Approximation by Radial Basis Functions with Finitely Many Centers

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Abstract: Interpolation by translates of "radial" basis functions Φ is optimal in the sense that it minimizes the pointwise error functional among all comparable quasi–interpolants on a certain "native" space of functions $\mathcal{F}_\Phi$. Since these spaces are rather small for cases where Φ is smooth, we study the behavior of interpolants on larger spaces of the form $\mathcal{F}_{\Phi_0}$ for less smooth functions Φ_0. It turns out that interpolation by translates of Φ to mollifications of functions f from $\mathcal{F}_{\Phi_0}$ yields approximations to f that attain the same asymptotic error bounds as (optimal) interpolation of f by translates of Φ_0 on $\mathcal{F}_{\Phi_0}$.

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1 Introduction

Given a continuous real–valued function Φ on $\mathbb{R}^d$ and a nonnegative integer m, we consider approximations by finitely many translates $\Phi(\cdot - x_j)$, $1 \leq j \leq N$, of Φ together with polynomials from the space $\mathbb{P}_m^d$ of d–variate polynomials of degree less than m. This defines the approximants, but we delay the definition of the functions f that are to be approximated. To comply with the theory of "radial" basis functions (see e.g. review articles by M.J.D. Powell [7], N. Dyn [2] and M. Buhmann [1]), we write the approximants as

$$p + g_\alpha := p + \sum_{j=1}^{N} \alpha_j \Phi(\cdot - x_j), \qquad p \in \mathbb{P}_m^d,\ \alpha \in \mathbb{R}^N, \tag{1.1}$$

for N pairwise distinct "centers" $x_1, \ldots, x_N \in \mathbb{R}^d$ and with the requirement

$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0 \qquad \text{for all } p \in \mathbb{P}_m^d \tag{1.2}$$

for the vector $\alpha \in \mathbb{R}^N$. Our main concern here is to keep m and Φ fixed and to study the approximation power of functions (1.1) when the number N of centers is large. Another interesting issue is the dependence of the approximation power on the location of the centers, but we do not pursue this question here.

Each function Φ that we shall treat here will implicitly introduce a "native" function space $\mathcal{F}$ with a seminorm $|\,.\,|$. But we shall use one function Φ_0 to define the space $\mathcal{F}_0$ of functions to be approximated, while the approximants are formed by (1.1) with another, possibly different function Φ_1. The corresponding seminorms will be $|\ |_0$ and $|\ |_1$. Error bounds are known so far only for interpolants with $\Phi_0 = \Phi_1$, and for $\Phi_0 \neq \Phi_1$ there are some interesting numerical observations (see [8]):

- For $\mathcal{F}_0 \supseteq \mathcal{F}_1$ the Φ_1–interpolants seem to have more or less the same error on the larger space $\mathcal{F}_0$ as the optimal Φ_0–interpolants (*quasi–optimality*).
- For $\mathcal{F}_0 \subseteq \mathcal{F}_1$ the Φ_1–interpolants seem to behave better on $\mathcal{F}_0$ than on $\mathcal{F}_1$ (*superconvergence*).

The results of this paper serve to support the first statement in case of approximation instead of interpolation.

Bounds for the interpolation error in case $\Phi_0 = \Phi_1$ are usually of the form

$$|f(x) - s_{f,X}(x)| \le |f|_0 \cdot P^*_{0,X}(x) \tag{1.3}$$

for all $x \in I\!R^d$, $f \in \mathcal{F}_0$, and all $X = \{x_1, \ldots, x_N\}$ with the nondegeneracy property

$$p(X) = \{0\},\ p \in I\!P^d_m \text{ implies } p \equiv 0. \tag{1.4}$$

Here $s_{f,X}$ is an interpolant to f on X of the form (1.1), and $P^*_{0,X}(x)$ is the *power function* that evaluates the norm of the error functional:

$$P^*_{0,X}(x) = \sup_{\substack{f \in \mathcal{F}_0 \\ |f|_0 \ne 0}} \frac{|f(x) - s_{f,X}(x)|}{|f|_0}.$$

Of course, the error bound (1.3) is large when x is far away from the centers. Therefore there are results that bound $P^*_{0,X}(x)$ nicely from above whenever x is surrounded by sufficiently many points from X. This is quantified by the "ρ–density"

$$h_{\rho,X}(x) := \sup_{\|y-x\|_2 \le \rho} \min_{z \in X} \|y - z\|_2 \tag{1.5}$$

of X around x. If X and x satisfy

$$h_{\rho,X}(x) \le h_0 \tag{1.6}$$

for a constant h_0 depending only on d, ρ and Φ, then error bounds of the form

$$\begin{aligned} P^*_{0,X}(x) &\le c \cdot (h_{\rho,X}(x))^k \\ P^*_{0,X}(x) &\le c \cdot \exp\left(-\frac{c}{(h_{\rho,X}(x))^k}\right) \end{aligned} \tag{1.7}$$

(see Madych/Nelson [6], [5] and Wu/Schaback [10]) are provided. Here and in the sequel we shall denote generic constants by c.

For approximation one should take x from a compact set $\Omega \subset I\!R^d$ and then consider all finite sets X such that (1.4) and

$$h_{\rho,X}(x) \le h \le h_0 \qquad \text{for all } x \in \Omega \tag{1.8}$$

hold. Thus h serves as a scaling parameter to control the approximation quality in terms of the density of points of X with respect to Ω. Note that this requires X to extend at least by a distance ρ out of Ω. But a closer look at the proof technique of [6], [5], and [10] reveals that this is not necessary, provided that the boundary of Ω satisfies a *uniform interior cone condition*, i.e. there must be a fixed positive angle α such that from each point of the boundary of Ω there is a cone of angle not less than α extending locally into the interior of Ω. The sup in (1.5) is then restricted to the cone instead of a ball. This has independently been observed by W. Light (private communication). In view of (1.7) we should look for bounds like

$$\begin{aligned} \|f - a_{f,X}\|_{\infty,\Omega} &\le c \cdot h^k \qquad \text{or} \\ \|f - a_{f,X}\|_{\infty,\Omega} &\le c \cdot \exp\left(-\frac{c}{h^k}\right) \end{aligned} \tag{1.9}$$

for all X satisfying (1.8), where the approximant $a_{f,X}$ is of the form (1.1). In this sense we can compare error orders for interpolation and approximation.

In all interesting cases we shall get that approximation of functions from $\mathcal{F}_0$ by functions (1.1) with Φ_1 attains the (optimal) orders of interpolation by Φ_0 on $\mathcal{F}_0$, provided that $\mathcal{F}_0 \supseteq \mathcal{F}_1$. This will be done by showing (1.9) for right–hand sides that are comparable to (1.7).

2 Basic assumptions

We assume Φ to be symmetric in the sense $\Phi(\cdot) = \Phi(-\cdot)$ and to be of at most polynomial growth at infinity. Then Φ has a generalized Fourier transform in the sense of tempered distributions, and we require this (possibly singular) distribution to coincide on $\mathbb{R}^d \setminus \{0\}$ with a *positive* continuous function φ in the sense of Jones [3]. The possible polynomial growth at ∞ then corresponds to a singularity of φ at the origin, and we assume

$$\varphi(\omega) \leq c \cdot \|w\|^{-d-s_0} \qquad \omega \in U_0 \tag{2.1}$$

for a fixed and minimal $s_0 \in \mathbb{R}$ in a neighborhood U_0 of zero. Then m and s_0 are related by the crucial requirement

$$2m > s_0, \tag{2.2}$$

and to make the Fourier transform correspondence between φ and Φ analytically sound, we need $\varphi \in L_1(U_\infty)$ for a neighborhood U_∞ of infinity. Details of this can be found in [4] and [9].

3 Native function spaces

Each pair Φ, φ as defined above will give rise to a "native" function space $\mathcal{F}_\Phi$. One way of introducing $\mathcal{F}_\Phi$ proceeds by taking generalized Fourier transforms of functions (1.1), resulting in tempered distributions that coincide with functions $S_\alpha \cdot \varphi$ on $\mathbb{R}^d \setminus \{0\}$, where

$$S_\alpha(\omega) := \sum_{j=1}^{N} \alpha_j e^{i\omega^T x_j}$$

is kind of a symbol function that satisfies

$$|S_\alpha(\omega)| \leq c \cdot \|\omega\|_2^m, \quad c = c(\alpha, x_1, \ldots, x_N, m, N) \tag{3.1}$$

due to (1.2). Then the integral

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{|S_\alpha(\omega) \cdot \varphi(\omega)|^2}{\varphi(\omega)} d\omega = (2\pi)^{-d} \int_{\mathbb{R}^d} \varphi(\omega) |S_\alpha(\omega)|^2 d\omega = |g_\alpha - p|_\Phi^2 = |g_\alpha|_\Phi^2$$

will exist due to (2.1) and (3.1) and will define a seminorm on the approximants from (1.1). The "native" function space for Φ will now be the largest space to which this seminorm can be properly extended. This will in general be a space of distributions, but for sake of simplicity we restrict ourselves here to the space $\mathcal{F}_\Phi$ of functions f in $C(\mathbb{R}^d)$ with a generalized Fourier transform $\hat{f}$ in the weighted L_2 space

$$\left\{ g : \int_{\mathbb{R}^d} \frac{|g(\omega)|^2}{\varphi(\omega)}\, d\omega < \infty \right\} \tag{3.2}$$

such that the Fourier inversion formula

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\omega) e^{i\omega^T x} d\omega \tag{3.3}$$

holds for all $x \in \mathbb{R}^d$. The seminorm in $\mathcal{F}_0 = \mathcal{F}_{\Phi_0}$ then is

$$|f|_0^2 := |f|_{\Phi_0}^2 := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2}{\varphi_0(\omega)}\, d\omega,$$

when φ_0 is the function that coincides with the generalized Fourier transform of Φ_0 on $\mathbb{R}^d \setminus \{0\}$. One of the most important spaces is Sobolew space $W_2^k(\mathbb{R}^d)$ of all functions $f \in L_2(\mathbb{R}^d)$ having distributional derivatives up to order k that coincide with functions in $L_2(\mathbb{R}^d)$. For $k > d/2$ this is the native space corresponding to $s_0 = -d$, $m = 0$ and

$$\begin{aligned} \varphi(\omega) &= (1 + \|\omega\|_2^2)^{-k} \\ \Phi(x) &= c \cdot \|x\|_2^{k-d/2} \cdot K_{k-d/2}(2\pi\|x\|_2) \end{aligned} \tag{3.4}$$

with the Macdonald or modified spherical Bessel function K_ν. Due to this observation we shall restrict ourselves to the approximation of functions f from a space $\mathcal{F}_0$ corresponding to a pair Φ_0, φ_0. But our *approximants* (1.1) will use a *different* pair Φ_1, φ_1.

4 Basic results

If f is from a "native" space $\mathcal{F}_0 := \mathcal{F}_{\Phi_0}$ with $\mathcal{F}_0$ larger than $\mathcal{F}_1 := \mathcal{F}_{\Phi_1}$, we first approximate f by a *regularization* $f_M \in \mathcal{F}_1$ obtained via truncation of the Fourier transform, i.e.:

$$\hat{f}_M := \hat{f} \cdot \chi_M,$$

χ_M being the characteristic function of the Euclidean ball around zero with radius $M > 0$. Then f_M can be defined via (3.3), and there is an easy uniform error bound:

Lemma 4.1 *For each function $f \in \mathcal{F}_0$ we have*

$$|f(x) - f_M(x)| \le |f|_0 \cdot c_0(M), \tag{4.1}$$

uniformly in $x \in \mathbb{R}^d$, where $|.|_0$ is the seminorm in $\mathcal{F}_0$ and

$$c_0^2(M) := (2\pi)^{-d} \int_{\|\omega\|_2 \ge M} \varphi_0(\omega) d\omega. \tag{4.2}$$

Proof: Use (3.3) to get

$$\begin{aligned} |f(x) - f_M(x)| &\le (2\pi)^{-d} \int_{\|\omega\|_2 \ge M} |\hat{f}(\omega)| d\omega \\ &\le \left((2\pi)^{-d} \int_{\|\omega\|_2 \ge M} \frac{|\hat{f}(\omega)|^2}{\varphi_0(\omega)} \right)^{1/2} \cdot \left((2\pi)^{-d} \int_{\|\omega\|_2 \ge M} \varphi_0(\omega) \cdot d\omega \right)^{1/2} \end{aligned}$$

via Cauchy–Schwarz. □

Note that the above proof could allow for an additional $o(1)$ factor in the bound (4.1) for $M \to \infty$, the precise $o(1)$ behavior being dependent on f.

Lemma 4.2 *If φ_0/φ_1 is bounded in a neighborhood of zero, then for all $M > 0$ and all $f \in \mathcal{F}_0$ the function f_M lies in $\mathcal{F}_1$ with seminorm*

$$|f_M|_1 \leq |f|_0 \cdot C_{01}(M),$$

where

$$C_{01}^2(M) := \sup_{||\omega||_2 \leq M} \frac{\varphi_0(\omega)}{\varphi_1(\omega)}\ . \qquad (4.3)$$

Proof: Just evaluate

$$\begin{aligned} |f_M|_1^2 &= (2\pi)^{-d} \int_{||\omega||_2 \leq M} \frac{|\hat{f}(\omega)|^2}{\varphi_1(\omega)} \frac{\varphi_0(\omega)}{\varphi_0(\omega)}\, d\omega \\ &\leq |f|_0^2 \cdot \sup_{||\omega||_2 \leq M} \frac{\varphi_0(\omega)}{\varphi_1(\omega)}\ . \end{aligned}$$

□

Note that for $M \to \infty$ the function $c_0(M)$ decreases to zero while $C_{01}(M)$ does not decrease. Thus

Lemma 4.3 *There is a positive constant c depending only on d, φ_0, and φ_1, such that for all $0 < \varepsilon \leq c$ we have an $M(\varepsilon)$ with*

$$C_{01}(M(\varepsilon)) \cdot \varepsilon \leq c_0(M(\varepsilon)), \qquad (4.4)$$

and $M(\varepsilon) \to \infty$ for $\varepsilon \to 0$. □

From the literature (see e.g. [8]) we cite

Lemma 4.4 *Given Φ_1 with φ_1 and m_1, there is an error bound of the form*

$$|f(x) - s_{f,X}(x)| \leq |f|_1 \cdot P_{1,X}^*(x)$$

for all functions in the native space $\mathcal{F}_1$ and interpolants $s_{f,X}$ to f by functions (1.1) on sets $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ with (1.4). The power function $P_{1,X}^*(x)$ is the norm of the error functional, i.e.:

$$P_{1,X}^*(x) = \sup_{|f|_1 \neq 0} \frac{|f(x) - s_{f,X}(x)|}{|f|_1}$$

and it is the minimum of all such norms, if quasi–interpolants

$$q_{f,X}(x) := \sum_{j=1}^{N} u_j(x) f(x_j)$$

with

$$p(x) = \sum_{j=1}^{N} u_j(x) p(x_j) \qquad \text{for all } p \in \mathbb{P}_m^d$$

are allowed instead of $s_{f,X}$.

Theorem 4.5 *Given two radial basis functions Φ_0, Φ_1 with associated functions φ_0, φ_1 such that φ_0/φ_1 is bounded around zero, there is a positive constant c, depending only on d, φ_0, and φ_1, such that for all points $x \in I\!R^d$ and all sets X of centers satisfying* (1.4) *and $P^*_{1,X}(x) \leq c$ there is for all functions $f \in \mathcal{F}_0$ an approximant $a_{f,X}$ of the form* (1.1) *with $\Phi = \Phi_1$, satisfying*

$$|f(x) - a_{f,X}(x)| \leq 2|f|_0 \cdot c_0(M(P^*_{1,X}(x))),$$

the function M taken from (4.4).

Proof: We pick $\varepsilon = P^*_{1,X}(x)$ and $M(\varepsilon)$ with (4.4). With this M and a given $f \in \mathcal{F}_0$ we apply Lemmas 4.1, 4.2, and 4.4 for $a_{f,X} = s_{f_M,X}$. Then the assertion follows from (4.4) and

$$\begin{aligned} |f(x) - a_{f,X}(x)| &\leq |f(x) - f_M(x)| + |f_M(x) - s_{f_M,X}(x)| \\ &\leq |f|_0 \cdot c_0(M) + |f_M|_1 \cdot P^*_{1,X}(x) \\ &\leq |f|_0 \cdot (c_0(M) + C_{01}(M) \cdot \varepsilon). \end{aligned}$$

□

Remarks: Under the hypotheses of the theorem, the approximant $a_{f,X}$ can be chosen *independent* of x, provided that X and x satisfy a uniform bound

$$P^*_{1,X}(x) \leq \varepsilon \leq c.$$

The error bound then is

$$|f(x) - a_{f,X}(x)| \leq 2|f|_0 \cdot c_0(M(\varepsilon)).$$

In all practical cases there is an error bound

$$P^*_{1,X}(x) \leq F_1(h_{\rho,X}(x))$$

for all x and X satisfying

$$h_{\rho,X}(x) \leq h_0, \qquad F_1(h_{\rho,X}(x)) \leq c$$

with a monotonic function $F_1 : I\!R_{>0} \to I\!R_{>0}$ satisfying $F(0) = 0$ (see [9] for details). Then for any h with $h \leq h_0$ and $F_1(h) \leq c$, for all compact sets $\Omega \subset I\!R^d$ and all functions $f \in \mathcal{F}_0$ we have for all X with

$$h_{\rho,X}(x) \leq h \qquad \text{for all } x \in \Omega$$

an error bound

$$|f(x) - a_{f,X}(x)| \leq 2|f|_0 \cdot c_0(M(F_1(h))) \tag{4.5}$$

uniformly for $x \in \Omega$. Applications of Theorem 4.5 and (4.5) proceed as follows: First, fix Φ_0 and φ_0 to determine c_0 of (4.2). Then, for any other Φ_1 and φ_1, calculate C_{01} of (4.3) and compare with c_0 to find the function M that optimally satisfies (4.4). Then Theorem 4.5 or (4.5) can be applied. The following two sections will proceed along these lines.

5 Approximation in Sobolew spaces

We now fix $\mathcal{F}_0 = W_2^k(I\!R^d)$ with $2k > d$ and study approximations of the form (1.1) with different functions Φ. Due to (3.4) we have

$$\begin{aligned} c_0^2(M) &= (2\pi)^{-d} \int_{\|\omega\|_2 \geq M} (1 + \|\omega\|_2^2)^{-k} d\omega \\ &= c(d) \int_M^\infty (1 + r^2)^{-k} \cdot r^{d-1} dr \\ &= c(d,k)(M^{d-2k} + o(M^{d-2k})) \quad \text{for } M \to \infty, \end{aligned}$$

while $C_{01}(M)$ will depend on the "radial" basis function Φ_1 that controls the approximation. Thin–plate splines $\Phi_1(x) = \|x\|^\beta$ or $\|x\|^\beta \log \|x\|$ for $\beta > 0, \beta \notin 2I\!N$ or $\beta \in 2I\!N$ will have

$$\varphi_1(\omega) = c(\beta, d)\|\omega\|^{-d-\beta}$$

and φ_0/φ_1 is bounded near zero. Furthermore, for $2k < d + \beta$ we have

$$C_{01}^2(M) = c(\beta, d, k) \cdot M^{d+\beta-2k}(1 + o(1)) \quad \text{for } M \to \infty$$

and (4.4) is satisfied for

$$M(\varepsilon) = c(\beta, d, k) \cdot \varepsilon^{-2/\beta}(1 + o(1)) \quad \text{for } \varepsilon \to 0$$

and we have

$$c_0(M(\varepsilon)) = c(\beta, d, k) \cdot \varepsilon^{(2k-d)/\beta}(1 + o(1)) \quad \text{for } \varepsilon \to 0.$$

Convergence results for optimal interpolants on native spaces guarantee the existence of constants such that for all X and x with $h_{\rho,X}(x) \leq h_0$ the bound

$$P_{1,X}^*(x) \leq c \cdot h_{\rho,X}^{\beta/2}(x) \leq c \cdot h^{\beta/2} \tag{5.1}$$

holds whenever X and x satisfy $h_{\rho,X}(x) \leq h$. For such X and x, we can set $\varepsilon = h^{\beta/2}$ and apply (4.5) to get

$$|f(x) - a_{f,X}(x)| \leq 2 \cdot |f|_0 \cdot c \cdot h^{k-d/2}.$$

The exponent of h thus is the same as in the optimal error bound

$$P_{0,X}^*(x) \leq c \cdot h_{\rho,X}^{k-d/2}(x) \leq c \cdot h^{k-d/2}$$

that is attainable on the native space $\mathcal{F}_0$ itself (see the technique of Wu/Schaback [10]).

We now turn to multiquadrics

$$\Phi_1(x) = (1 + \|x\|_2^2)^{\beta/2}$$

for $\beta \notin 2Z\!\!Z$, where

$$\varphi_1(\omega) \leq c(d, \beta) \cdot \left\{ \begin{array}{ll} \|\omega\|^{-d-\beta} & \omega \text{ near } 0 \\ e^{-\|\omega\|}\|\omega\|^{-\frac{d+\beta+1}{2}} & \omega \text{ near infinity} \end{array} \right\}$$

for γ fixed, and where the exponent β in φ_1 is best possible. Then, up to constants and for $M \to \infty$, we have

$$C_{01}(M) = M^{-k} e^{M/2} M^{(d+\beta+1)/4} \, (1 + o(1)),$$

and φ_0/φ_1 is bounded near zero. Furthermore, $M(\varepsilon)$ has to satisfy

$$\varepsilon \cdot M^{-k} M^{k-d/2} e^{M/2} M^{(d+\beta+1)/4} = const$$

which can be done by solving the equation

$$\tfrac{1}{2}\, M(\varepsilon) = const - \log\varepsilon + \left(\tfrac{d}{4} - \tfrac{\beta}{4} - \tfrac{1}{4} \right) \log M(\varepsilon)$$

for sufficiently small ε and large $M(\varepsilon)$. Now the optimal error bound on $\mathcal{F}_1$ is

$$P^*_{1,X}(x) \le c \cdot e^{-\frac{\delta}{h_{\rho,X}(x)}} \le c \cdot e^{-\frac{\delta}{h}} =: \varepsilon \tag{5.2}$$

for some $\delta > 0$ due to Madych/Nelson [6], and we get $\log\varepsilon \approx -\dfrac{\delta}{h}$ and

$$\begin{aligned} \tfrac{1}{2}\, M(\varepsilon)(1+o(1)) &= \tfrac{\delta}{h}\,, \\ c_0(M(\varepsilon)) &\le c(d,k,\beta,\delta) h^{k-d/2}, \end{aligned}$$

for $h \to 0$ such that (4.5) again equalizes the rate of optimal interpolation in Sobolew space $W_2^k(\mathbb{R}^d)$.

For Gaussians

$$\Phi_1(x) = \exp(-\beta \|x\|_2^2)$$

we find

$$\varphi_1(\omega) = c \cdot \exp(-\|\omega\|_2^2/4\beta)$$

and φ_0/φ_1 will always be bounded. Clearly

$$C_{01}(M) = M^{-k} e^{+M^2/8\beta} \, (1 + o(1))$$

for $M \to \infty$ and $M(\varepsilon)$ has to satisfy

$$\varepsilon \cdot M^{-k} M^{k-d/2} e^{+M^2/8\beta} = const,$$

or

$$\frac{M^2(\varepsilon)}{8\beta} = const - \log\varepsilon + \tfrac{d}{2}\, \log M(\varepsilon)$$

for $\varepsilon \to 0$ and $M \to \infty$. Due to Madych/Nelson [6] the optimal error bound is

$$P^*_{1,X}(x) \le c \cdot e^{-\frac{\delta}{h^2_{\rho,X}}(x)} \le c \cdot e^{-\frac{\delta}{h^2}}, \tag{5.3}$$

and again $\log\varepsilon = (1+o(1)) \left(-\dfrac{\delta}{h^2} \right)$ yields the optimal order via

$$c_0(M(\varepsilon)) \le c \cdot h^{k-d/2}.$$

6 Approximation by Gaussians in spaces defined by multiquadrics

Finally, let us consider approximations of functions from the native space $\mathcal{F}_0$ for multiquadrics $\Phi_0(x) = (1^2 + \|x\|_2^2)^{\beta/2}$ by Gaussians $\Phi_1(x) = e^{-\alpha\|x\|^2}$. Then

$$c_0^2(M) \approx \int_M^\infty e^{-r} r^{d-1} dr^{-(d+\beta+1)/2} = c \cdot e^{-M} M^{(d-\beta-3)/2} \cdot (1 + o(1))$$

for $M \to \infty$ and

$$\begin{aligned} C_{01}^2(M) &\leq c \cdot M^{-d-\beta} e^{M^2/4\alpha}(1 + o(1)), \\ -\log \varepsilon &= const + \frac{M^2(\varepsilon)}{8\alpha} + \frac{M(\varepsilon)}{2} - \left(\frac{d-1}{2} + \frac{d+\beta}{2}\right) \log M(\varepsilon) \\ &= \frac{\delta}{h^2}(1 + o(1)) \qquad \text{(see (5.3))}. \end{aligned}$$

for $h, \varepsilon \to 0$. This yields

$$0 < \frac{\delta}{h^2} - \frac{\zeta}{h} \leq \frac{M^2(\varepsilon)}{2\alpha} \leq \frac{\delta}{h^2} \qquad \text{for a suitable } \zeta > 0$$

and

$$c_0(M) \approx \left(\frac{\sqrt{2\alpha\delta}}{h}\right)^{d-1} \exp\left(-\sqrt{2\alpha\left(\frac{\delta}{h^2} - \frac{\zeta}{h}\right)}\right), \tag{6.4}$$

which is still an exponential bound, but not of the form (5.2).

7 Final remarks

Unfortunately, the above cases had to be handled individually as special cases of the results of section 4. There would be a fairly general theorem stating quasi–optimality of "radial" basis function approximants on larger spaces, if some additional things, as suggested by the previous section, would hold true. If the interpolation error for Φ_i on its own native space is

$$P_{i,X}^*(x) \leq F_i(h_{\rho,X}(x))$$

for $i = 0, 1$, then we can observe that bounds like

$$\begin{aligned} c_0\left(\frac{c}{h}\right) &\leq c \cdot F_0(h) \\ C_{01}\left(\frac{c}{h}\right) \cdot F_1(h) &\leq c \cdot F_0(h) \end{aligned} \tag{7.5}$$

hold for $0 < h \leq h_0$ in all of the above cases. The technique in Wu/Schaback [10] ties the interpolation error to the behaviour of the Fourier transform around infinity, as required for such bounds as (7.5), but the bounds do not follow from there. There is a gap by a factor of h^{1-d} between (6.4) and (5.2), but the constant δ of (5.2) is not necessarily optimal. The precise form of optimal exponential error bounds for multiquadrics and Gaussians still is unknown: the results of [9] suggest that there might well be a factor of the form h^k occurring in the optimal bounds.

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