Convergence Analysis of Methods for Solving General Equations

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Abstract

The unsymmetric collocation method by E. Kansa has been very successfully used in many applications, though there theoretically exist rare situations in which it will fail. This contribution modifies the method somewhat and then proves convergence and error bounds for the modified technique. The results will be presented within a general framework for methods that solve operator equations by minimizing residuals. Thus the modifications may also help to put other methods on a solid foundation.

1 Linear Operator Equations

We consider a linear problem

$$D(u) = w \tag{1}$$

which has a solution u that has to be recovered from data $w = D(u)$ under a linear mapping D. This setting covers many differential or integral equation problems in weak or strong form. For convenience, we restrict ourselves here to examples of problems in strong form. In case of a Poisson boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

the data $w = D(u)$ for a *strong* problem formulation will be

$$w = (f, \varphi) = D(u) := (-\Delta, Id_{\partial\Omega})(u).$$

The data $D(u)$ should depend linearly and continuously on the solution u, which means that there is a bound of the form

$$\|D(u)\|_W \leq \|D\| \cdot \|u\|_U \tag{3}$$

if U and W are suitably normed spaces.

Solving the operator equation (1) means inversion of $D : U \to W$ on its range W. We assume further that the problem is well–posed, i.e. uniquely and stably solvable. This means that the inverse D^{-1} of D is a linear and bounded map $D^{-1} : W \to U$. Then there is an *analytic a-priori inequality*

$$\|u\|_U \leq \|D^{-1}\| \cdot \|D(u)\|_W = \|D^{-1}\| \cdot \|w\|_W \tag{4}$$

for the solutions u of problems $D(u) = w$ of (1). Such bounds are derived in the literature (see e.g. [3] and examples in [4]). We call $\|D^{-1}\|$ the *analytic constant* for convenience, because the constant is a matter of mathematical analysis and independent of numerical techniques. It arises for every well–posed linear problem, but it is only rarely known exactly. It bounds the possible stability of any conceivable solution algorithm. In the following we shall assume that we have an upper bound C_a for the analytic constant, being enough for our purposes. Furthermore, we weaken the above requirement somewhat by going into a larger space $\mathcal{U} \supseteq U$ into which U can be continuously embedded via

$$\|u\|_{\mathcal{U}} \leq C_U \|u\|_U \text{ for all } u \in U.$$

In applications, the norm $\|.\|_{\mathcal{U}}$ will just be the L_2 or L_∞ norm. Then we use

$$\|u\|_{\mathcal{U}} \leq C_U \|D^{-1}\| \|D(u)\|_W \leq C_U C_a \|D(u)\|_W = C_U C_a \cdot \|w\|_W \tag{5}$$

instead of (4). We call a linear operator equation (1) *well-posed and stably solvable*, if the assumptions made in this section are satisfied. So far we have not mentioned any numerical algorithm, and we did not distinguish between different problem formulations, e.g. strong and weak formulations of linear PDE problems.

2 Minimizing Residuals

If some numerical method has produced an approximate solution $\tilde{u}$ to the problem (1), one can calculate the data $\tilde{w} = D(\tilde{u})$ and get the *a-posteriori* error bound

$$\|u - \tilde{u}\|_{\mathcal{U}} \leq C_U C_a \cdot \|D(u) - D(\tilde{u})\|_W = C_U C_a \cdot \|w - \tilde{w}\|_W \qquad (6)$$

for free, since the *residual* $w - \tilde{w}$ is explicitly known. It means that errors in the solution are bounded by the norm of the residuals, multiplied with the analytic constant.

Theorem 1 *Let a well-posed and stably solvable linear problem be given. Any numerical technique that produces approximate solutions with small residuals will automatically guarantee small errors in the solution, the blow–up factor being the analytic constant.*

□

We now look at techniques that construct approximate solutions $\tilde{u}_h$ from some *trial space* $U_h \subseteq U \subseteq \mathcal{U}$. Note that this still includes plenty of methods, with or without meshes, like finite elements, Petrov-Galerkin schemes, spectral methods, and unsymmetric collocation. It is trivial that the choice of the trial space should be such that the true solution u can be approximated easily by functions from the trial space. In case of solutions with singularities, like for Poisson problems on domains with incoming vertices, one should make sure that the trial space should contain the expected singular functions.

Since we know from (6) that small residuals will lead to small solution errors, a straightforward choice of method would be to solve the *residual minimization problem*

$$\tilde{u}_h = \arg\min\{\|w - D(u_h)\|_W \ : \ u_h \in U_h\}. \qquad (7)$$

Theorem 2 *Residual minimization techniques guarantee asymptotically optimal approximation orders for the selected trial spaces. This means that they achieve the error of the best approximation to the solution from the trial space, up to a fixed multiplicative constant.*

Proof: Let $\hat{u}_h$ be the optimal approximation to the true solution u in the trial space, i.e.

$$\hat{u}_h = \arg\min\{\|u - u_h\|_U \ : \ u_h \in U_h\}.$$

Then

$$\begin{aligned}
\|u-\tilde{u}_h\|_U &\leq \|u-\hat{u}_h\|_U + \|\hat{u}_h-\tilde{u}_h\|_U \\
&\leq \|u-\hat{u}_h\|_U + C_a\|D(\hat{u}_h)-D(\tilde{u}_h)\|_W \\
&\leq \|u-\hat{u}_h\|_U + C_a\|w-D(\hat{u}_h)\|_W + C_a\|w-D(\tilde{u}_h)\|_W \\
&\leq \|u-\hat{u}_h\|_U + 2C_a\|w-D(\hat{u}_h)\|_W \\
&\leq (1+2C_a\|D\|)\|u-\hat{u}_h\|_U.
\end{aligned}$$

□

The above proof never used that $\hat{u}_h$ is in fact the optimal approximation. Thus we also have

Theorem 3 *If for each $u \in U$ there is some $\hat{u}_h \in U_h$ with a small value of $\|u-\hat{u}_h\|_U$, then the residual minimization technique will have at most the error*

$$\|u-\tilde{u}_h\|_U \leq (1+2C_a\|D\|)\|u-\hat{u}_h\|_U.$$

This is independent of how $\hat{u}_h$ is defined or constructed.

□

The above theorem allows to separate the operator equation (1) from the technique that provides $\hat{u}_h \in U_h$ for each $u \in U$. The latter can be some plain interpolation or approximation process, using data of u which are quite different from the data $D(u)$ necessary to solve (1).

We can get even more if we can approximate the data well on U:

Theorem 4 *If for each $u \in U$ there is some $\hat{u}_h \in U_h$ with an error bound of the form*

$$\|D(u)-D(\hat{u}_h)\|_W \leq C(h)\|u\|_U \text{ for all } u \in U, \tag{8}$$

then the residual minimization technique will have at most the error

$$\|u-\tilde{u}_h\|_{\mathcal{U}} \leq C_U C_a C(h)\|u\|_U.$$

Proof: From (6), we directly get

$$\begin{aligned}
\|u-\tilde{u}_h\|_{\mathcal{U}} &\leq C_U C_a\|D(u)-D(\tilde{u}_h)\|_W \\
&\leq C_U C_a\|D(u)-D(\hat{u}_h)\|_W \\
&\leq C_U C_a C(u)\|u\|_U.
\end{aligned}$$

□

This is again independent of how $\hat{u}_h$ is defined or constructed. The error bound (8), which only uses the data map of the operator equation (1), carries over to the error of the approximate solution $\tilde{u}_h$ of the operator equation, up to a fixed factor. Bounds of this type are available in the literature [7].

The methods described here will usually take the form of an optimization on the finite-dimensional trial space U_h. The type of optimization depends on the norm chosen for the residuals. If residuals are normed by $\|.\|_\infty$, the problem turns into a semi-infinite linear programming problem. If the L_2 norm is chosen for the residuals, the result is a least-squares optimization. Details are in [4].

Note that we did not assume an analytical optimization problem here, as is usually assumed for elliptic PDE problems, in which the solution minimizes a certain quadratic form.

Error bounds providig information on the accuracy of first and second order derivatives for linear systems can be obtained by specializing the spaces U and $\mathcal{U}$ and their norms appropriately. For instance, trial spaces generated from smooth radial basis functions will always imply error bounds and convergence of higher derivatives. Typical error bounds are in [7].

3 Test Spaces

The counterpart of the trial space U_h of the previous section is a *test space*. But we do not use a space of functions or solutions here. Neither do we follow the lines of [6] to introduce test functionals, but we keep quite close to that paper. We view the linear data map D of the first section as infinitely many linear conditions $D(u) = w$ which the solution u has to satisfy. A "test" is defined by a subset of these conditions, and thus by a linear map D_h which maps U onto a finite-dimensional space W_h called *test space* in this context. The true solution u satisfies the linear equations described by $D_h u =: w_h$. Thus w_h is a vector of known discrete data.

In the standard situation, discrete data get "dense" for $h \to 0$. Thus we postulate an error bound of the form

$$\|v\|_{\mathcal{U}} \le \delta(h)\|v\|_U \text{ for all } v \in U \text{ with } D_h(v) = 0. \tag{9}$$

New results [5] for functions with zeros in Sobolev spaces provide such bounds. This is independent of the trial space, it just depends on the data and the test space.

Many methods then construct a function $\tilde{u}_h \in U_h$ such that

$$D_h \tilde{u}_h = w_h = D_h u,$$

i.e. a function $\tilde{u}_h$ in the trial space U_h is constructed to satisfy only part of the necessary conditions for the full solution. Such methods include collocation and Petrov-Galerkin techniques, with various possibilities for trial and

test spaces. To provide error bounds, one must then bound $u - \tilde{u}_h$ using the equation $D_h(u - \tilde{u}_h) = 0$. If no additional information is available, this will fail, because there will usually be nonzero solutions $v \in U$ of the homogeneous equation $D_h(v) = 0$ which spoil the error bound, since there is no way to prevent $\tilde{u}_h$ being closer to e.g. $u + 10^{12} \cdot v$ than to u.

There are various possibilities for escape here. The previous section always used the full data $D_h = D$ and the well-posedness of the analytical problem. Then there is no nonzero function v with $D_h(v) = D(v) = 0$. This comes at the price of having to solve an optimization problem on the *full* data.

Another way out is to use (9) to get

$$\|u - \tilde{u}_h\|_{\mathcal{U}} \leq \delta(h)\|u - \tilde{u}_h\|_U$$

because of $D_h(u - \tilde{u}_h) = 0$. This does not help, unless we impose additional conditions which ensure a bound like

$$\|\tilde{u}_h\|_U \leq K\|u\|_U \tag{10}$$

on the norm of all approximate solutions. Then we would arrive at

$$\|u - \tilde{u}_h\|_{\mathcal{U}} \leq \delta(h)(1 + K)\|u\|_U$$

which now provides a convergence rate which is determined by approximation properties of the *test* space. Note that Theorem 4 has a convergence rate which depends on the *trial* space.

4 Kansa's Method

Let us now use a trial space U_h spanned by translates of a radial basis function ϕ, where the centers $x_1, \ldots, x_N$ of translation lie in a bounded domain $\Omega \subset \mathbb{R}^d$ such that we have a small *fill distance*

$$h := \sup_{y \in \Omega} \min_{1 \leq j \leq N} \|y - x_j\|_2.$$

This is the trial space used by Kansa's method [1, 2]. Then there always are error bounds of the form (8), where U is the native Hilbert space for ϕ or a suitable Sobolev space, and where $\hat{u}_h$ is the interpolant to u on the centers $x_1, \ldots, x_N$ using only the data $u(x_1), \ldots, u(x_N)$. The factor $C(h)$ will behave like a positive power of h or even like $\exp(-c/h)$, depending on the smoothness of ϕ. See details in the recent book [7].

This is independent of the operator equation (1) for which the trial space may eventually be used. By Theorem 4, the error bound carries over to the solution of every well–posed linear problem, if residual minimization is applied as a numerical technique using the Kansa trial space. Two such techniques, based on L_∞ and L_2 residual minimization, respectively, are described in [4].

However, the original technique of Kansa does *not* minimize residuals. It uses a finite discrete subset of the problem data $D(u)$ and collocates on U_h. This means that it makes a finite subset of residuals equal to zero, not caring for the rest of the residuals. It is a special instance of linear methods that replace a set of infinitely many linear conditions by a finite subset. Thus it falls into the class of methods of the previous section. The map D_h just takes a subset of the data, and we always have $\|D_h(u)\|_\infty \leq \|D(u)\|_\infty$.

There, we suggested to use a second technique for proving convergence. It requires to keep a bound of the form (10) on the Kansa-type solution $\tilde{u}_h$. An easy way to have an a-posteriori check on convergence would be to monitor the norm $\|\tilde{u}_h\|_U$ while performing a series of calculations for decreasing h. If the norm stays bounded, the considerations of the previous section will guarantee convergence. Thus we can prove convergence of a "monitored" Kansa technique:

Theorem 5 *If the traditional Kansa technique is carried out without change, and if it provides approximate solutions $\tilde{u}_h$ whose norms $\|\tilde{u}_h\|_U$ are bounded above independent of h if the collocation data get dense for $h \to 0$, the process will be convergent.* □

If the norms get large, one has to modify the method. The linear equations $D_h(\tilde{u}_h) = D_h(u) = w_h$ should be posed in a larger Kansa-type trial space U_h with additional degrees of freedom that can be used to keep the norm of the approximant in the larger space at bay. For instance, one could minimize $\|\tilde{u}_h\|_U^2$ under linear restrictions

$$\|D_h(\tilde{u}_h) - w_h\|_\infty \leq \kappa(h), \tag{11}$$

arriving at a quadratic discrete optimization problem on U_h. If $\kappa(h)$ is not too small, the unknown interpolant u_h to the true solution u at the Kansa centers will lie in this space. This follows from

$$\begin{aligned} \|D_h(u_h) - w_h\|_\infty &= \|D_h(u_h) - D_h(u)\|_\infty \\ &\leq \|D(u_h) - D(u)\|_\infty \\ &\leq \lambda(h, D, U)\|u_h - u\|_U \\ &\leq \kappa(h) \end{aligned}$$

if $\kappa(h) \geq \lambda(h, D, U)\|u_h - u\|_U$. Error bounds of the required form

$$\|D(u_h) - D(u)\|_\infty \leq \lambda(h, D, U)\|u_h - u\|_U$$

which are similar to (8), are available in the literature on radial basis functions [7]. Now, if the above assumption on $\kappa(h)$ is satisfied, the minimization problem and the minimum-norm property of the standard interpolant imply

$$\|\tilde{u}_h\|_U^2 \leq \|u_h\|_U^2 \leq \|u\|_U^2$$

which is the required bound for $\|\tilde{u}_h\|_U^2$.

Theorem 6 *Assume that* (9) *can be generalized to*

$$\|v\|_{\mathcal{U}} \leq \delta(h)\|v\|_U + \lambda(h)\|D_h(v)\|_\infty \text{ for all } v \in U. \tag{12}$$

If the Kansa technique is modified into a quadratic optimization problem along the lines described above, the method is convergent for $h \to 0$. An error bound is provided by

$$\|u - \tilde{u}_h\|_{\mathcal{U}} \leq \delta(h)\|u - \tilde{u}_h\|_U + \lambda(h)\kappa(h).$$

Proof: We use the above inequality for $u - \tilde{u}_h$ to get

$$\begin{aligned} \|u - \tilde{u}_h\|_{\mathcal{U}} &\leq \delta(h)\|u - \tilde{u}_h\|_U + \lambda(h)\|D_h(u - \tilde{u}_h)\|_\infty \\ &\leq \delta(h)\|u - \tilde{u}_h\|_U + \lambda(h)\kappa(h). \end{aligned}$$

□

The error bound (12) can be derived similarly to (9) following the lines of [5]. This will be contained in a follow-up to the cited paper.

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