A Nonlinear Discretization Theory for Meshfree Collocation Methods applied to Quasilinear Elliptic Equations

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Abstract

We generalize our earlier results concerning meshfree collocation methods for semilinear elliptic second order problems to the quasilinear case. The stability question, however, is treated differently, namely by extending a paper on uniformly stable discretizations of well-posd linear problems to the nonlinear case. These two ingredients allow a proof that all well-posed quasilinear elliptic second-order problems can be discretized in a uniformly stable way by using sufficient oversampling, and then the error of the numerical solution behaves like the error obtainable by direct approximation of the true solution by functions from the chosen trial space, up to a factor induced by being forced to use a Hölder-type theory for the nonlinear PDE. We apply our general technique to prove convergence of meshfree methods for quasilinear elliptic equations with Dirichlet and non-Dirichlet boundary conditions. This is achieved for bifurcation and center manifolds of elliptic partial differential equations and their numerical methods as well.

1 Overview

Similarly to our previous papers, we examine the convergence behavior of suitable collocation-based meshfree methods. In [11, 12] we had studied exactly one of the simplest and then one of the most complicated nonlinear elliptic boundary value problems of order two, the fully nonlinear Monge–Ampère equation. The positive numerical experience with these cases calls for a generalization to a whole class of quasi-linear problems.
For application aspects of these problems, there are many numerical results with meshfree methods, however, without discussing the convergence [14, 32, 28, 13, 20, 19].

The book [7] contains a lot of new, but also summarizes many known results for the non-meshfree case. For example, for Finite Element Methods Zeidler [31] and Skrypnik [27], for difference methods Schumann and Zeidler [26]. Most interesting, however, are the contributions to Discontinuous Galerkin Methods. They start with Riviére and Wihler [23] and continue e.g. with Süli and his colleagues Houston, Robson, and Wihler [17, 18, 16].

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This paper starts with the essential prerequisites of the nonlinear discretization theory from [11], namely

1. well-posedness of the PDE problem,
2. approximation in trial spaces,
3. testing by equations and solving by optimization, and finally
4. Stability.

However, it takes a different path in Section 5 concerning stability. There, the general nonlinear discretization theory of [11] is combined with a result of [24] for linear problems. The latter allows to construct uniformly stable monotone, refinable, and dense discretizations of well-posed linear problems that are uniformly stable, the numerical stability being only dependent on the stability of the original linear PDE problem. The region of the local validity of this discretization does not vary with the refinement of the discretization. This strengthens a result in [11], and is applicable to all PDE problems that satisfy the well-posedness assumptions of [11]. In section 7, we prove that these assumptions are satisfied for general elliptic quasilinear equations of order two. Extensions will allow other nonlinear well-posed PDE problems, provided that the specific well-posedness assumptions are satisfied. Finally, Section 8 seems to be the first dealing with the case of non-Dirichlet boundary conditions for meshless methods solving quasilinear elliptic problems.

2 Well-posed nonlinear problems

We consider boundary value problems on bounded Lipschitz domains Ω in $\mathbb{R}^d$ and formulate them strongly as

$$\begin{array}{rcccc} Gu & = & f_1 & \text{on} & \Omega \\ Bu & = & f_2 & \text{on} & \partial\Omega \end{array}$$

with differential and boundary operators G and B, respectively. These are mappings defined as

$$\begin{array}{ccccccc} G & : & \mathcal{D}(G) & \subseteq & \mathcal{U} & \to & \mathcal{V}_1 \\ B & : & \mathcal{D}(B) & \subseteq & \mathcal{U} & \to & \mathcal{V}_2 \end{array}$$

on a Banach space $\mathcal{U}$ of functions on $\overline{\Omega}$, and map that space into Banach spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ on the domain and the boundary, respectively. We combine the two maps and simplify the problem by

$$\begin{array}{rcl} F & = & (G, B) \\ F\ :\ \mathcal{D}(F) & \subseteq & \mathcal{U} \to \mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2, \\ Fu & = & f := (f_1, f_2) \end{array} \tag{1}$$

for notational convenience.

The space $\mathcal{U}$ should contain a locally unique true solution denoted by u^* that we want to approximate numerically.

Around this local solution, we require some form of well-posedness of the problem. One way is to let the linearization F' of F near u^* be boundedly invertible as a map $F' : \mathcal{U} \to \mathcal{V}$. The other, less popular one is to ask for an inequality of the form

$$c_F^{-1}\|u-v\|_{\mathcal{U}} \leq \|Fu - Fv\|_{\mathcal{V}} \leq C_F\|u-v\|_{\mathcal{U}} \tag{2}$$

for all u, v in a neighborhood of u^*. Fortunately, (2) follows from the usual well-posedness condition:

Theorem 1. *[11] Let F be Fréchet-differentiable in each point of a neighborhood $\mathcal{N}(u^*)$ of u^* and let the Fréchet derivatives $F'(u)$ at u be bounded and Lipschitz continuous, i.e.*

$$\|F'(v) - F'(u)\|_{\mathcal{V}} \leq C\|u-v\|_{\mathcal{U}} \text{ for all } u, v \in \mathcal{N}(u^*).$$

Finally, let $F'(u^*)$ have a bounded inverse. Then (2) holds in a neighborhood of u^*, and all Fréchet derivatives are uniformly bounded and have uniformly bounded inverses there.

3 Approximation in Trial Spaces

Our meshless numerical approximations are taken from a scale $\{U_r\}_{r>0}$ of linear finite-dimensional nested *trial* spaces $U_r \subset \mathcal{U}$ with $U_{r'} \subset U_r$ for $r' < r$. Our substitute for *consistency* is the assumption that the true solution u^* can be approximated well by elements u_r of the trial spaces U_r in the sense

$$\inf_{u_r \in U_r} \|u^* - u_r\|_{\mathcal{U}} \leq \epsilon(r, u^*) \tag{3}$$

for all $r > 0$, with small $\epsilon(r, u^*)$ tending to zero for $r \to 0$. Since we have the norm of $\mathcal{U}$ on the left-hand side, and since we want a good convergence rate of the approximations, we shall usually have to assume that the true solution u^* and the trial spaces U_r lie in a *regularity subspace* $\mathcal{U}_R$ of $\mathcal{U}$ that determines the convergence rate.

Since we did not choose $\epsilon(r, u^*)$ minimally in (3), we can assume that there are elements $u_r^* \in U_r$ with

$$\inf_{u_r \in U_r} \|u^* - u_r\|_{\mathcal{U}} \leq \|u^* - u_r^*\|_{\mathcal{U}} \leq \epsilon(r, u^*) \tag{4}$$

that realize the optimal convergence rate of approximation. Our goal is that the numerical solution $\tilde{u}_r \in U_r$ of the PDE problem, if we use a proper numerical algorithm for PDE solving based on the trial space U_r, converges to the true solution u^* as fast as the approximation u_r^* in the trial space converges. However, as convergence is usually an interplay of consistency and stability, there may be factors arising from instabilities that deteriorate that convergence rate.

4 Testing and Solving

The trial space discretizes the *domain* of F, but *testing* discretizes the *range*. This is done by a scale $\{T_s\}_{s>0}$ of linear *test maps* that takes functions $f = (f_1, f_2)$ in $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ into *data values* $T_s(f) \in V_s$ in some finite-dimensional space V_s, e.g.

$$T_s(f) = (f_1(x_1), \dots, f_1(x_M), f_2(y_1), \dots, f_2(y_N)) \in V_s := \mathbb{R}^{M+N}$$

with $M,\ N$, and the *collocation points* $x_i \in \Omega,\ y_j \in \partial\Omega$ implicitly depending on s.

The *discretized* problem replacing $Fu = f$ now consists in solving

$$T_s(Fu_r) \approx T_s(f) \tag{5}$$

for some trial function $u_r \in U_r$. From Section 3 we know that there are good approximations $u_r^* \in U_r$ to u^*, and therefore we are satisfied with finding a numerical solution $\tilde{u}_r \in U_r$ with

$$\|T_s(F\tilde{u}_r) - T_s(f)\|_{V_s} \leq 2\|T_s(Fu_r^*) - T_s(f)\|_{V_s}.$$

This can, for instance, be accomplished by an approximate solution of the finite-dimensional nonlinear optimization problem

$$\min_{u_r \in U_r} \|T_s(Fu_r) - T_s(f)\|_{V_s}$$

under a suitable parametrization of the trial space. Note that we do not solve the linearized problem, in contrast to many standard algorithms for nonlinear problems.

5 Stability

Clearly, solving (5) in the linear case will run into problems if $\dim U_r > \dim V_s$, and in general it will stabilize the problem when we *oversample*, i.e. take $\dim U_r$ smaller than $\dim V_s$. We can mimic the logic of well-posedness of Section 2 by asking for an inequality

$$\|u_r - v_r\|_{\mathcal{U}} \leq C_{SF}(r,s)\|T_sFu_r - T_sFv_r\|_{V_s} \tag{6}$$

for all u_r, v_r in U_r, but true stability would mean that the constant $C_{SF}(r,s)$ has a fixed upper bound. This can be achieved [24] for well-posed problems by letting the test strategy *oversample*, i.e. by letting the test maps T_s depend on the U_r such that $C_{SF}(r, s(r))$ is uniformly bounded.

We want to play this back to the linearization, using [11] and [24]. The latter paper proves that for all well-posed linear problems one can find well-designed *monotonic refinable dense* (MRD) discretizations that are uniformly stable. In the notation of this paper, the basic requirement for MRD discretizations can be shortly rephrased as

$$\|v\|_{\mathcal{V}} := \sup_{s>0} \|T_s v\|_{V_s} \text{ for all } v \in \mathcal{V}, \tag{7}$$

defining a norm on $\mathcal{V}$. This holds if for $s \to 0$ the discrete norms approximate the continuous norm from below (*monotonicity*) by getting finer and finer (*refinability*) and *dense* in the limit. Note that this property is independent of PDEs and trial spaces. It just expresses how discrete norms approximate continuous norms. The simplest case arises for the sup norm on $\mathcal{V}$ and pointweise collocation, i.e. strong discretization of the data by pointwise evaluation on finite sets getting dense in the domain. But it holds also for weak discretizations and the L_2 norm.

Unfortunately, a problem for the applications in Sections 7 and 8 to quasilinear systems arises with (7), because the PDE theory there needs $\mathcal{V}$ to carry a Hölder norm. But in [24] and in the numerical paper [12], the range space should be $\mathcal{V} = C(\overline{\Omega}) \times C(\partial\Omega)$, making (7) obvious. This is why we have to change the argument in [24] accordingly.
We assume a linear operator equation of the form $Au = f$ with $A : \mathcal{U} \to \mathcal{V}$ and a well-posedness inequality

$$\|u\|_{\mathcal{U}} \leq C_S \|Au\|_{\mathcal{V}} \text{ for all } u \in \mathcal{U}.$$

The space $\mathcal{V}$ should carry a norm $\|.\|_{\infty,\mathcal{V}}$ that maybe weaker than the norm in $\mathcal{V}$, and that allows MRD discretizations as in [24], i.e.

$$\|v\|_{\infty,\mathcal{V}} = \sup_{s>0} \|T_s v\|_{\infty,V_s} \text{ for all } v \in \mathcal{V}$$

instead of (7). This is true if $\mathcal{V}$ carries a Hölder norm of type C^γ with $\gamma > 0$ to satisfy the requirements of PDE theory, and if collocation is used on the numerical side, leading to the weaker norm $\|.\|_{\infty,\mathcal{V}} = \|.\|_\infty$.
For any given finite-dimensional trial subspace $U_r \subset \mathcal{U}$, we form the finite-dimensional subspace $W_r := A(U_r)$ and follow the argument in [24] to find a MRD discretization with sup norms on the spaces V_s with

$$\|w_r\|_{\infty,\mathcal{V}} \leq 2\|T_s w_r\|_{\infty,V_s} \text{ for all } w_r \in W_r = A(U_r).$$

On the finite-dimensional subspace W_r, we have a norm-equivalence relation

$$c_{r,V}^{-1}\|w_r\|_{\mathcal{V}} \leq \|w_r\|_{\infty,\mathcal{V}} \leq C_V\|w_r\|_{\mathcal{V}} \text{ for all } w_r \in W_r,$$

where the left-hand constant may depend on r. Then, using (7), we get

$$\begin{array}{rcl}
\|u_r\|_{\mathcal{U}} & \leq & C_S\|A(u_r)\|_{\mathcal{V}} \\
 & \leq & c_{r,V}C_S\|A(u_r)\|_{\infty,\mathcal{V}} \\
 & \leq & 2c_{r,V}C_S\|T_s(A(u_r))\|_{\infty,V_s} \\
 & \leq & 2c_{r,V}C_S\|A(u_r)\|_{\infty,V} \\
 & \leq & 2c_{r,V}C_SC_V\|A(u_r)\|_{\mathcal{V}} \text{ for all } u_r \in U_r,
\end{array}$$

i.e. the linear problem has a stability bound that depends only on the trial space, and in a controllable way. On the downside, the final convergence rates will be decreased by the behavior of $c_{r,V}$.

Theorem 2. *Assume a linear operator equation of the form $Au = f$ with $A : \mathcal{U} \to \mathcal{V}$ and a well-posedness inequality*

$$\|u\|_{\mathcal{U}} \leq C_S \|Au\|_{\mathcal{V}} \text{ for all } u \in \mathcal{U}.$$

Then for each trial space $U_r \subset \mathcal{U}$ there is an MRD discretization by uniformly bounded test maps T_s such that the linear problem has a stability bound in the sense of (6) with

$$C_{SF}(r, s(r)) \leq 2c_{r,V} C_S C_V. \quad \square$$

Note that there is no ellipticity assumption. The basic proof ingredient in [24] is a covering argument in $\mathcal{V}$ for the unit ball of the finite-dimensional subspace $A(U_r)$. On the downside, Approximation Theory shows that a strong amount of oversampling may be needed for badly chosen collocation points, up to $\dim T_{s(r)} \geq c\,(\dim U_r)^2$.

In the special situation of this paper, we have the model situation

$$c_{r,\gamma}^{-1} \|w_r\|_{C^\gamma} \leq \|w_r\|_\infty \leq C_\gamma \|w_r\|_{C^\gamma} \text{ for all } w_r \in W_r$$

for the transition between Hölder space $C^\gamma(\Omega)$ and $C(\Omega)$ with the sup norm. There does not seem to be any literature on this for $0 < \gamma < 1$, not even for simple trial spaces, while for positive integer γ, such inequalities are of Markov type, see [22] for the univariate polynomial case. But by monotonicity arguments, one can replace the constant $c_{r,\gamma}$ for $0 < \gamma < 1$ by $c_{r,1}$ at a certain loss that needs further research.

However, we add a case for kernel-based trial spaces. Assume a scale $\{W_r\}_{r>0}$ of trial spaces W_r consisting of translates of the Whittle-Matérn kernel generating Sobolev space $W_2^m(\mathbb{R}^d)$ with $m > d/2 + \gamma \geq 1 + \gamma$, and let the translates be formed by sets $X_r := \{x_1, \ldots, x_{M(r)}\} \subset \overline{\Omega}$ of $M(r)$ centers that are *asymptotically uniformly distributed*, i.e. the *fill distance*

$$h_r := \sup_{y \in \Omega} \min_{x_j \in X_r} \|y - x_j\|_2$$

and the *separation distance*

$$q_r := \frac{1}{2} \min_{x_j \neq x_k \in X_r} \|x_j - x_k\|_2$$

satisfy

$$0 < c\, q_r \leq h_r \leq C\, q_r$$

with constants that are independent of r. Up to a factor, this implies $M(r) \approx h_r^{-d}$.

Theorem 3. *Under the above assumptions, and if the domain has a C^1 boundary,*

$$c_{r,\gamma} \leq Ch_r^{-\gamma-d/2}$$

with a constant independent of r.

Proof. The paper [25] proves that all trial functions $w_r \in W_r$ satisfy an *inverse inequality*

$$\|w_r\|_{W_2^m(\Omega)} \leq Ch_r^{-m+d/2}\|w_r\|_{2,X_r} \leq Ch_r^{-m}\|w_r\|_{\infty,X_r}$$

with generic constants depending on m and the domain, but not on r and the position of centers in X_r.

By Morrey's embedding theorem, Hölder spaces $C^{0,\gamma}(\Omega)$ for $0 < \gamma < 1$ are continuously embedded into Sobolev space $W_p^1(\Omega)$ if $p = d/(1-\gamma) \in (d, \infty)$ and if the domain has a C^1 boundary.

Then we invoke a sampling inequality [3]

$$|u|_{W_p^1(\Omega)} \leq C\left(h_r^{m-1-d(1/2-1/p)_+}|u|_{W_2^m(\Omega)} + h_r^{-1}\|w\|_{\infty,X_r}\right) \text{ for all } u \in W_2^m(\Omega)$$

and get

$$|w_r|_{W_p^1(\Omega)} \leq Ch_r^{-1-d(1/2-1/p)_+}\|w_r\|_{\infty,X_r} \text{ for all } w_r \in W_r.$$

For $d \geq 2$ we have $1/p \leq 1/2$ and thus finally arrive at

$$|w_r|_{C^{0,\gamma}(\Omega)} \leq Ch_r^{-\gamma-d/2}\|w_r\|_{\infty,X_r} \text{ for all } w_r \in W_r.$$

□

An extension to nonlinear problems is

Theorem 4. *Assume a well-posed nonlinear problem satisfying Theorem 6 and consider it in a neighborhood of a solution u^*. Then for each trial space $U_r \subset \mathcal{U}$, there is an MRD discretization by uniformly bounded test maps T_s in the sense of [24] such that the nonlinear problem is uniformly stable in the sense of (6) with an upper bound as in Theorem 2.*

Proof. In view of Theorem 6 we can assume that all local inverses of the Fréchet derivatives have a uniform upper bound, and we can reuse the constant c_F of (2) to get

$$\|v\|_{\mathcal{U}} \leq c_F\|F'(u^*)v\|_{\mathcal{V}} \text{ for all } v \in \mathcal{U}$$

for the linearization $F'(u^*)$ of F in u^*. If this linear problem is given an MRD discretization in the sense of [24], we have for each trial space U_r a scale of uniformly bounded test maps T_s such that

$$\|u_r\|_{\mathcal{U}} \leq 2c_{r,V}c_FC_V\|T_sF'(u^*)u_r\|_{V_s} \text{ for all } u_r \in U_r$$

holds. Note, however, that [24] lets T_s depend on the linear PDE problem, while U_r is fixed. Thus we have to fix T_s by using the fixed linearization in u^*. Note further that Definition 3 and (10) in [24] imply that the maps T_s are uniformly bounded, if an MRD discretization is chosen.

We now extend the above bound to linearizations at other functions. With $K(r) := c_{r,V}c_F C_V$ we get

$$\begin{array}{rcl}\|u_r\|_{\mathcal{U}} & \leq & 2K(r)\|T_sF'(u^*)u_r\|_{V_s}\\ & \leq & 2K(r)\|T_s(F'(u^*)-F'(u))u_r\|_{V_s} + 2K(r)\|T_sF'(u)u_r\|_{V_s}\\ & \leq & 2K(r)C\|T_s\|\|u-u^*\|_{\mathcal{U}}\|u_r\|_{\mathcal{U}} + 2K(r)\|T_sF'(u)u_r\|_{V_s}\end{array}$$

to arrive at

$$\|u_r\|_{\mathcal{U}} \leq 4K(r)\|T_sF'(u)u_r\|_{V_s} \text{ for all } u_r \in U_r$$

and all u in a neighborhood of u^*, using uniform boundedness of the test maps T_s again. We finally repeat the proof of Theorem 4 of [11], noting that due to uniform boundedness of the T_s we have uniformly bounded $C''(s)$ by equation (25) and well behaving R by the first formula below (27) there. Then, up to a constant, the stability property of the linearization carries over to the nonlinear case. □

Standard cases of MRD discretizations are collocation methods where the operator values are sampled in sufficiently many points. But [24] also treats more sophisticated situations that we do not pursue here.

6 Error Bounds and Convergence Rates

Since the previous section guaranteed stability inequalities by a suitable test discretization for any choice of trial spaces, [11] implies that the error for the numerical solution along the lines of Section 4 inherits the behavior (3) of the approximation error up to the factor $c_{r,V}$. This applies to many different situations, depending on the trial space and the smoothness assumptions on the solution, the domain, and the PDE problem as a whole. In cases where the true solution has a rapidly convergent expansion into trial functions, this convergence rate, measured in the norm of the well-posedness property of the problem, carries over to the numerical solution, provided that an MRD test strategy is chosen, including sufficient oversampling, and if a nonlinear optimization like in Section 4 is carried out. Typical numerical examples were provided in [11] and [12], the latter focusing on the Monge-Ampére equation that is not covered by this paper unless a result like Theorem 6 is provided.

To make this paper self-contained, we state the final result of [11] adapted to the situation here, including Theorem 4:

Theorem 5. *For a well-posed nonlinear problem (1) in the sense of Section 2 with a unique true solution $u^* \in \mathcal{U}$, and for all finite-dimensional trial spaces*

$U_r \subset \mathcal{U}$ there is a testing strategy using a test map T_s such that the solution technique of Section 4 leads to a numerical solution $\tilde{u}_r \in U_r$ with an error bound

$$\|u^* - \tilde{u}_r\|_{\mathcal{U}} \leq C\epsilon(r, u^*)c_{r,V}$$

where $\epsilon(r, u^*)$ is the error of a good approximation $u_r^* \in U_r$ to u^* as given in (4), where C depends only on the well-posedness of F near u^*m, and where $c_{r,V}$ is the instability factor induced by being forced to a Hölder-type treatment of the PDE. In short, the convergence rate of the approximation error in U_r determines the convergence rate of the solution to the PDE problem up to the factor $c_{r,V}$. $\square$

Note that the above result allows various convergence rates, depending on the smoothness of u^* and the trial space U_r, up to spectral convergence.

7 Quasilinear Equations of Order Two

We now show that the above assumptions are satisfied for quasilinear elliptic second order equations in strong form on a bounded Lipschitz domain, and we start with the case of Dirichlet boundary conditions.

We use results from Gilbarg and Trudinger's book [15], Chapters 10 and 15, as summarized in [7], subsection 2.5.4, Theorems 2.61 and 2.64 on existence, uniqueness, and regularity. We skip over the nonuniform elliptic cases, cf. (2.217), (2.218) there, noting that subsections 2.6.4, 2.6.6,.2.6.7 of [7] would allow strong extensions to systems of q equations of order $2m$ with higher technical complications.

The spaces and operators are

$$\begin{array}{rcl} \mathcal{U} & := & C^{2,\gamma}(\overline{\Omega}, \mathbb{R}), \\ \mathcal{V} & := & V_1 \times V_2 = C^{\gamma}(\overline{\Omega}) \times C^{\gamma}(\partial\Omega), \\ Gu & := & \displaystyle\sum_{i,j=0}^{d} a_{ij}(x, u, \nabla u)\partial^i\partial^j u, \\ Bu & := & u_{|\partial\Omega} \end{array} \tag{8}$$

where we used $-\partial^0 := Id$ and impose the compatibility condition

$$\mathcal{D}(G) := \left\{u \in \mathcal{U} \ : \ (x, u, \nabla u) \in \mathcal{D}(a_{ij}), 0 \leq i, j \leq d, Gu \in C^{\gamma}(\overline{\Omega})\right\}$$

to make everything well-defined. Furthermore, the domain boundary should satisfy $\partial\Omega \in C^{2,\gamma}$.

The linearization around a function u can be formally written as an operator

$$\begin{array}{rcl} G'(x, z, p)v & = & \displaystyle\sum_{i,j=0}^{d} a_{ij}(x, z, p)\partial^i\partial^j v \\ & & \displaystyle +\sum_{i,j=0}^{d} \partial^i\partial^j u \left(v\frac{\partial}{\partial z} + \sum_{k=1}^{d} \partial^k v \frac{\partial}{\partial p_k}\right) a_{ij}(x, z, p) \end{array} \tag{9}$$

for all points (x, z, p) in a neighborhood of $(x, u^*(x), \nabla u^*(x))$, and we assume the principal part to be uniformly elliptic there. Furthermore, the coefficients a_{ij}, $\frac{\partial}{\partial z}a_{ij}$, $\frac{\partial}{\partial p_k}a_{ij}$ should be Lipschitz continuous in z, p near the locally unique solution u^*, i.e. $(x, u^*(x), \nabla u^*(x))$, and finally satisfy the conditions around [7, Thm. 2.61].

Theorem 6. *Under these assumptions, the following holds:*

1. *There exists a solution $u^* \in C^{2,\gamma}(\overline{\Omega})$ of the problem $Fu^* = f$.*
2. *The principal part of the linearization G' is coercive near u^*.*
3. *If we write $F = (G, B)$ as in (1), and if zero is not an eigenvalue of $F'(u^*)$, then F' is boundedly invertible near u^* and $u^*$is a locally unique solution of (1).*
4. *In a neighborhood of u^*, the linearization in u is Lipschitz continuous in u.*

More detailed existence, uniqueness and regularity results are listed in [7] near Theorems 2.61 and 2.64.

8 Non-Dirichlet Boundary Conditions

In this section we show that for quasilinear elliptic differential equations Dirichlet and generalized Neumann, Robin, or mixed boundary conditions define well–posed problems. Then, by Theorem 5, suitably oversampled meshless methods are convergent. This generalization is highly nontrivial, because uniqueness might be missing in the basic theory. Furthermore, the boundary operators have to match the differential operators by *complementing conditions*. These have been intensively studied, e.g. by Agmon/ Douglis/Nirenberg [1], Lions, Magenes [21], Wloka [29], Zeidler [30] and Amann [2] p. 21.

To describe what is possible, we go back to the notation in Section 7. We assume that A, mimicking the principal part of $G'(u^*)$ in the first line of (9), is a strongly elliptic linear differential operator.

Neumann/Robin boundary operators take first derivatives of $u \in C^{2,1}(\overline{\Omega})$ on the $C^{1,1}$ boundary, and we can parametrize them into a normal component $\frac{\partial}{\partial \nu}$ and $d-1$ tangential components $\frac{\partial}{\partial t^i}$, $1 \leq i \leq d-1$ to arrive at

$$B_{NR}u := b_\nu \frac{\partial u}{\partial \nu} + \sum_{i=1}^{d-1} b_i \frac{\partial u}{\partial t^i} + b_0 u \tag{10}$$

with continuous functions b_ν and b_i, $0 \leq i \leq d-1$. A crucial assumption then is that b_ν is strictly positive.

Mixed conditions can be written as

$$B_\delta u := \delta B_{NR} u + (1-\delta) B_D u$$

with $\delta : C(\partial\Omega) \to \{0,1\}$ being a piecewise constant switch function on the boundary with constant value $\delta(\Gamma)$ on different components $\Gamma \subset \partial\Omega$. If A is now any strongly elliptic linear differential operator, the operator pairs (A, B_D), (A, B_{NR}), and (A, B_δ) are coercive.

This can be transferred to the nonlinear situation. We summarize [9] into

Theorem 7. *If the principal part A of the linearization $G'(u^*)$ is uniformly elliptic, and if the above conditions on the boundary operators are satisfied, then the nonlinear problems $(G, B_D), (G, B_{NR})$, and (G, B_δ) are well-posed in the sense of Section 2, provided that $\lambda = 0$ is not an eigenvalue of the linearized problems.*

Theorem 8. *Numerical Liapunov-Schmidt and center manifold methods for the above wide range of nonlinear elliptic (and parabolic) problems (G, B) in a bifurcation point u_0 are defined by bordering the nonlinear elliptic problem by a few rows and columns incorporating $\mathcal{N}(G'(u_0)$ and $(\mathcal{R}(G'(u_0))^\perp$. Thus these extended systems are well posed, cf. [7, 10, 4, 6, 5, 8]. Assume the conditions in Theorem 7. Then again the previous results for quasilinear equations apply and prove stable and convergent meshfree methods for Liapunov-Schmidt and center manifold techniques.*

This builds on results of [2] for parabolic problems, but we have to refer the reader to [9] for details. An extension to nonlinear boundary operators is obvious. Just replace in (10) the b_ν and b_i by $b_\nu(u)$ and $b_i(u)$ and thee previous $F = (G, B_{NR})$ by $F(u) = (G(u), B_{NR}(u)u)$. Then all results prevail.

9 Summary and Outlook

Our nonlinear discretization theory splits the necessary work for error bounds and convergence rates into two parts:

1. into PDE theory for establishing explicit results about well-posedness,
2. into Approximation theory for proving rates of convergence in trial spaces.

Once these ingredients are provided, there are numerical methods that have a certain stability, though at the expense of oversampling and nonlinear optimization. They guarantee that the numerical solution of the PDE solution converges at the rate of the trial space approximation reduced by a factor that arises from the Hölder theory of the PDE. Furthermore, they apply to all possible trial spaces and usually do not require any background integration.

However, there are several shortcomings that need additional work:

1. The amount of oversampling is strongly problem-dependent and cannot easily be addressed [24].

2. Playing the algorithmic part back to optimization [11, 12] ignores numerical efficiency.

3. There is no mention of sparsity considerations.

4. The convergence rates will in many cases not be competitive with sophisticated finite element techniques. They require high smoothness of the true solution to be effective. But, on the positive side, the optimization techniques are very easy to implement compared to finite element methods [11, 12], making them attractive for users who want quick answers without too much hassle.

5. There may be a different workaround for the stability complications arising from the Hölder theory of the PDE. If the well-posedness norm in $\mathcal{U}$ is taken weaker than the full norm on $\mathcal{U}$, the approach gets closer to [24] and yields uniform stability in that weaker norm, but then the linearization arguments of [11] have to be replaced.

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