(\tilde{f}(\mathcal{R}^{-q}))}{x_{(e,l)}(R_q, \mathcal{R}^{-q})} \geq \text{cost}_q(\mathcal{R})$$

and underestimating auxiliary edge-based cost functions

$$\tilde{\text{cost}}_q^{\text{lb}}(\mathcal{R}) := \text{cost}_q(\mathcal{R}^{-q}) = \sum_{(e,l) \in R_q} \frac{\text{cost}_{(e,l)}(f(\mathcal{R}^{-q}))}{x_{(e,l)}(R_q, \mathcal{R}^{-q})} \leq \text{cost}_q(\mathcal{R}_q).$$

We define over- and underestimated versions of the individual objective functions

$$\begin{aligned} \tilde{h}_q^{\text{ub}}(R_q, \mathcal{R}^{-q}) &:= \alpha_1 \cdot c(\mathcal{R}) + \alpha_2 \cdot \tilde{\tau}^{\text{ub}}(\mathcal{R}^{-q}) + \beta \cdot \text{transfer}_q(R_q) + \tilde{\text{cost}}_q^{\text{ub}}(R_q, \mathcal{R}^{-q}), \\ \tilde{h}_q^{\text{lb}}(R_q, \mathcal{R}^{-q}) &:= \alpha_1 \cdot c(\mathcal{R}) + \alpha_2 \cdot \tilde{\tau}^{\text{lb}}(\mathcal{R}^{-q}) + \beta \cdot \text{transfer}_q(R_q) + \tilde{\text{cost}}_q^{\text{lb}}(R_q, \mathcal{R}^{-q}) \end{aligned}$$

and obtain

$$\tilde{h}_q^{\text{lb}}(R_q, \mathcal{R}^{-q}) \leq h_q(R_q, \mathcal{R}^{-q}) \leq \tilde{h}_q^{\text{ub}}(R_q, \mathcal{R}^{-q}).$$

Given a passenger q and a set of strategies $\mathcal{R}^{-q}$ for the remaining passengers, the auxiliary frequencies allow us to define weights for the arcs in the CGN which depend only on the strategy choices of the remaining passengers $\mathcal{R}^{-q}$. This observation is summarized in the following lemma.

Lemma 4.9. *For arc weights*

$$\tilde{w}_a^{ub}(\mathcal{R}^{-q}) := \begin{cases} \alpha_1 c(e) + \frac{\text{cost}_{(e,l)}(\tilde{f}(\mathcal{R}^{-q}))}{x_a(\mathcal{R}^{-q})+1} & \forall a = (e,l) \in \mathcal{A}_{line} \\ \frac{1}{f_l(\mathcal{R}^{-q}) + f_{l'}(\mathcal{R}^{-q})} + \beta & \forall a = ((v,l),(v,l')) \end{cases}$$

$$\text{or } \tilde{w}_a^{lb}(\mathcal{R}^{-q}) := \begin{cases} \alpha_1 c(e) + \gamma \frac{\text{cost}_{(e,l)}(f(\mathcal{R}^{-q}))}{x_a(\mathcal{R}^{-q})+1} & \forall a = (e,l) \in \mathcal{A}_{line} \\ \frac{1}{\tilde{f}_l(\mathcal{R}^{-q}) + \tilde{f}_{l'}(\mathcal{R}^{-q})} + \beta & \forall a = ((v,l),(v,l')) \end{cases}$$

we have

$$\tilde{h}_q^{ub}(R_q, \mathcal{R}^{-q}) = \sum_{a \in P_q} \tilde{w}_a^{ub}(\mathcal{R}^{-q}) \text{ and } \tilde{h}_q^{lb}(R_q, \mathcal{R}^{-q}) = \sum_{a \in P_q} \tilde{w}_a^{lb}(\mathcal{R}^{-q})$$

(where P_q denotes the path in the CGN corresponding to R_q) and the routing problem can be solved in polynomial time.

Here, the last statement follows from Lemma 4.4.

Note that the use of the auxiliary objective functions $\tilde{h}_q$ does not guarantee the existence of an equilibrium: In fact, in the counter example shown in Section 4.2.1 we have $f_l(\mathcal{R}) = \tilde{f}_l(\mathcal{R}^{-q})$ for all choices of q and R_q. Hence, this example also proves the possibility that no equilibrium for objective functions $\tilde{h}_q^{\mathrm{lb}}$ exists.

4.4.2 Auxiliary arc weights

Since the heuristic from section 4.4.1 does not always lead to an equilibrium, we consider a further heuristic simplification which guarantees the existence of an equilibrium and the convergence of the best-response-algorithm.

Consider a set of passenger routes $\mathcal{R}$ and a transfer edge $a = ((v, l), (v, l'))$. Then the frequency of l and l', respectively, is at least $\left\lceil \frac{x_a(\mathcal{R})}{B} \right\rceil$, since at least all passengers transferring from l to l' have to use l and l', respectively. Additionally, all frequencies are at most $\left\lceil \frac{|\mathcal{Q}|}{B} \right\rceil$ since no more than all passengers can use any given line. This leads to the following approximate arc weight functions:

$$\bar{w}_a^{\mathrm{lb}}(x) := \begin{cases} \alpha_1 c(e) + \frac{\gamma_1 k_l^1 + \gamma_2 k_l^2 \lceil \frac{x}{B} \rceil}{x} \cdot \frac{c(e)}{\sum_{e \in l} c(e)} & \text{if } a = (e, l) \in \mathcal{A}_{\text{line}} \\ \frac{\alpha_2 T}{2 \cdot \lceil \frac{|\mathcal{Q}|}{B} \rceil} + \beta & \text{if } a \in \mathcal{A}_{trans} \end{cases} \tag{7}$$

and

$$\bar{w}_a^{\mathrm{ub}}(x) := \begin{cases} \alpha_1 c(e) + \frac{\gamma_1 k_l^1 + \gamma_2 k_l^2 \lceil \frac{|\mathcal{Q}|}{B} \rceil}{x} \cdot \frac{c(e)}{\sum_{e \in l} c(e)} & \text{if } a = (e, l) \in \mathcal{A}_{\text{line}} \\ \frac{\alpha_2 T}{2 \cdot \lceil \frac{x}{B} \rceil} + \beta & \text{if } a \in \mathcal{A}_{trans}, \end{cases} \tag{8}$$

where $\bar{w}_a$ is defined as in (5).
With $\bar{h}_q^{\mathrm{lb}}(R_q, \mathcal{R}^{-q}) := \sum_{a \in P_q} \bar{w}_a^{\mathrm{lb}}(x_a)$ and $\bar{h}_q^{\mathrm{ub}}(P_q, \mathcal{R}^{-q}) := \sum_{a \in P_q} \bar{w}_a^{\mathrm{ub}}(x_a)$ (where P_q is the path in the CGN corresponding to R_q), we obtain:

Lemma 4.10. *For every passenger $p \in \mathcal{Q}$ with route R_q and $\mathcal{R} = (R_q, \mathcal{R}^{-q})$ we have*

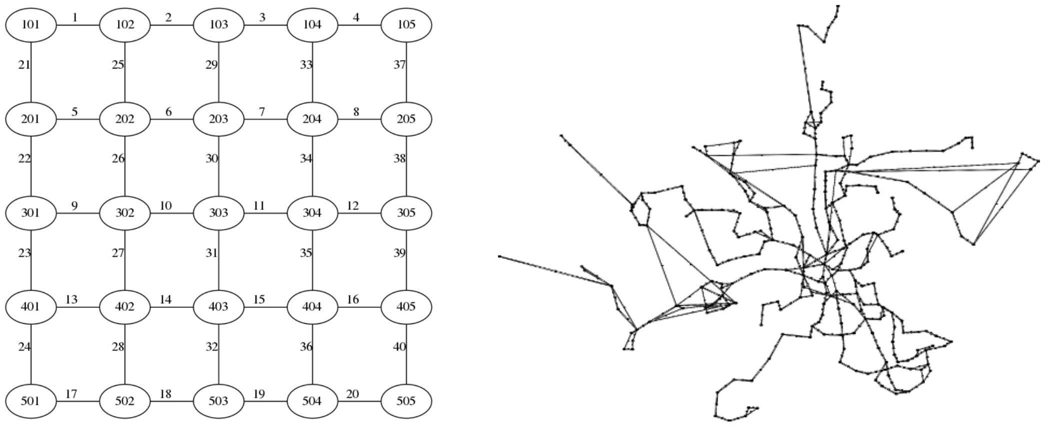
$$\bar{h}_q^{\mathrm{lb}}(R_q, \mathcal{R}^{-q}) = \sum_{a \in P_q} \bar{w}_a^{\mathrm{lb}}(x_a) \leq h_q(R_q) \leq \sum_{a \in P_q} \bar{w}_a^{\mathrm{ub}}(x_a) = \bar{h}_q^{\mathrm{ub}}(R_q, \mathcal{R}^{-q}).$$

From Lemma 4.4, Lemma 4.5, Lemma 4.6, and Lemma 4.8 we conclude:

Corollary 4.11. *For individual objective functions $\bar{h}_q^{\mathrm{lb}}$ and $\bar{h}_q^{\mathrm{ub}}$, the routing step of Algorithm 1 can be executed in polynomial time using arc weights $\bar{w}_a^{\mathrm{lb}}(x_a)$ or $\bar{w}_a^{\mathrm{ub}}(x)$, respectively, in the CGN. With respect to* these objective functions *equilibria exist and Algorithm 1 converges towards an equilibrium. The price of anarchy is at most $|\mathcal{Q}|$, and when starting with an empty state, this quality is already reached after computing the best response once for every passenger.*

5 Experiments

In this section we describe a first experimental evaluation of our routing game approach. We tested the best response strategy with the five different variants for solving the routing problem



(a) Underlying public transportation network for instance GRID

(b) Underlying public transportation network for instances GOE

Figure 6: Infrastructure networks

described in this paper: solving the routing problem exactly (abbreviated as *BR*), using the auxiliary frequency (*AF*) heuristic with overestimated (*ub*)/ underestimated (*lb*) transfer times, and using the auxiliary arc weight (*AW*) heuristic with overestimated (*ub*)/ underestimated (*lb*) transfer times. We furthermore compare it to the exact solution of the non-linear integer program (LPQC) which we solved as a semidefinite quadratic problem with Gurobi 7 [Gur16] (abbreviated as *MP*). Note that this is only possible for $\alpha_2 = 0$ (because otherwise it is a non-semidefinite quadratic program).

We tested the different approaches on two different instances. The first instance *GRID* is based on a 5×5-grid instance which was introduced in [FHSS17a] with a modified line pool. The PTN is depicted in Figure 6a. It consists of 25 stations and 40 edges, the line pool has 13 lines and there are 1927 passengers in 567 OD pairs. The second instance, *GOE*, is taken from the LinTim toolbox [Lin14]. The PTN, shown in Figure 6b, is derived from the bus-network in Göttingen, Germany. The instance consists of 257 stations, 548 edges, 6114 OD pairs and 6321 passengers. A line pool consisting of 44 lines was generated for these experiments. All experiments were done on a CPU of 16 cores with 2.4GHz and 132GB of RAM. The standard parameter set $P_3 = (1/1, 10, 3/3)$ was chosen to represent a realistic assessment of the generalized costs, provided by practical public transport planners. The parameter sets P_1 and P_2 are simplifications for the presented algorithms which are chosen to approximate P_3.

Table 1 shows the objective values with respect to the (LPQC), running times, and number of iterations for running a best-response strategy, compared to the mathematical program MP, on the instance GRID with parameter set $P_1 = (1/0, 20, 3/3)$. Note that BR is computed according to $P_2 = (1/0, 20, 6/0)$ in order to be able to solve the routing problem exactly but it is evaluated according to P_1. Objective values are reported relative to the optimal solution/best solution found. We see that BR and our heuristics converge to equilibria after 6 or 7 iterations, but that these equilibria are not identical to the system optimum, i.e., the solution found by the (LPQC). We also observe that the running times of the best response strategies are only 3.6% to 7.7% of the running time of MP where BR and the simpler heuristics AW ub/lb are faster than AF ub/lb. In turn, the more complicated heuristics AF ub/lb yield on average better solutions than AW lb/ub. Note that the heuristics cannot utilize their full potential in this experiment, since transfer times are neglected here.

In Section 3.3 we describe how in an extreme case, (LPQC) can find a solution which has a better centralized objective value, but is unrealistic in the sense that some passengers have to choose

	relative objective P_1	runtime	# iterations
MP	1	5:36	-
BR	1.391	0:14	7
AF ub	1.357	0:23	6
AF lb	1.481	0:26	7
AW ub	2.329	0:14	7
AW lb	1.391	0:12	6

Table 1: Comparison of solutions for (LPQC) under parameter settings $P_1 = (1/0, 20, 3/3)$ for MP and the heuristics, $P_2 = (1/0, 20, 6/0)$ for BR on instance GRID. Runtime in min:sec.

much longer routes than others. Table 2 shows that, to a lesser degree, this is also the case for the experiment presented here.

	MP	BR&heuristics
average standard deviation drive time	0.002	0
average standard deviation transfer time	0.067	0
average standard deviation number of transfers	0	0

Table 2: Comparison of solutions for (LPQC) under parameter settings $P_1 = (1/0, 20, 3/3)$ for MP and the heuristics, $P_2 = (1/0, 20, 6/0)$ for BR on instance GRID

Here, we compute a more balanced solution using BR instead of MP. Using MP, passengers for the same OD pair are assigned paths of different quality. While the number of transfers does not deviate within an OD pair, the average standard deviation over the number of passengers of the drive time of passengers belonging to the same OD pair is 0.002 with a maximum of 0.227 and the average standard deviation of the transfer time is 0.067 with a maximum of 7.071. Such a system optimal solution may not be possible to implement in reality, similarly as described in the example from Section 3.3. This problem does not occur when applying BR, where all passengers can choose a path of identical quality.

In Table 3 we see a comparison of the different variants of the best-response strategy with respect to the objective value of the (LPQC), running times, and number of iterations for the parameter set $P_3 = (1/1, 10, 3/3)$ on instance GRID. For MP the solution is computed with parameter set $P_1 = (1/0, 20, 3/3)$ and for BR with parameter set $P_2 = (1/0, 20, 6/0)$ (compare Table 1), since we can only apply these methods for $\alpha_2 = 0$ and $\gamma_2 = 0$ in case of BR. Preliminary experiments have indicated that among the parameter sets with $\alpha_2 = 0$, P_1 approximates P_3 best and among those with $\alpha_2 = \gamma_2 = 0$, P_2 approximates P_3 best.

	rel. objective P_1	rel. objective P_3	runtime	# iterations
MP*	1	1	5:36	-
BR$^\square$	1.391	1.168	0:14	7
AF ub	1.362	1.147	0:26	7
AF lb	1.405	1.152	0:24	6
AW ub	1.977	1.484	0:10	5
AW lb	1.645	1.3	0:12	6

Table 3: Comparison of solutions for (LPQC) under parameter settings $P_1 = (1/0, 20, 3/3)$ for MP, $P_2 = (1/0, 20, 6/0)$ for BR and $P_3 = (1/1, 10, 3/3)$ for the heuristics on instance GRID. Runtime in min:sec.

We see that in all versions of the best-response strategy, convergence to the equilibrium is reached

after 5 to 7 iterations. When comparing the solutions based on the objective value of the (LPQC) we see that the MP, executed with the parameter set P_1, still outperforms the best-response heuristics, although this parameter set neglects the transfer times. However, among the best response strategies, we see that the inclusion of transfer times seems to yield a benefit, since multiple heuristics find better solution than BR w.r.t. P_3.
To further investigate the different heuristics when transfer times are taken into account, Table 4 shows a comparison for different parameter sets on the instance GRID. We see that the more complex heuristics AF ub/lb always find the best solutions and often both outperform AW ub/lb. The simpler algorithms BR, AW ub/lb are faster than the more complex ones AF ub/lb which in turn are much faster than the optimization model MP.

	P_4			P_5			P_6			P_7			P_8		
	obj	time	it	obj	time	it	obj	time	it	obj	time	it	obj	time	it
AF ub	1.005	32	9	1.008	43	12	1.026	33	9	**1**	23	6	**1**	29	8
AF lb	**1**	28	8	**1**	37	10	**1**	29	8	1.055	19	5	1.065	26	7
AW ub	1.532	20	5	1.448	**12**	6	1.576	**12**	6	1.002	**11**	5	1.372	15	8
AW lb	1.169	**14**	7	1.283	**12**	6	1.173	17	9	1.022	12	6	1.05	**12**	5

Table 4: Heuristic solutions on GRID. $P_4 = (2/1, 10, 3/3)$, $P_5 = (1/2, 10, 3/3)$, $P_6 = (1/1, 20, 3/3)$, $P_7 = (1/1, 10, 6/3)$, $P_8 = (1/1, 10, 3/6)$ relative objective values w.r.t best solution. Runtime in seconds.

Additionally to instance GRID, we tested our algorithms on the larger instance GOE as shown in Table 5. Here, the solution found by BR is only 8.3% worse than the one found by MP and the solution quality of mosts heuristics is similarly good. The runtime of BR and the heuristics range between 12.6% and 28.6% of the runtime of MP, again showing that BR and the simpler heuristics AW ub/lb are significantly faster than AF ub/lb while the more complex heuristics perform better.

	relative objective P_1	runtime	# iterations
MP	1	1:03:32	-
BR	1.083	0:08:34	5
AF ub	1.132	0:18:10	5
AF lb	1.143	0:14:20	4
AW ub	1.577	0:08:01	5
AW lb	1.156	0:11:42	7

Table 5: Comparison of solutions for (LPQC) under parameter settings $P_1 = (1/0, 20, 3/3)$ for MP and the heuristics, $P_2 = (1/0, 20, 6/0)$ for BR on instance GOE, runtime in h:min:sec

6 Conclusions and further research

We presented a new idea to approach line planning by solving a routing game where the passengers are the players who aim at minimizing a weighted sum of their travel time, transfer penalties, and a cost share. Under strong assumptions on the objective function (transfer time is not taken into account and line costs can be assigned to edges and are independent of frequencies) equilibria of this game can be found using the described best-response algorithm. In case that the objective function does not fulfill these properties, applicability and convergence of the best-response approach can be achieved by a slight modification of the individual objective functions.

A logical next step will be to evaluate whether the line planning routing game, besides being an interesting object of study in itself, does indeed lead to a good heuristic for line planning.
First, more experiments of the type presented in Section 5 on instances of realistic size (in particular also with respect to passenger numbers) may lead to more insights on the performance of

the different approaches presented in Section 4.4. A positive effect of increasing passenger numbers is that the approximate frequencies $f(\mathcal{R}^{-q})$ and $\tilde{f}(\mathcal{R}^{-q})$ become better estimates of actual frequencies $f(\mathcal{R})$. However, in the current version of the best-response strategy, in each iteration a shortest path for each passenger has to be found, hence running time increases with increasing number of passengers. For large passenger numbers it may thus make sense to use flow equilibration techniques in the inner loop instead of shortest path computations for each individual passenger.

Second, line planning solutions obtained with the routing game approach should be compared to state-of-the-art exact and heuristic solution methods for line planning with respect to objective value, running time, and practicability of the found solution (in the sense of Section 3.3).

While the terms for travel time and transfers are quite intuitive, many different choices are possible for the cost-sharing among passengers. It remains an interesting question how to divide operational costs among passengers such that, on the one hand, the algorithmic approach is still viable, and on the other hand, cost shares are comparable to real-world travel costs. Furthermore, it would be interesting to investigate whether the routing game approach can also be applied to line planning with additional constraints and other planning problems which can be considered integrated network design and routing problem like, e.g., timetabling or delay management with integrated routing.

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Appendix

NP-hardness of the routing problem with transfer times

Theorem 4.3. *The routing problem as in Definition 4.1 is NP-hard, even if transfer penalties and operational costs are not taken into account, i.e.,* $\beta = 0$ *and* $\gamma_1 = \gamma_2 = 0$*.*

Proof. Similarly to the proof of Theorem 4.2 we prove this theorem by reduction from SCP. Let $(\mathcal{M}, \mathcal{C}, K)$ denote an instance of SCP and denote $n := |\mathcal{M}|$. Our PTN consists of two parts: The first part is used to ensure that at most K sets are chosen from $\mathcal{C}$. The second part is similar to the construction in the proof of Theorem 4.2 and is used to determine whether the chosen sets cover $\mathcal{M}$.

The first part of the PTN consists of vertices v_i for $i = 1, \ldots, 2K+1$ and edges $e_i = (v_i, v_{i+1})$, $i = 1, \ldots, 2K$ with $c(e_i) = 0$. For every edge e_{2i-1} with an odd index we introduce a line $\bar{l}_{2i-1}$ which consists of this edge only.

The second part of the PTN consists of vertices w_i for $i = 1, \ldots, 2n+1$ and edges $a_i = (w_i, w_{i+1})$ for $i = 1, \ldots, 2n$ with $c(a_i) = 0$. Furthermore, we add edges $\bar{a}_{ij}$ which connect all pairs of vertices w_i and w_j with $i < j$ and whose length is $c(\bar{a}_{ij}) := K' + 1$, where $K' := \frac{2K+2n}{3}$. For each $i = 1, \ldots, n$ we introduce a line $\tilde{l}_{2i-1}$ which covers the edge a_{2i-1}. We connect both parts of the PTN by a transition edge $t = (v_{2K+1}, w_1)$.

For every $C \in \mathcal{C}$ we create a line $l_C \in \mathcal{L}$ containing all edges $\{e_{2i} : m_i \in C\}$ from the first part of the PTN, the transition edge t, and the edges a_{2i} with $m_i \in C$ from the second part of the PTN. We add additional edges with lengths $K' + 1$ wherever needed to ensure that the lines are connected paths in the PTN.

In contrast to the proof of Theorem 4.2, in this proof we have $|\mathcal{C}| + 1$ passengers. Each passenger q_C with $C \in \mathcal{C}$ has origin v_1 and destination v_{2K+1} and his route $\mathcal{R}_{q_C}$ is identical to line l_C from v_1 to v_{2K+1}. The passenger q for which we have to solve the routing problem has origin v_1 and destination w_{2n+1}. We set the capacity in each vehicle to $B := 1$. For the objective function we use the parameters $\alpha_1 = \alpha_2 = 1$, $\beta = \gamma_1 = \gamma_2 = 0$ and $T = 1$. Note that line costs can be set to arbitrary values, since $\gamma_1 = \gamma_2 = 0$.

We now show that there is a solution to the considered instance of SCP if and only if there is a solution R_q to the routing problem (RP_q) with individual objective value $h_q(R_q, \mathcal{R}^{-q}) \leq K'$.

First note that any such route R_q in the first part of the PTN will use the lines $\bar{l}_i$ on edges with an odd index and some lines l_C on the ones with an even index, because otherwise $h_q(R_q, \mathcal{R}^{-q}) > K'$. Note that whenever the passenger uses a line l_C, the frequency of this line is set to $f_{l_C} := 2$. Consequently, for all of these paths the contribution from the first part of the PTN to the transfer time component τ_q in the individual objective function is $\frac{2K}{3}$, since the length of every used edge is 0 and on each such path there is a transfer at each station between a line $\bar{l}_{2i-1}$ with frequency 1 (used only by passenger q) and a line l_C with frequency 2. In the second part of the PTN, only edges a_i can be used in such a route R_q, because otherwise $h_q(R_q, \mathcal{R}^{-q}) > K'$. Hence $c_q(R_q) = 0$. Now consider the contribution to τ_q of route R_q in the second part of the PTN. At each node in the second part of the PTN a transfer has to take place, between a line $\tilde{l}_{2i-1}$ and a line l_C. Thereby, transfer time is $\frac{1}{2}$ if passenger q did not use line l_C in the first part of the PTN, $\frac{1}{3}$ if he used it. Since there are $2n$ such transfers, any path with individual objective value less or equal to K' uses on edge a_{2i} a line that was already used in the first part of the PTN (because otherwise $h_q(R_q, \mathcal{R}^{-q}) > K'$).

Due to the construction of the lines l_C, this means that if there is a route R_q with $h_q(R_q, \mathcal{R}^{-q}) \leq K'$, for each element $m_i \in \mathcal{M}$ at least one line l_C with $C \ni m_i$ is used in the first part of the PTN. Since not more than K such lines can be used in R_q, there must be a solution to the considered instance of SCP.

On the other hand, if there is a solution $\mathcal{C}' = \{C_1, \ldots, C_k\}$ with $k \leq K$ to the considered instance of SCP, using line l_{C_i} on edge e_{2i} for $i = 1, \ldots, k$ (and arbitrary lines on e_{2i} for $i = k+1, \ldots, K$) allows the passenger to choose a path with transfer time $\frac{n}{3}$ in the second part of the PTN and thus yields an individual objective value of at most K'. $\square$

Non-existence of equilibria

We now describe the example for non-existence of equilibria from Section 4.2.1 more formally and prove that no equilibrium exists.
We consider the PTN from Figure 3 with 12 nodes and 18 edges. Every edge is served by one directed line which contains only this edge, so that we have a one-to-one correspondence between edges and lines. The capacity of a vehicle is $B = 1$. There are three *main* passengers q_1 from u_1 to v_1, q_2 from u_2 to v_2, and q_3 from u_3 to v_3 and six sets of *auxiliary* passengers: $\mathcal{Q}_i^j$ for $i = 1, \ldots, 3$ and $j = 1, 2$ contains M passengers from v_i^j to v_i (where M is a sufficiently large number, e.g., $M > 12$). We denote by $\mathcal{Q}'$ the union of the auxiliary passengers.
In our objective function we take only the transfer time into account, i.e., $\alpha_1 = \beta = \gamma_1 = \gamma_2 = 0$ and $h_q(\mathcal{R}) := \tau_q(R_q, \mathcal{R}^{-q})$. We set $T = 1$.
Note that for the auxiliary passengers there is only one route from origin to destination, hence, each of them only has one strategy. Let $\mathcal{R}'$ denote the set of these strategies. Each of the main passengers q_i has two different strategies: to take the route R_i^1 starting with edge (u_i, v_i^1) or to take the route R_i^2 starting with edge (u_i, v_i^2).
We now show that there does not exist an equilibrium in the described situation. Assume that $\mathcal{R}$ is an equilibrium of the described line planning routing game. Denote by $R_i^{j_i}$ the strategy chosen by q_i. Without loss of generality, assume that $j_1 = 1$. Then

$$g_2(R_1^1, R_2^1, R_3^{j_3}, \mathcal{R}') = \begin{cases} \frac{1}{1+2} + \frac{1}{2+2} + \frac{1}{2+M+1} = & \frac{7}{12} + \frac{1}{M+3} \quad \text{if } j_3 = 1 \\ \frac{1}{1+2} + \frac{1}{2+1} + \frac{1}{1+M+1} = & \frac{8}{12} + \frac{1}{M+2} \quad \text{if } j_3 = 2 \end{cases}$$

and

$$g_2(R_1^1, R_2^2, R_3^{j_3}, \mathcal{R}') = \begin{cases} \frac{1}{1+1} + \frac{1}{1+1} + \frac{1}{1+M+1} = & \frac{12}{12} + \frac{1}{M+2} \quad \text{if } j_3 = 1 \\ \frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{2+M+1} = & \frac{10}{12} + \frac{1}{M+2} \quad \text{if } j_3 = 2 \end{cases}$$

Since $\mathcal{R}$ is an equilibrium, we conclude that $j_2 = 1$, i.e., $R_2^{j_2} = R_2^1$.
Now

$$g_3(R_1^1, R_2^1, R_3^1, \mathcal{R}') = \frac{1}{1+2} + \frac{1}{2+1} + \frac{1}{1+M+1} = \frac{4}{6} + \frac{1}{M+2}$$

and

$$g_3(R_1^1, R_2^1, R_3^2, \mathcal{R}') = \frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{2+M+1} = \frac{5}{6} + \frac{1}{M+3}.$$

Since $\mathcal{R}$ is an equilibrium, we conclude that $j_3 = 1$, i.e., $R_3^{j_3} = R_3^1$.
Now we have a look at the strategies for q_1:

$$g_1(R_1^1, R_2^1, R_3^1, \mathcal{R}') = \frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{2+M+1} = \frac{5}{6} + \frac{1}{M+3}$$

and

$$g_1(R_1^2, R_2^1, R_3^1, \mathcal{R}') = \frac{1}{1+2} + \frac{1}{2+1} + \frac{1}{1+M+1} = \frac{4}{6} + \frac{1}{M+2}.$$

Thus, $g_1(R_1^1, R_2^1, R_3^1, \mathcal{R}') > g_1(R_1^2, R_2^1, R_3^1, \mathcal{R}')$. This is a contradiction to R_1^1 being part of an equilibrium.
Due to the symmetry of the construction of the instance, the assumption that R_1^2 is part of an equilibrium leads to a contradiction in the same way.

Proof of existence of potential functions for games with arc weight functions

Lemma 4.5. *Let $I := (G, \mathcal{L}, \mathcal{Q}, \{h_q : q \in \mathcal{Q}\})$ be an instance of the LPRG such that arc weight functions as specified in* (4) *exist. Then*

1. $\Phi(\mathcal{R}) := \sum_{a\in\mathcal{A}} \sum_{i=1}^{x_a(\mathcal{R})} \bar{w}_a(i)$ *is a potential function for I,*
2. *there exists an equilibrium to I,*
3. *Algorithm 1 converges to an equilibrium in a finite number of steps,*
4. *each of the steps can be executed in polynomial time.*

Proof. This proof follows standard arguments for convergence of atomic routing games, compare, e.g., [Rou07].

1. Let $\mathcal{R}$ and $\mathcal{R}'$ be two route sets. We denote with P_q and P'_q the corresponding paths for passenger q in the CGN and with $x_a := x_a(\mathcal{R})$ and $x'_a := x_a(\mathcal{R}')$ the corresponding flows on edge a of the CGN. We first observe that

$$\begin{aligned}\Phi(R_q, \mathcal{R}^{-q}) - \Phi(R'_q, \mathcal{R}^{-q}) &= \sum_{a\in P_q\setminus P'_q} \bar{w}_a(x_a) - \sum_{a\in P'_q\setminus P_q} \bar{w}_a(x'_a)\\ &= h_q(R_q, \mathcal{R}^{-q}) - h_q(R'_q, \mathcal{R}^{-q}),\end{aligned}$$

   hence Φ indeed is a potential function by (1).

2. Hence, every optimum of Φ is an equilibrium of the game. Since the number of solutions is finite, there exists at least one optimum of Φ/equilibrium of I.
3. Since in each step of Algorithm 1 there is a non-zero improvement in the individual objective function and thus also in the potential function, and the number of solutions is bounded, Algorithm 1 converges to an optimum of Φ which is an equilibrium.
4. We set $w^q_a(\mathcal{R}^{-q}) := \bar{w}_a(x_a(\mathcal{R}^{-q}) + 1)$. Then

$$\begin{aligned}h_q(R_q, \mathcal{R}^{-q}) &= \sum_{a\in P_q} \bar{w}_a(x_a(\mathcal{R}))\\ &= \sum_{a\in P_q} \bar{w}_a(x_a(\mathcal{R}^{-q}) + 1)\\ &= \sum_{a\in P_q} w^q_a(\mathcal{R}^{-q}).\end{aligned}$$

   The proposition follows from Lemma 4.4.

□

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