Convergence Analysis of General Spectral Methods

R. Schaback[b,*], M. Mohammadi[a]

[a] Department of Mathematical Sciences. Isfahan University of Technology, Iran
[b] Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Lotzestr. 16-18, D-37083 Göttingen, Germany

Abstract

If a spectral numerical method for solving ordinary or partial differential equations is written as a biinfinite linear system $b = Za$ with a map $Z : \ell_2 \to \ell_2$ that has a continuous inverse, this paper shows that one can discretize the biinfinite system in such a way that the resulting finite linear system $\tilde{b} = \tilde{Z}\tilde{a}$ is uniquely solvable and is unconditionally stable, i.e. the stability can be made to depend on Z only, not on the discretization. Convergence rates of finite approximations $\tilde{b}$ of b then carry over to convergence rates of finite approximations $\tilde{a}$ of a. Spectral convergence is a special case. Some examples are added for illustration.

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1. Introduction

In previous papers [10, 11] a convergence theory for a fairly general class of linear PDE solving techniques was presented, including unsymmetric kernel–based collocation and meshless Petrov–Galerkin methods. Its basic ingredients were

1. a well–posed and solvable PDE problem,
2. a trial space that approximates the solution well,

*Corresponding author
Email addresses: schaback@math.uni-goettingen.de (R. Schaback), m_mohammadi@math.iut.ac.ir (M. Mohammadi)

3. a test discretization that is fine enough to guarantee a stability inequality, and finally
4. an optimization routine serving as a solver.

The final step is necessary because the arising linear systems are not necessarily square and not necessarily solvable, though they have a good approximate solution. This discretization theory was extended to nonlinear problems in a recent paper [3], while the extension to spectral methods is the goal of this paper.

To this end, linear PDE problems and the standard versions of spectral methods (Galerkin, Tau, pseudospectral and Petrov–Galerkin) are presented in sections 2 and 3, with a common framework described in section 3.6 that allows a general convergence theory in section 4 that starts from biinfinite linear systems and considers solvability of discrete subsystems along the steps described above. Among other things, it is proven that well–posed biinfinite linear systems have stable and consistent discretizations, if the latter are properly chosen. The theory is applied to several numerical examples in section 6.

2. Linear PDE Problems

We consider a standard setup for time–independent problems as

$$\begin{array}{rcll} Lu &=& f & \text{in } \Omega, \\ Bu &=& g & \text{in } \Gamma := \partial\Omega \end{array} \tag{1}$$

with a linear differential operator L and a linear boundary operator B. They map between spaces as

$$\begin{array}{rcl} L &:& U \to F \\ B &:& U \to G \end{array} \tag{2}$$

where U and F are Hilbert spaces of functions on $\overline{\Omega}$ and G is a Hilbert trace space.

The problem (1) is assumed to be *well–posed* in the sense that the operators L and B are bounded maps in (2) and there is a constant C such that

$$\|u\|_U \leq C(\|Lu\|_F + \|Bu\|_G) \text{ for all } u \in U. \tag{3}$$

For elliptic problems, this usually holds in scales of Sobolev spaces depending on regularity assumptions, but we assume no details here.

3. Spectral Methods and Others

For all variations of spectral and pseudospectral methods [7, 6, 4, 5, 8], the starting point is to write solutions u of (1) as a series expansion

$$u = \sum_{j \in \mathbb{N}} \alpha_j u_j \tag{4}$$

in terms of *trial* functions u_j that in special spectral methods is assumed to be a complete orthonormal system in U. Then

$$\|u\|_U^2 = \sum_{j \in \mathbb{N}} |\alpha_j|^2 < \infty,$$

and an implicit assumption behind all of this is that the $|\alpha_j|$ decay quickly for increasing j.

If such methods were *meshless*, they should express their trial functions "*entirely in terms of values at nodes*" [2].

Also, the orthogonality of the trial functions is not essential at this point. One can think of finite elements as trial functions as well, but then there is no decay of weights. But since at various places we compare expansions, linear independence will be necessary. An extension to frames is open.

Another common feature of spectral methods and others is that they generate conditions on the α_j by *testing* the residuals $Lu - f$ and $Bu - g$ for solution candidates u. This can be carried out in various ways that we describe now.

3.1. Galerkin Methods

Here, the boundary conditions should be homogeneous, and the trial functions should automatically satisfy them. Then one can drop B completely and change the definition of the spaces U and F accordingly to care for boundary conditions.

Galerkin methods assume $U \subset B$ and then they test the residual $Lu - f$ against the u_j themselves, i.e.

$$\begin{aligned} (Lu - f, u_k)_F &= 0, \\ (Lu, u_k)_F &= (f, u_k)_F, \\ \sum_{j \in \mathbb{N}} \alpha_j \underbrace{(Lu_j, u_k)_F}_{=:L_{jk}} &= \underbrace{(f, u_k)_F}_{\phi_k}, \end{aligned}$$

leading to a biinfinite linear system

$$\sum_{j\in\mathbb{N}} \alpha_j L_{jk} \;=\; \phi_k,\ k \in \mathbb{N} \tag{5}$$

that will appear also in other methods to follow below.

Under coercivity assumptions on L one can prove that finite subsystems

$$\sum_{j=1}^{N} \alpha_j L_{jk} = \phi_k,\ 1 \le k \le N$$

are uniquely solvable.

3.2. Tau Methods

Here, the boundary conditions can be general, and the trial functions do not automatically satisfy them. One can take an orthonormal system of functions g_k in G and expand the boundary data g as

$$g = \sum_{k\in\mathbb{N}} \beta_k g_k.$$

Applying the boundary operator implies

$$\begin{aligned} Bu &= g, \\ \sum_{j\in\mathbb{N}} \alpha_j B u_j &= \sum_{k\in\mathbb{N}} \beta_k g_k \end{aligned}$$

and it is reasonable to expand all Bu_j into the g_k as well, i.e.

$$Bu_j = \sum_{k\in\mathbb{N}} B_{jk} g_k$$

to get

$$\sum_{j\in\mathbb{N}} \alpha_j B_{jk} = \beta_k,\ k \in \mathbb{N}. \tag{6}$$

This gives two simultaneous linear systems

$$\begin{aligned} \sum_{j\in\mathbb{N}} \alpha_j L_{jk} &= \phi_k, \\ \sum_{j\in\mathbb{N}} \alpha_j B_{jk} &= \beta_k. \end{aligned} \tag{7}$$

that have to be discretized properly. Unique solvability of finite subsystems now is a nontrivial problem, but if existence of a true solution of the PDE problem is assumed, the full biinfinite system is uniquely solvable.

3.3. Pseudospectral Methods

Here, the residual $Lu - f$ is evaluated at points $x_k \in \Omega$ to arrive at conditions

$$\begin{aligned} (Lu)(x_k) &= f(x_k), \\ \sum_{j\in\mathbb{N}} \alpha_j (Lu_j)(x_k) &= f(x_k), \end{aligned}$$

which results in a system (5) again, but with different coefficients now being defined as

$$L_{jk} = (Lu_j)(x_k), \tag{8}$$
$$\phi_k = f(x_k). \tag{9}$$

This is a *collocation* technique, and one can also collocate the boundary conditions by evaluating at points y_k on the boundary. Then

$$\begin{aligned} (Bu)(y_k) &= g(x_k), \\ \sum_{j\in\mathbb{N}} \alpha_j (Bu_j)(y_k) &= g(x_k), \end{aligned}$$

which results in a system (6) again, but with different coefficients now being defined as

$$B_{jk} = (Bu_j)(y_k), \tag{10}$$
$$\beta_k = g(x_k). \tag{11}$$

3.4. Petrov–Galerkin Methods

Like in the tau method, one can use an orthonormal basis of functions f_k in F to expand

$$f = \sum_{k\in\mathbb{N}} \phi_k f_k$$

and to expand

$$Lu_j = \sum_{k\in\mathbb{N}} L_{j,k} f_k, \ j \in \mathbb{N}.$$

This gives another case of the system (5) again, but with different coefficients defined above by expansion.

3.5. General Methods

All of these techniques can be subsumed into the general strategy of hitting $Lu = f$ with functionals $\lambda_k \in F^*$ and $Bu = g$ with functionals $\mu_k \in G^*$. This yields the combined system (7) again, but with

$$\begin{array}{rcl} \phi_k & := & \lambda_k(f), \\ \beta_k & := & \mu_k(g), \\ L_{jk} & := & \lambda_k(Lu_j), \\ B_{jk} & := & \mu_k(Lu_j). \end{array}$$

These functionals can be chosen to be orthonormal bases in the dual, or just be *total* in the sense that the intersection of their kernels is zero.

Note that all variations of the *Meshless Local Petrov Galerkin* method of S.N. Atluri [1] and his collaborators are subsumed here, if trial functions and test functionals are adequately chosen.

Similarly, extended finite element methods fit into here, and various mixtures of numerical techniques.

But note that the specific choice of functionals will have a strong influence on the properties of the biinfinite system (7), and we shall have to care for that.

3.6. Summary

We now assume a general biinfinite coupled system (7) to be given, and we assume that it is a well–posed rewriting of (1) in terms of certain coefficients. We can mix both parts into one new biinfinite system

$$\sum_{j\in\mathbb{N}} \alpha_j Z_{jk} = \beta_k,\ k \in \mathbb{N} \tag{12}$$

that models (1) and its well–posedness. We write this biinfinite system as

$$Za = b$$

and assume that the well–posedness of (1) is built into the system by

$$\|a\|_2 \ \leq \ C\|Za\|_2 \tag{13}$$

such that Z^{-1} is a bounded linear map $\ell_2 \to \ell_2$.

In most of what follows, we shall not need that Z itself is bounded as a map $\ell_2 \to \ell_2$. If we take the standard basis in both instances of ℓ_2, the numbers Z_{jk} for varying k are the expansion coefficients of Ze_j, thus square summable over k. But for letting Z be continuous as a map $\ell_2 \to \ell_2$, we would need that *all* elements Z_{jk} are square summable.

At this point, it is clear that our assumptions on Z are satisfied in case of Galerkin and Tau methods, if the discretizations there are made via orthonormal systems. For pseudospectral techniques, this also follows if we can rewrite the discretization as one in an orthonormal system. But this is possible if we take the u_j orthonormal in U and the $f_k \in F$ and the $g_k \in G$ to be Newton bases [9] for a positive definite kernel that generates F and G. The functionals λ_k and μ_k should then be the unique orthonormal data functionals associated to the Newton bases. In that case, they exactly generate the right expansion coefficients. Then the pseudospectral method is just a Tau method, implemented for special bases and special functionals.

However, Petrov–Galerkin methods without orthonormal expansions will not directly fit in here, e.g. the variations of MLPG. It will need additional arguments to show that certain biinfinite systems arising from meshless local discretizations have an associated biinfinite matrix that is a map on ℓ_2 to ℓ_2 with a continuous inverse.

4. Discretization in Theory

We assume (12) in the form $Za = b$ to be given, and we want to derive theoretical conditions under which a discretized system

$$\sum_{j \in M \subset \mathbb{N}} \alpha_j Z_{jk} = \beta_k,\ k \in N \subset \mathbb{N} \tag{14}$$

is solvable for two finite subsets N and M of $\mathbb{N}$, at least in the least–squares sense. We use tildes for truncated matrices and vectors throughout, and thus rewrite (14) in matrix form

$$\tilde{Z}\tilde{a} = \tilde{b},\ \tilde{b} \in \mathbb{R}^N,\ \tilde{a} \in \mathbb{R}^M,\ \tilde{Z} \in \mathbb{R}^{N \times M}$$

with the standard notation of A^B for the set of maps $B \to A$.

Lemma 1. *Assume that a biinfinite system (12) is well–posed in the sense of (13) with a fixed constant C. Then for each set $M \subset \mathbb{N}$ there is a set $N \subset \mathbb{N}$ such that the discrete system (14) is well–posed as well, with*

$$\|\tilde{\alpha}\|_2 \leq 2C\|\tilde{Z}\tilde{a}\|_2 \text{ for all } \tilde{\alpha} \in \mathbb{R}^M. \tag{15}$$

Furthermore, the truncated matrix $\tilde{Z}$ has full rank and all singular values are bounded below by $1/(4C^2)$.

Proof. For each $j \in M$ the numbers Z_{jk} for varying k are the square–summable expansion coefficients of Ze_j. Thus we can pick a large set $N \subset \mathbb{N}$ depending on M such that

$$C^2 \sum_{j\in M} \sum_{k\notin N} Z_{jk}^2 < 3/4. \tag{16}$$

Now we take an arbitrary truncated vector $\tilde{a}$ and proceed via

$$\begin{aligned}
\|\tilde{a}\|_2^2 &\leq C^2\|Z\tilde{a}\|_2^2 \\
&= C^2 \sum_{k\in\mathbb{N}} \left(\sum_{j\in M} \tilde{\alpha}_j Z_{jk}\right)^2 \\
&= C^2 \sum_{k\in N} \left(\sum_{j\in M} \tilde{\alpha}_j Z_{jk}\right)^2 + C^2 \sum_{k\notin N} \left(\sum_{j\in M} \tilde{\alpha}_j Z_{jk}\right)^2 \\
&\leq C^2\|\tilde{Z}\tilde{a}\|_2^2 + C^2\|\tilde{a}\|_2^2 \sum_{j\in M}\sum_{k\notin N} Z_{jk}^2
\end{aligned}$$

to prove (15). The matrix $\tilde{Z}$ clearly has full rank, and its singular values are bounded below by $1/(4C^2)$ due to

$$\|\tilde{\alpha}\|_2^2 \leq 4C^2 \tilde{\alpha}^T \tilde{Z}^T \tilde{Z} \tilde{\alpha} \text{ for all } \tilde{\alpha} \in \mathbb{R}^M.$$

Lemma 2. *Assume that the hypotheses of Lemma 1 hold and the system (12) is solvable by some a^*. Then the discrete system (14) has a unique least–squares solution $\tilde{a}$ with the error bound*

$$\|\tilde{a} - \tilde{a}^*\|_2 \leq 4C\|Za^* - Z\tilde{a}^*\|_2$$

where $\tilde{a}^*$ is the truncation of a^*. If ϵ is defined via the choice of M by

$$\epsilon^2 := \sum_{j\notin M} |\alpha_j^*|^2 = \|a^* - \tilde{a}^*\|_2^2 = \|a^*\|_2^2 - \|\tilde{a}^*\|_2^2,$$

then

$$\|\tilde{a} - a^*\|_2 \leq 4C\|Za^* - Z\tilde{a}^*\|_2 + \epsilon$$

holds, where the extension of $\tilde{a}$ by zeros is denoted by $\tilde{a}$ again.

Proof: By (15) and least–squares minimization, we get

$$\begin{aligned}\|\tilde a-\tilde a^*\|_2 &\le 2C\|\tilde Z(\tilde a-\tilde a^*)\|_2\\ &\le 2C\|\tilde Z\tilde a-\tilde b\|_2+2C\|\tilde b-\tilde Z\tilde a^*\|_2\\ &\le 4C\|\tilde b-\tilde Z\tilde a^*\|_2\\ &\le 4C\|b-Z\tilde a^*\|_2\\ &= 4C\|Za^*-Z\tilde a^*\|_2,\end{aligned}$$

and

$$\begin{aligned}\|\tilde a-a^*\|_2 &\le \|\tilde a-\tilde a^*\|_2+\|\tilde a^*-a^*\|_2\\ &\le 4C\|Za^*-Z\tilde a^*\|_2+\epsilon\end{aligned}$$

proceeding like in [11]. □

Note that C is still independent of the discretization.

The quantity $\|Za^*-Z\tilde a^*\|_2$ depends on how well Za^* is approximated the Z–image $Z\tilde a^*$ of the truncation of a^*. In many cases, this has a very good error bound provided by approximation theory, even if Z models derivatives.

In applications with specific expansions into orthonormal systems, choosing a large set M results in an arbitrarily small ϵ, using known results on the rates of approximation by such systems.

If, in addition, Z is continuous, we get

$$\|\tilde a-a^*\|_2 \le (1+4C\|Z\|)\epsilon.$$

5. Discretization in Practice

If confronted with a PDE problem like in section 2, users should postpone choosing a numerical method of section 3. Instead, they should first select basis functions $u_j \in U$ with indices forming a set M such that the true solution u^* can be expected to have a good approximation by these functions. This will later become a selection of columns of Z, but at this point users might not have chosen a method yet, and there is no matrix Z yet. Independent of which method is chosen, the discretized linear system will then be inexactly solvable with small residuals, and the ϵ of the theory in the previous section, though not known exactly, can be expected to be small.

Then a method of section 3 should be chosen, and this choice may be guided by various reasons, in particular computational efficiency. Having chosen a method, one has to choose the equations to set up, i.e. one has to choose the set N. The condition (16) is not available in practice, but users can collect more and more test equations until they find numerically that an inequality like (15) is valid, i.e. the smallest singular value σ_N^2 of $\tilde{Z}$ is positive and acts within the theory like $1/(4C^2)$. The error bound of the previous section then holds with

$$\|\tilde{a} - \tilde{a}^*\|_2 \leq \frac{2\|Z\|}{|\sigma_N|}\epsilon$$

though $\|Z\|$ and ϵ are not explicitly known.

At least, the user can safely calculate the least–squares solution $\tilde{a}$ of the discretized system and then form the approximate solution $\tilde{u}$ with these expansion coefficients. As a replacement for a strict error bound on $u^* - \tilde{u}$, users can then evaluate residuals $L\tilde{u} - f$ and $B\tilde{u} - g$ at fine point sets and thus conclude to have an exact solution $\tilde{u}$ of a PDE problem with small (and roughly known) perturbations in f and g. If the problem is known to be well–posed, users can be satisfied at that point, though they do not know the constant C controlling the well–posedness. The previous section suggests to look at $1/(2|\sigma_N|)$ to get a rough estimate of C.

6. Examples

One of the simplest cases are elliptic problems of the type $Lu = f$ with zero boundary conditions moved into the trial space U, where the operator L has orthonormal eigenfunctions u_j in U with eigenvalues $\lambda_j > 0$ which typically satisfy $\lambda_j \to \infty$ for $j \to \infty$. The problem

$$-u'' = f \in [0, 1], u(0) = u(1) = 0$$

is of this type with $u_j(x) = \sin(\pi j x)$.

The trial function is expanded into (4) and the right–hand side similarly, with coefficients f_k. Then the uniquely solvable infinite linear system is

$$\alpha_j \lambda_j = f_j \text{ for all } j$$

and practical solutions will use a finite subsystem wit indices $j \in M$.

To account for minimal possible regularity, the coefficients should satisfy

$$\|f\|_F := \sum_k \frac{f_k^2}{\lambda_k^2} < \infty$$

and the space F should formally carry the associated inner product. This implies well–posedness with $C_W = 1$ due to $\|u\|_U = \|Lu\|_F$.

The error analysis then is

$$\begin{aligned}\|u^* - \tilde{u}\|_U^2 &= \sum_{j\notin M} \frac{f_j^2}{\lambda_j^2}\\ &= \|f - \tilde{f}\|_F^2\end{aligned}$$

with exactly the same truncation strategy. In this form, the convergence speed for increasing M is depending on the expansion of f, and it is a good idea to use nonlinear approximation in the sense of choosing indices j with large $|f_j|$.

The biinfinite Z matrix of section 4 will be diagonal with the λ_j in the diagonal. Any superset N of M will work, because then the double sums in (16) are always zero. The factor 2 in (15) is not necessary. The technique there, if carried out literally, would lead to

$$\|u^* - \tilde{u}\|_U^2 \leq 4\|f - \tilde{f}\|_F^2.$$

To exemplify a Tau method, we take $\Omega = [-\delta, \delta]$ and pose the problem

$$-u'' = f \in \Omega,\ u(+\delta) = f_1,\ u(-\delta) = f_0$$

there. We assume analyticity and use expansions into power series that we assume to be absolutely convergent in $[-1, +1]$. Starting from

$$f(x) = \sum_{k=0}^{\infty} b_{k+2} x^k,$$

we see that the expansion coefficients of

$$u(x) = \sum_{j=0}^{\infty} a_j x^j$$

satisfy the biinfinite system

$$\begin{aligned}a_{k+2}(k+1)(k+2) &= b_{k+2},\quad \text{for all } k \geq 0,\\ \sum_{j=0}^{\infty} a_j \delta^j &= b_1,\\ \sum_{j=0}^{\infty} a_j (-\delta)^j &= b_0\end{aligned}$$

which is of the form (12) for biinfinite vectors a and b,

The simplest numerical method would be to solve

$$\begin{aligned}
a_{k+2} &= b_{k+2}(k+1)(k+2),\ 0 \le k \le M-2, \\
b_1 &= \sum_{j=0}^{M} a_j \delta^j, \\
b_0 &= \sum_{j=0}^{M} a_j (-\delta)^j,
\end{aligned}$$

using the two final equations to solve for a_0 and a_1.

Of course, one can base a simple error analysis on this toy case, but we want to show that it fits into this paper by proving a well–posedness inequality (13). Clearly.

$$\sum_{k=0}^{\infty} a_{k+2}^2 \le \sum_{k=0}^{\infty} b_{k+2}^2 \le \|b\|_2^2$$

and due to

$$\begin{aligned}
a_1 &= \frac{b_1 - b_0}{2} + \sum_{j\ge 2,\ j \text{ odd}} a_j \delta^j \\
a_0 &= \frac{b_1 + b_0}{2} + \sum_{j\ge 2,\ j \text{ even}} a_j \delta^j \\
\sum_{j=2}^{\infty} |a_j| \delta^j &\le (1-\delta^2)^{-1/2} \left(\sum_{k=0}^{\infty} a_{k+2}^2 \right)^{1/2} \le (1-\delta^2)^{-1/2} \left(\sum_{k=0}^{\infty} b_{k+2}^2 \right)^{1/2}
\end{aligned}$$

we get

$$\begin{aligned}
|a_1| &\le \frac{1}{2}|b_0| + \frac{1}{2}|b_1| + (1-\delta^2)^{-1/2} \left(\sum_{k=0}^{\infty} b_{k+2}^2 \right)^{1/2} \\
&\le \sqrt{b_0^2 + b_1^2} + (1-\delta^2)^{-1/2} \left(\sum_{k=0}^{\infty} b_{k+2}^2 \right)^{1/2} \\
&\le (1-\delta^2)^{-1/2} \|b\|_2
\end{aligned}$$

and the same bound for a_0. This combines into

$$\|a\|_2^2 \le (1 + 2(1-\delta^2)^{-1}) \|b\|_2^2$$

and proves (13).

If, for some $M \geq 2$, we solve for $a_0, \dots, a_M$ only, we get that the above inequalities also hold for the truncated series describing the errors. The error in coefficients then has the bound

$$\sum_{j>M} a_j^2 \quad \leq \quad \sum_{j>M} b_j^2,$$

i.e. the decay rate of the coefficients b_j of f, whatever it is, carries over to the decay rate of the coefficients a_j of u.

The error at $x \in [-\delta, \delta]$ will then be bounded by

$$\sum_{j>M}^{\infty} |a_j||x|^j \quad \leq \quad \delta^{M+1} \sum_{j>M}^{\infty} |b_j|$$

and this is exponentially decaying for $M \to \infty$, proving spectral convergence.

For a pseudospectral technique along this line, a first possibility is to use collocation in the sense

$$u''(x_k) = f(x_k), \ 1 \leq k \leq M-1$$

on points $-\delta \leq x_1 < \dots < x_{M-1} \leq \delta$ together with the two boundary conditions, in order to fix a polynomial $\tilde{u}$ of degree M. The full system is

$$\begin{array}{rcl} f(x_k) & = & \sum_{j=2}^{M} a_j j(j-1) x_k^{j-2}, \ 1 \leq k \leq M-1 \\ u(\delta) & = & \sum_{j=0}^{M-1} a_j \delta^{j-2}, \\ u(-\delta) & = & \sum_{j=0}^{M-1} a_j (-\delta)^{j-2}, \end{array}$$

and the numerical solution is exactly what we had before, because the truncation $\tilde{f}$ of f coincides with the interpolant of degree M to f.

While the numerical procedure is valid and efficient, the theory of this paper does not apply. The problem with pseudospectral methods for PDE problems in strong form is that the infinite problem cannot carry the L_2 norm in the range of Z, because each component is a function value, not an expansion coefficient. This calls for a reformulation of the theory of this paper, allowing the L_∞ norm in the range of Z.

But we can treat the above special case more thoroughly, showing how the general case may be handled. Our trial space consists of polynomials of degree M, and we take $\delta = 1$ now. The conditions

$$\begin{array}{rcl} u(+1) & = & u^+ \\ u(-1) & = & u^- \\ u''(x_k) & = & f(x_k),\ 1 \le k \le N \end{array}$$

for points

$$-1 = x_0 \le x_1 < x_2 < \ldots < x_N \le x_{N+1} = +1$$

will overdetermine a polynomial of degree M, if $N + 1 > M$, but we assume that the true solution u^* has a good polynomial approximation $\hat{u}$ of degree M that will satisfy all conditions approximately. Thus there is some polynomial $\tilde{u}$ of degree M that solves the linear optimization problem

$$\begin{array}{rclcl} & & \text{Minimize } \epsilon & & \\ -\epsilon & \le & u^+ - \tilde{u}(+1) & \le & \epsilon \\ -\epsilon & \le & u^- - \tilde{u}(-1) & \le & \epsilon \\ -\epsilon & \le & f(x_k) - \tilde{u}''(x_k) & \le & \epsilon, \end{array}$$

and since $\hat{u}$ is admissible, it satisfies the above conditions with some $\hat{\epsilon}$ that is not smaller than the $\tilde{\epsilon}$ for $\tilde{u}$.

The overall analytic problem is easily proven to be well-posed in the sense that

$$|u(x)| \le C \left(|u(-1)| + |u(+1)| + \|u''\|_{\infty.[-1,+1]}\right)$$

holds for all $u \in C^2[-1, +1]$. This implies

$$|(\tilde{u} - u^*)(x)| \le C \left(2\hat{\epsilon} + \|(\tilde{u} - u^*)''\|_{\infty.[-1,+1]}\right),$$

but we need a bound on $\|(\tilde{u} - u^*)''\|_{\infty.[-1,+1]}$. If $N \ge M$, we know that for all polynomials p of degree $M - 2$ we have a bound

$$\|p\|_{\infty.[-1,+1]} \le S(M, N) \max_{1 \le k \le N} |p(x_k)|$$

with a certain *stability* constant $S(M, N)$ that we shall consider later. We insert it and get

$$\begin{array}{rcl} |(\tilde{u} - u^*)(x)| & \le & C \left(2\hat{\epsilon} + S(M, N) \max\limits_{1 \le k \le N} |(\tilde{u} - u^*)''(x_k)|\right) \\ & \le & \hat{\epsilon}\, C\, (2 + S(M, N)). \end{array}$$

Depending on the smoothness of u^*, we can use standard results of Approximation Theory to get small bounds on $\hat{\epsilon}$. If we ignore the constant $C(M,N)$, this would mean the the L_∞ error of the boundary value problem has the same decay rate when M increases as the L_∞ approximation error for second derivatives of u^*.

But it is well-known that for equidistant points x_k and $N = M$ the constant $C(M,M)$ will grow exponentially with M, while for $M = N$ and Chebyshev points it still grows with $\log M$. This is not too bad, but still not sufficient to keep the approximation quality of second derivatives. For Chebyshev points, this requires oversampling with $N \geq \lceil 2\pi M \rceil$, while the equidistant case even requires $N = \mathcal{O}(M^2)$.

7. Conclusions

Well-posed spectral methods that can be written as biinfinite systems have uniformly stable discretizations obtainable by choosing finite subsystems. Convergence rates are then played back to approximation errors committed by truncation of expansions of the true solutions. Some examples show that the theory is applicable in various situations, but it is left open how large the systems must be to be uniformly stable.

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