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# Convergence Order Estimates of Meshless Collocation Methods using Radial Basis Functions

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We study meshless collocation methods using radial basis functions to approximate regular solutions of systems of equations with linear differential or integral operators. Our method can be interpreted as one of the emerging meshless methods, cf. [1]. Its range of application is not confined to elliptic problems. However, the application to the boundary value problem for an elliptic operator, connected with an integral equation, is given as an example. Although the method has been used for special cases for about ten years, cf. [11], there are no error bounds known. We put the main emphasis on detailed proofs of such error bounds, following the general outline described in [6].

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## 1 Introduction

We will treat systems of linear equations, each of the form

$$Lu = f \quad \text{on} \quad \Omega, \tag{1.1}$$

where $\Omega$ is a domain in $\mathbb{R}^d$ and $L$ is a linear operator, acting on complex valued functions on $\Omega$. The function $f$ is given ("data") and each equation should have a nonempty space of solutions.

## 1.1 Collocation Method for Single Equations

Collocation is a well-known method to approximate the solution of equations of the form (1.1). The idea is to approximate the requested solution $u$ by a function

$$s_u = \sum_{j=1}^{n} \Phi_j \, \alpha_j \in \text{span}\,\{\Phi_j\}_{j=1,\ldots,n} \,, \tag{1.1.1}$$

where the $\Phi_j : \Omega \to \mathbb{C}$ are certain linearly independent *basis functions*. The connection between $u$ and $s_u$ shall be the pointwise equality

$$(Ls_u)(x) = (Lu)(x) = f(x) \quad \text{for all } x \in X_\Omega$$

or in short form

$$(Ls_u)(X_\Omega) = f(X_\Omega) \tag{1.1.2}$$

on a finite, ordered set $X_\Omega \subset \Omega$ of *centers*. Since $X_\Omega$ is ordered, $f(X_\Omega)$ is a vector.

The set (1.1.2) of generalized *interpolation conditions* leads to the linear *interpolation system*

$$\Big( (L\Phi_j)(x) \Big)_{x \in X_\Omega,\, j=1,\ldots,n} \cdot \alpha = f(X_\Omega) \,, \tag{1.1.3}$$

using (1.1.1) for collocation. Now there are two main questions:

- Does the inverse of the *interpolation matrix* $\Big( (L\Phi_j)(x) \Big)_{x \in X_\Omega,\, j=1,\ldots,n}$ exist?
- Are there any upper bounds for $\|s_u - u\|$ or $\|Ls_u - Lu\|$ ?

Both questions will be answered here, with focus on the last one.

## 1.2 Extension to Systems of Linear Equations

Just a few modifications are necessary to treat a *system* of linear equations by collocation. Let $L_\nu$ be a linear operator, acting on complex valued functions defined on $\Omega_\nu$. Let

$$\bigcup_{\nu=1}^{N} \Omega_\nu = \overline{\Omega} \tag{1.2.1}$$

be a decomposition of the closure $\overline{\Omega}$ of $\Omega$. Overlap is permitted. We suppose the system

$$L_\nu u = f_\nu \quad \text{on} \quad \Omega_\nu, \; \nu = 1, \ldots, N \tag{1.2.2}$$

to have a solution for the given set $\{f_\nu\}$ of data functions. We shall see that there is no essential difference between problems with $N = 2$ or $N > 2$. A typical example of the first case is the *boundary value problem*

$$\begin{array}{rcl} L\,u &=& f_\Omega \quad \text{on} \quad \Omega_1 := \Omega \\ B\,u &=& f_{\partial\Omega} \quad \text{on} \quad \Omega_2 := \partial\Omega \end{array} . \tag{1.2.3}$$

We mainly focus on systems of this form, comprising several operators of different type. In general, we do not need to restrict the types of operators involved; in particular we are not confined to elliptic boundary value problems.

Let $X_{\overline{\Omega}}$ denote the collection of all ordered sets $X_{\Omega_\nu}$, for $\nu = 1, \ldots, N$. We subsequently can allow a point $x$ to appear in several of the $X_{\Omega_\nu}$, such that different collocation conditions $L_\nu u(x) = f_\nu(x)$ can be imposed at a single point. Now the interpolation system (1.1.3) takes the form of the $N$ simultaneous equation systems

$$\left( (L_\nu \Phi_j)(X_{\Omega_\nu}) \right)_{j=1,\ldots,n} \cdot \alpha \;=\; f_\nu(X_{\Omega_\nu}) \quad \text{for } \nu = 1, \ldots, N\,. \tag{1.2.4}$$

Several ideas exist to choose the basis functions $\Phi_j$, $j = 1, \ldots, n$ depending on $X_{\overline{\Omega}}$ and the operators $L_\nu$, $\nu = 1, \ldots, N$. The theory of interpolation by radial basis functions has good reasons to let every $\Phi_j$ depend on a center $x_j \in X_{\overline{\Omega}}$. Kansa had numerical success using no dependence on the $L_\nu$ and using multiquadrics for $\Phi_j$ (cf. [11]). But since the resulting matrix is not symmetric, no one has been able to prove its nonsingularity so far. Several authors (cf. [8, 19] and [4]) suggest the following method, which lets the basis functions also depend on the operators. It yields a symmetric interpolation matrix at the expense of a second application of the linear operators to the basis functions. This method will be used here.

Let $\Phi(x, y)$ be a *feasible basis function* (see the precise definition in section 2). Then we define the set of basis functions via

$$\Phi_j(x) \;:=\; \delta^y_{x_j} \circ \overline{L_\nu}^y \Phi(x, y) \quad \text{for } x_j \in X_{\Omega_\nu};\; \nu = 1, \ldots, N\,.$$

For a fixed $x$, $\delta_x$ denotes the linear functional mapping a function $f$ to $f(x)$. For a given map $T$, the complex conjugate map $\overline{T}$ is defined by $\overline{T}(f) := \overline{T(\overline{f})}$, and the superscript $y$ of $T^y$ denotes the application of $T$ to $\Phi(x, y)$ with respect to the variable $y$.

To simplify the notation, and to connect the theory with *Hermite-Birkhoff interpolation*, we define the functionals

$$\lambda_j \;:=\; \delta_{x_j} \circ L_\nu \quad \text{for } x_j \in X_{\Omega_\nu};\; \nu = 1, \ldots, N\,,$$

and denote the ordered set of all of these by

$$\Lambda := \left(\delta_{X_{\Omega_\nu}} \circ L_\nu\right)_{\nu=1,\dots,N} . \tag{1.2.5}$$

Next we define the vector on the right hand side of the interpolation system (1.2.4) to be $z := (\, f_\nu(X_{\Omega_\nu})\,)_{\nu=1,\dots,N}$. Now this system takes the simple form

$$\Lambda^x \overline{\Lambda}^y \Phi(x,y) \cdot \alpha \;=\; z \, . \tag{1.2.6}$$

Since $\Phi$ is semisymmetric, the interpolation matrix $\Lambda^x \overline{\Lambda}^y \Phi(x,y)$ is Hermitian. In [19], Theorem 3.1, it is shown to be positive definite under reasonable conditions, and then it is invertible. We will reuse parts of the proof of the abovementioned theorem in an extended context. To express the dependence of the approximating function $s_u$ on $\Lambda$ and $\Phi$, we subsequently denote it by $s_{u,\Lambda,\Phi}$. It takes the form

$$s_{u,\Lambda,\Phi}(x) \;=\; \sum_{\nu=1}^{N} \sum_{x_j \in X_{\Omega_\nu}} \left(\delta_{x_j} \circ \overline{L_\nu}\right)^y \Phi(x,y)\, \alpha_j$$

or short

$$s_{u,\Lambda,\Phi}(x) \;=\; \overline{\Lambda}^y \Phi(x,y) \cdot \alpha \, . \tag{1.2.7}$$

Here, $\overline{\Lambda}^y \Phi(x,y)$ is regarded as a $1 \times n$ matrix of functions in $x$.

## 2 Error Analysis and Native Spaces

Let $\mathcal{L}$ be a linear space of complex valued functionals, being defined for functions on $\Omega$. For example

$$\mathcal{L} \;:=\; \text{span}\left(\{\,\delta_x \circ L\,\}_{x\in\Omega} \cup \{\,\delta_x \circ B\,\}_{x\in\partial\Omega}\right)$$

is useful in the case of the boundary value problem (1.2.3).

**Definition 2.1** *(Feasible Basis Functions and Native Spaces) A* feasible basis function *with respect to $\mathcal{L}$ is a function $\Phi : \Omega \times \Omega \to \mathbb{C}$, being*

- semisymmetric: $\Phi(x,y) \;=\; \overline{\Phi(y,x)}$ *for all $x, y \in \Omega$*
- positive definite: *for every finite, nonempty, linearly independent ordered set $\Lambda \subset \mathcal{L}$, $\Lambda \neq 0$, the matrix $\Lambda^x \overline{\Lambda}^y \Phi(x,y)$ is positive definite and*
- sufficiently smooth *to apply two functionals $\lambda, \mu$ of $\mathcal{L}$ to each variable and to let the functionals commute: $\lambda^x \mu^y \, \Phi(x,y) = \mu^y \lambda^x \, \Phi(x,y)$.*

*For any feasible basis function* $\Phi$*, the term*

$$\langle \lambda \,|\, \mu \rangle_{\mathcal{F}_\Phi^*} \;:=\; \lambda^x \,\overline{\mu}^y \,\Phi(x,y) \tag{2.1}$$

*defines a scalar product on the* dual native space

$$\mathcal{F}_\Phi^* \;:=\; \left\{ \lambda : \; \langle \lambda \,|\, \lambda \rangle_{\mathcal{F}_\Phi^*} < \infty \right\} \quad \textit{with} \quad \mathcal{L} \subseteq \mathcal{F}_\Phi^* \; . \tag{2.2}$$

*Subsequently it is assumed to be a* Hilbert space*. The dual* $\mathcal{F}_\Phi^{**}$ *of* $\mathcal{F}_\Phi^*$ *is identified with the* native space

$$\mathcal{F}_\Phi \;:=\; \left\{ f : \; \|f\|_{\mathcal{F}_\Phi} < \infty \right\} \tag{2.3}$$

*of (generalized) functions* $f$ *which allow to be evaluated by all* $\lambda \in \mathcal{F}_\Phi^*$*. The norm is defined by*

$$\|f\|_{\mathcal{F}_\Phi} \;:=\; \sup_{\lambda \in \mathcal{F}_\Phi^*, \lambda \neq 0} \frac{|\lambda(f)|}{\|\lambda\|_{\mathcal{F}_\Phi^*}} \; . \tag{2.4}$$

*Since* $\mathcal{F}_\Phi^*$ *is a Hilbert space, so is* $\mathcal{F}_\Phi$*.*

**Remark 2.2** *The term* $\lambda^x \overline{\mu}^y \Phi(x,y)$ *in equation (2.1) defines a scalar product, since it is sesquilinear and* $\Phi$ *is assumed to be positive definite. Methods for proving the positive definiteness and examples are given in section 3.*

**Remark 2.3** *We assumed* $\mathcal{F}_\Phi^*$ *to be a Hilbert space, i.e. it has to be complete. The construction of the maximal dual native space to a given class* $\mathcal{L}$ *of functionals and a basis function having a Fourier transform is described in [10]. In addition,* conditionally positive definite *basis functions are allowed there.*

**Remark 2.4** *The approximating function* $s_{u,\Lambda,\Phi}$ *is an element of* $\mathcal{F}_\Phi$*, since we have* $\mu(s_{u,\Lambda,\Phi}) = \langle \mu \,|\, \Lambda \cdot \overline{\alpha} \rangle_{\mathcal{F}_\Phi^*}$ *for every* $\mu \in \mathcal{F}_\Phi^*$*.*

**Theorem 2.5** *Let* $\Phi$ *be a feasible basis function,* $\Lambda \subset \mathcal{F}_\Phi^*$ *be linearly independent and* $u \in \mathcal{F}_\Phi$*. Let* $s_{u,\Lambda,\Phi} \in \text{span}\, \overline{\Lambda}^y \Phi(\cdot, y)$ *be the* reconstruction *of* $u$*, i.e.* $\Lambda(s_{u,\Lambda,\Phi}) = \Lambda(u)$*. Then the bound*

$$|\lambda(u) - \lambda(s_{u,\Lambda,\Phi})| \;\leq\; P_{\Phi,\Lambda}(\lambda) \cdot \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} \tag{2.5}$$

*for the* reconstruction error $|\lambda(u) - \lambda(s_{u,\Lambda,\Phi})|$ *holds for every* $\lambda \in \mathcal{F}_\Phi^*$*, where*

$$P_{\Phi,\Lambda}(\lambda) \;:=\; \inf_{\mu \in \text{span}\, \Lambda} \|\lambda - \mu\|_{\mathcal{F}_\Phi^*} \tag{2.6}$$

*is called the* power function *of* $\lambda$*. Optimization theory yields*

$$\|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} \;=\; \inf_{s \in \text{span}\, \overline{\Lambda}^y \Phi(\cdot,y)} \|u - s\|_{\mathcal{F}_\Phi} \; , \tag{2.7}$$

*cf. [20], Theorem 1 and [13], Theorem 4.1.*

**Proof.** Since we have $\Lambda(s_{u,\Lambda,\Phi}) = \Lambda(u)$, we know $\mu\,(u - s_{u,\Lambda,\Phi}) = 0$ for all $\mu \in \text{span}\ \Lambda$. Therefore we can estimate

$$|\lambda(u) - \lambda(s_{u,\Lambda,\Phi})| \;=\; |(\lambda-\mu)(u - s_{u,\Lambda,\Phi})| \;\leq\; \|\lambda-\mu\|_{\mathcal{F}_\Phi^*} \cdot \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} \;.$$

■

The equations (2.5)–(2.7) show how the influences of a test functional $\lambda$ and of the function $u$ on the error are separable. This separation results in two approximation errors: The power function is the error of approximation of $\lambda$ in span $\Lambda$, while the second factor is the error concerning $u$ and span $\overline{\Lambda}^y\Phi(\cdot,y)$. This factor is independent of $\lambda$.

To adapt the previous theorem to systems of linear equations, we need some information about the behaviour of the power function concerning ordered subsets $\Lambda'$ of $\Lambda$, which contain functionals of the same type.

**Theorem 2.6** (Splitting Theorem) *The inclusion $\Lambda' \subseteq \Lambda \subseteq \mathcal{F}_\Phi^*$ implies*

$$\begin{array}{rrcll} & P_{\Phi,\Lambda}(\lambda) & \leq & P_{\Phi,\Lambda'}(\lambda) & \textit{for all } \lambda \in \mathcal{F}_\Phi^* \\ \textit{and} & \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} & \leq & \|u - s_{u,\Lambda',\Phi}\|_{\mathcal{F}_\Phi} & \textit{for all } u \in \mathcal{F}_\Phi \;. \end{array}$$

**Proof.** $\Lambda' \subseteq \Lambda$ implies span $\Lambda' \subseteq$ span $\Lambda$. Therefore we find

$$\inf_{\mu \in \text{span}\ \Lambda'} \|\lambda-\mu\|_{\mathcal{F}_\Phi^*} \;\geq\; \inf_{\mu \in \text{span}\ \Lambda} \|\lambda-\mu\|_{\mathcal{F}_\Phi^*}$$

for the infima. The second assertion is proved analogously, using equation (2.7).

■

The Splitting Theorem allows us to focus our attention on just one $\Omega_\nu$ (denoted by $\Omega$ to save subscripts), its centers $X := X_{\Omega_\nu}$ and its operator $L := L_\nu$, as they are given in section 1.2.

**Theorem 2.7** (Transformation Theorem) *Let $\Phi(x,y)$ be a feasible basis function. Let the linear functionals $(\delta_z \circ L)^x$ and $\mu^y$ applied to $\overline{L}^y\Phi(x,y)$ commute for all $z \in \Omega$ and $\mu \in \mathcal{F}_\Phi^*$. If*

$$\Psi_L(x,y) \;:=\; (\delta_x \circ L)^u\,(\delta_y \circ \overline{L})^v\,\Phi(u,v) \tag{2.8}$$

*is also a feasible basis function, and $\mathcal{F}_{\Psi_L}^*$, $\mathcal{F}_{\Psi_L}$ and $P_{\Psi_L,\Lambda}$ are defined for it according to definition 2.1 and equation (2.6), then*

$$\|\lambda\|_{\mathcal{F}_{\Psi_L}^*} \;=\; \|\lambda \circ L\|_{\mathcal{F}_\Phi^*} \tag{2.9}$$

*for all $\lambda \in \mathcal{F}^*_{\Psi_L}$. The equation*

$$P_{\Psi_L,\Lambda}(\lambda) \;=\; P_{\Phi,\Lambda\circ L}(\lambda \circ L) \tag{2.10}$$

*holds for every finite $\Lambda \subset \mathcal{F}^*_{\Psi_L}$ and $\lambda \in \mathcal{F}^*_{\Psi_L}$.*

*If the operator $L : \mathcal{F}_\Phi \to \mathcal{F}_\Phi$ has a norm $\|L\|_\Phi < \infty$, and if all involved functionals $\lambda$ and $\Lambda$ are in $\mathcal{F}^*_\Phi \cap \mathcal{F}^*_{\Psi_L}$, then*

$$P_{\Phi,\Lambda\circ L}(\lambda \circ L) \;\le\; P_{\Phi,\Lambda}(\lambda) \cdot \|L\|_\Phi \;. \tag{2.11}$$

*But note that in general, differential operators will not map the native space $\mathcal{F}_\Phi$ into itself.*

**Proof.** Since $\mu^y$ and $(\delta_x \circ L)^u$ commute, the equations (2.8) and (2.1) yield for all $\lambda, \mu \in \mathcal{F}^*_{\Psi_L}$

$$\begin{aligned} \langle \lambda \,|\, \mu \rangle_{\mathcal{F}^*_{\Psi_L}} &= \lambda^x \,\overline{\mu}^y \,(\delta_x \circ L)^u \,(\delta_y \circ \overline{L})^v \,\Phi(u,v) \\ &= (\lambda \circ L)^u \,(\overline{\mu} \circ \overline{L})^v \,\Phi(u,v) \\ &= \langle \lambda \circ L \,|\, \mu \circ L \rangle_{\mathcal{F}^*_\Phi} \;. \end{aligned}$$

Equation (2.9) follows immediately. Moreover, we see $(\lambda - \mu) \circ L \in \mathcal{F}^*_\Phi$ for $\lambda \in \mathcal{F}^*_{\Psi_L}$ and $\mu \in \text{span}\,\Lambda$. Therefore $P_{\Psi_L,\Lambda}(\lambda)$ is well-defined. Using equation (2.6), we calculate

$$\begin{aligned} P_{\Psi_L,\Lambda}(\lambda) &= \inf_{\mu \in \text{span}\,\Lambda} \|\lambda - \mu\|_{\mathcal{F}^*_{\Psi_L}} \\ &= \inf_{\mu \in \text{span}\,\Lambda} \|\lambda \circ L - \mu \circ L\|_{\mathcal{F}^*_\Phi} \\ &= \inf_{\mu \in \text{span}\,\Lambda\circ L} \|\lambda \circ L - \mu\|_{\mathcal{F}^*_\Phi} \quad = \quad P_{\Phi,\Lambda\circ L}(\lambda \circ L) \;. \end{aligned}$$

Inequality (2.11) is proved by

$$\begin{aligned} P_{\Phi,\Lambda\circ L}(\lambda \circ L) &= \inf_{\mu \in \text{span}\,\Lambda} \|\lambda \circ L - \mu \circ L\|_{\mathcal{F}^*_\Phi} \\ &\le \inf_{\mu \in \text{span}\,\Lambda} \|\lambda - \mu\|_{\mathcal{F}^*_\Phi} \cdot \|L\|_\Phi \;. \end{aligned}$$

$\blacksquare$

In the important example (1.2.3), using $\lambda = \delta_x$ and $\Lambda = \delta_X$, the benefit of this theorem is to reduce the unknown power function $P_{\Phi,\delta_X \circ L}(\delta_x \circ L)$ according to the basis function $\Phi$ to the classical power function $P_{\Psi_L,\delta_X}(\delta_x)$ of $\Psi_L$.

Now we compare two different approximations. First, there is the one we have been dealing with, namely:

$$s_{u,\delta_X \circ L,\Phi}(x) \;=\; \overline{L}^y \Phi(x,X)\cdot\alpha \quad \text{satisfying} \quad L s_{u,\delta_X \circ L,\Phi}(X) \;=\; Lu(X)$$

is a recovery of $u$ from its data $\delta_X \circ Lu$. Here, $\alpha$ is given via

$$L^x \overline{L}^y \Phi(X,X)\cdot\alpha \;=\; Lu(X)\ .$$

But on the other hand, we have the reconstruction of $Lu$ with respect to $\Psi_L$ by

$$s_{Lu,\delta_X,\Psi_L}(x) \;=\; \Psi_L(x,X)\cdot\alpha' \quad \text{satisfying} \quad s_{Lu,\delta_X,\Psi_L}(X) \;=\; Lu(X)\ .$$

The corresponding interpolation system is

$$\Psi_L(X,X)\cdot\alpha' \;=\; Lu(X)\ .$$

Using the definition (2.8) of $\Psi_L$, we find $\alpha' = \alpha$, and therefore we get the following theorem:

**Theorem 2.8** *The identity*

$$L\, s_{u,\delta_X \circ L,\Phi} \;=\; s_{Lu,\delta_X,\Psi_L} \tag{2.12}$$

*holds for any $u$ which allows the operation $L$.*

To demonstrate the use of the theorems of this section, we give a typical application. Regard $\Phi$ and $\Lambda$ as given. The ordered subsets $\Lambda_\nu$ of $\Lambda$ should have the form of (1.2.5), i.e. $\Lambda_\nu = \delta_{X_\nu} \circ L_\nu$ with $X_\nu := X_{\Omega_\nu} \subset \Omega_\nu$ finite. Then from Theorem 2.5 and the Splitting Theorem 2.6, we get the error bound

$$\begin{aligned} |\lambda\,(u - s_{u,\Lambda,\Phi})| &\le P_{\Phi,\Lambda}(\lambda)\cdot\|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} \\ &\le \min_{\nu=1,\dots,N} P_{\Phi,\Lambda_\nu}(\lambda) \;\cdot\; \min_{\nu=1,\dots,N} \|u - s_{u,\Lambda_\nu,\Phi}\|_{\mathcal{F}_\Phi} \end{aligned} \tag{2.13}$$

for any $\lambda \in \mathcal{F}_\Phi^*$. Taking $\lambda := \delta_x \circ L_\nu$ as test functionals at $x \in \Omega_\nu$, we can use the Transformation Theorem 2.7 to treat the first factor: $P_{\Phi,\Lambda_\nu}(\lambda) = P_{\Psi_{L_\nu},\delta_{X_\nu}}(\delta_x)$. Using equation (2.11), they can also be bounded by $P_{\Phi,\delta_{X_\nu}}(\delta_x)\cdot\|L_\nu\|_\Phi$, which is a multiple of the classical power function. We use the Primal Transformation Theorem 3.5 below and Theorem 2.8 to rewrite the last factor of equation (2.13) in the form

$$\left\| u - s_{u,\delta_{X_\nu}\circ L_\nu,\Phi} \right\|_{\mathcal{F}_\Phi} \;=\; \left\| L_\nu u - L_\nu s_{u,\delta_{X_\nu}\circ L_\nu,\Phi} \right\|_{\mathcal{F}_{\Psi_L}} \;=\; \left\| L_\nu u - s_{L_\nu u,\delta_{X_\nu},\Psi_L} \right\|_{\mathcal{F}_{\Psi_L}}\ .$$

The right hand side is the classical approximation error of $L_\nu u$ in $\mathcal{F}_{\Psi_L}$.

# 3 Applications of Fourier Transforms

Subsequently, we assume the basis function $\Phi$ not only to be feasible, but to be *translation invariant*, i.e.

$$\Phi(x,y) \;=\; \Phi(x-z,y-z) \quad \text{for all } x,y\in\Omega \text{ and } z\in\mathbb{R}^d\ .$$

This condition is equivalent to the existence of a function $\Phi_0$ with

$$\Phi(x,y) \;=\; \Phi_0(x-y) \quad \text{for all } x,y\in\Omega\ . \tag{3.1}$$

Moreover, we assume $\Phi_0$ to be the inverse Fourier transform of some $\varphi$:

$$\Phi_0(x) \;=\; \mathrm{FT}^{-1}(\varphi)(x) \;:=\; (2\pi)^{-d}\int_{\mathbb{R}^d}\varphi(\omega)\,e^{i\,\omega^{\mathrm{tr}}x}\,d\omega\ . \tag{3.2}$$

Under certain circumstances one has to resort to the generalized distributional Fourier transform, but we omit these technical details here. Further we will assume that $\varphi$ is nonnegative and positive almost everywhere. This ensures $\Phi$ to be positive definite with respect to $\delta_x$-functionals, cf. [2]. To take advantage of the representation of $\Phi$ by (3.1) and (3.2), we have to restrict the dual native space $\mathcal{F}_\Phi^*$ to functionals commuting with the Fourier transform. That means, we assume

$$\mathcal{F}_\Phi^* \;\subseteq\; \left\{\ \lambda:\ \lambda^x\left(\int\chi(\omega)\,e^{i\,\omega^{\mathrm{tr}}x}\,d\omega\right) \;=\; \int\chi(\omega)\,\mathrm{sym}_\lambda(\omega)\,d\omega\ \right\} \tag{3.3}$$

for $\chi=\varphi$ and $\chi=\varphi\cdot\mathrm{sym}_\mu$, $\mu\in\mathcal{F}_\Phi^*$ arbitrary, where

$$\mathrm{sym}_\lambda(\omega) \;:=\; \lambda^x\left(e^{i\,\omega^{\mathrm{tr}}x}\right) \tag{3.4}$$

is called the *symbol function* of the functional $\lambda$. It may even be a generalized function, defined by the equality in (3.3). In addition, we require any function $f$ in $\mathcal{F}_\Phi$ to have a representation via the inverse Fourier transform:

$$\text{For any } f\in\mathcal{F}_\Phi \text{ there is a function } \hat{f} \text{ such that } f=\mathrm{FT}^{-1}\left(\hat{f}\right)\ . \tag{3.5}$$

Where necessary, we assume $\hat{f}=\mathrm{FT}(f)$ that is: the Fourier transform is bijective on $\mathcal{F}_\Phi$. Apparently this is a lot of assumptions, but we know several kinds of feasible basis functions which allow our construction.

**Remark 3.1** *It is also common to define the symbol function by* $\lambda^\omega\left(e^{-i\,\omega^{\mathrm{tr}}x}\right)$, *using the negative sign in the exponent. If* $\lambda$ *is a* regular distribution, *i.e. a* *functional of the form* $\lambda_l(g) = \int l(\omega)\, g(\omega)\, d\omega$ *with a representing function* $l$*, the so defined symbol function coincides with the representer of the Fourier transform*

$$\mathrm{FT}(\lambda_l)(f) \;:=\; \lambda_l(\mathrm{FT}(f)) \;=\; \int f(x)\, \lambda_l^{\omega}\left(e^{-i\,\omega^{\mathrm{tr}} x}\right)\, dx$$

*of* $\lambda_l$ *by the use of Fubini's Theorem. Therefore, this definition of the symbol function justifies to denote it by* $\widehat{\lambda}$*. But we will not use this notation, because it requires* $\lambda$ *to be regular or to use generalized Fourier transforms.*

Assuming (3.3), we calculate the scalar product

$$\begin{aligned} \langle \lambda \,|\, \mu \rangle_{\mathcal{F}_\Phi^*} &= \lambda^x \overline{\mu}^y \,(2\pi)^{-d} \int_{\mathbb{R}^d} \varphi(\omega)\, e^{i\,\omega^{\mathrm{tr}}(x-y)}\, d\omega \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \varphi(\omega)\, \lambda^x e^{i\,\omega^{\mathrm{tr}} x}\, \overline{\mu^y e^{i\,\omega^{\mathrm{tr}} y}}\, d\omega \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \varphi(\omega)\, \mathrm{sym}_\lambda(\omega)\, \overline{\mathrm{sym}_\mu(\omega)}\, d\omega \;, \end{aligned} \tag{3.6}$$

and find

$$\|\lambda\|_{\mathcal{F}_\Phi^*}^2 \;=\; (2\pi)^{-d} \int_{\mathbb{R}^d} \varphi(\omega)\; |\mathrm{sym}_\lambda(\omega)|^2\; d\omega \;. \tag{3.7}$$

To prove the feasibility of a given $\Phi$ (in particular the positive definiteness, which has been stated in definition 2.1), it suffices to show that the integrand $\varphi(\omega)\; |\mathrm{sym}_\lambda(\omega)|^2$ is nonnegative and positive almost everywhere (cf. a classical result from [2]).

**Theorem 3.2**

- *The equations (3.3) and (3.5) yield*

$$\lambda(f) \;=\; (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot \mathrm{sym}_\lambda(\omega)\, d\omega \tag{3.8}$$

  *for all* $\lambda \in \mathcal{F}_\Phi^*$ *and* $f \in \mathcal{F}_\Phi$*.*

- *If the linear operator* $L$ *is* translation invariant*, i.e.*

$$L^x(\, f(x-z)\,) \;=\; (Lf)(x-z) \tag{3.9}$$

  *for all* $z \in \mathbb{R}^d$ *and* $f$ *in the domain of* $L$*, we have*

$$\mathrm{sym}_{\delta_x \circ L}(\omega) \;=\; \mathrm{sym}_{\delta_0 \circ L}(\omega) \cdot e^{i\,\omega^{\mathrm{tr}} x} \quad \textit{for all } x,\, \omega \in \mathbb{R}^d \;. \tag{3.10}$$

**Proof.** Only equation (3.10) requires a proof:

$$\begin{aligned}
\operatorname{sym}_{\delta_{x_0+z}\circ L}(\omega) &= \left(L^x\left(e^{i\,\omega^{\mathrm{tr}}x}\right)\right)(x_0+z) \\
&= \left(L^x\left(e^{i\,\omega^{\mathrm{tr}}(x+z)}\right)\right)(x_0) \qquad \text{using (3.9)} \\
&= \left(L^x\left(e^{i\,\omega^{\mathrm{tr}}x}\cdot e^{i\,\omega^{\mathrm{tr}}z}\right)\right)(x_0) \;=\; \left(L^x\left(e^{i\,\omega^{\mathrm{tr}}x}\right)\right)(x_0)\cdot e^{i\,\omega^{\mathrm{tr}}z} \\
&= \operatorname{sym}_{\delta_{x_0}\circ L}(\omega)\cdot e^{i\,\omega^{\mathrm{tr}}z}\;.
\end{aligned}$$

Now we substitute $x_0$ by 0 and $z$ by $x$. ∎

We now want to represent native space norms via Fourier Transforms.

**Lemma 3.3** *Let $\mathcal{F}_\Phi^*$ and $\mathcal{F}_\Phi$ satisfy the equations (3.3) and (3.5), respectively. Then for every $f \in \mathcal{F}_\Phi$, the equation*

$$\widehat{f}(\omega) \;=\; \varphi(\omega)\cdot\overline{\operatorname{sym}_{\lambda_f}(\omega)}$$

*is valid, where $\lambda_f \in \mathcal{F}_\Phi^*$ is the* Riesz representer *of the function $f$.*

**Proof.** Due to the Theorem of Riesz, for any given $f \in \mathcal{F}_\Phi$, there is a unique $\lambda_f \in \mathcal{F}_\Phi^*$ satisfying $\lambda(f) = \langle\,\lambda\,|\,\lambda_f\,\rangle_{\mathcal{F}_\Phi^*}$ for all $\lambda \in \mathcal{F}_\Phi^*$ . Using the equations (3.6) and (3.8), we find

$$\begin{aligned}
\lambda\left(\mathrm{FT}^{-1}\left(\widehat{f}\right)\right) &= \lambda(f) \;=\; \langle\,\lambda\,|\,\lambda_f\,\rangle_{\mathcal{F}_\Phi^*} \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\varphi(\omega)\operatorname{sym}_\lambda(\omega)\,\overline{\operatorname{sym}_{\lambda_f}(\omega)}\,d\omega \\
&= \lambda\left(\mathrm{FT}^{-1}\left(\varphi\,\overline{\operatorname{sym}_{\lambda_f}}\right)\right)
\end{aligned}$$

for every $\lambda \in \mathcal{F}_\Phi^*$. Thus $\mathrm{FT}^{-1}\left(\widehat{f}\right) \;=\; \mathrm{FT}^{-1}\left(\varphi\,\overline{\operatorname{sym}_{\lambda_f}}\right)$. ∎

**Theorem 3.4** *Let $\Phi$ be a feasible basis function which has a representation via the equations (3.1) and (3.2). Let $\mathcal{F}_\Phi^*$ and $\mathcal{F}_\Phi$ satisfy the equations (3.3) and (3.5), respectively. Then the equation*

$$\langle\,f\,|\,g\,\rangle_{\mathcal{F}_\Phi} \;=\; (2\pi)^{-d}\int_{\mathbb{R}^d}\widehat{f}(\omega)\,\overline{\widehat{g}(\omega)}/\,\varphi(\omega)\,d\omega \tag{3.11}$$

*holds for every $f, g \in \mathcal{F}_\Phi$. Recall that $\varphi$ can only vanish on a set of measure 0.*

Equation (3.11) is the analog of equation (3.6) for the scalar product in $\mathcal{F}_\Phi$. More about this representation is found in [20], section 4.

**Proof.** Due to Lemma 3.3, we know $\overline{\mathrm{sym}_{\lambda_f}(\omega)} = \widehat{f}(\omega)/\varphi(\omega)$ almost everywhere for any given function $f$, where $\lambda_f \in \mathcal{F}_\Phi^*$ is the Riesz representer of $f$. We use its property $\langle \lambda_g \,|\, \lambda_f \rangle_{\mathcal{F}_\Phi^*} = \langle f \,|\, g \rangle_{\mathcal{F}_\Phi}$ and equation (3.6) to calculate

$$\begin{aligned} \langle f \,|\, g \rangle_{\mathcal{F}_\Phi} &= \langle \lambda_g \,|\, \lambda_f \rangle_{\mathcal{F}_\Phi^*} = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{sym}_{\lambda_g}(\omega)\, \overline{\mathrm{sym}_{\lambda_f}(\omega)} \cdot \varphi(\omega)\, d\omega \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \overline{\widehat{g}(\omega)}\, \widehat{f}(\omega)/\varphi(\omega)\, d\omega \,. \end{aligned}$$

■

As we saw above, the symbol function is an essential tool for proving positive definiteness of the basis functions $\Phi$ and $\Psi_L$. Moreover, we will use it to establish a connection between the norms of their native spaces $\mathcal{F}_\Phi$ and $\mathcal{F}_{\Psi_L}$ and between their power functions, respectively. For this reason, we assume the operator $L$ to be translation invariant. Since $\Phi$ is translation invariant, so is $\Psi_L$, and a function $\Psi_{L,0}$ with $\Psi_L(x,y) = \Psi_{L,0}(x-y)$ exists. If in addition there is a function $\psi_L$ with $\Psi_{L,0} = \mathrm{FT}^{-1}(\psi_L)$, we find with equation (3.3) for almost every $x, y$ in $\mathbb{R}^d$

$$\int \psi_L(\omega) \cdot e^{i\,\omega^{\mathrm{tr}}(x-y)}\, d\omega = \int \varphi(\omega) \cdot L^x\left(e^{i\,\omega^{\mathrm{tr}} x}\right) \cdot \overline{L^y\left(e^{i\,\omega^{\mathrm{tr}} y}\right)}\, d\omega \,,$$

where

$$L^x\left(e^{i\,\omega^{\mathrm{tr}} x}\right) := \delta_x^z\, L^z\left(e^{i\,\omega^{\mathrm{tr}} z}\right) = \mathrm{sym}_{\delta_x \circ L}(\omega)$$

denotes the symbol function for functionals of the form $\lambda = \delta_x \circ L$ with a linear operator $L$. Using equation (3.10), we calculate

$$L^x\left(e^{i\,\omega^{\mathrm{tr}} x}\right) \cdot \overline{L^y\left(e^{i\,\omega^{\mathrm{tr}} y}\right)} = \left|\mathrm{sym}_{\delta_0 \circ L}(\omega)\right|^2 e^{i\,\omega^{\mathrm{tr}}(x-y)} \,.$$

Therefore, the functions $\psi_L$ and $\varphi$ are connected by

$$\psi_L(\omega) = \varphi(\omega) \cdot \left|\mathrm{sym}_{\delta_0 \circ L}(\omega)\right|^2 \quad \text{for a. e. } \omega \in \mathbb{R}^d \,. \tag{3.12}$$

If $\left|\mathrm{sym}_{\delta_0 \circ L}(\omega)\right|^2$ is positive almost everywhere like $\varphi$, so is $\psi_L$. Then $\Psi_L$ is positive definite, and it is a feasible basis function. The following theorem is the analog of the Transformation Theorem 2.7.

**Theorem 3.5** (Primal Transformation Theorem) *Let $L$ be translation invariant and let it commute with the Fourier transform integral for $f \in \mathcal{F}_\Phi$. Let $\Phi$ and $\Psi_L$ be feasible basis functions. Let $\mathcal{F}_\Phi^*$ and $\mathcal{F}_\Phi$ satisfy the equations (3.3) and (3.5), respectively. Then the connection between $\|\cdot\|_{\mathcal{F}_{\Psi_L}}$ and $\|\cdot\|_{\mathcal{F}_\Phi}$ is*

$$\|f\|_{\mathcal{F}_\Phi} \;=\; \|Lf\|_{\mathcal{F}_{\Psi_L}} \quad \textit{for all } f \in \mathcal{F}_\Phi \;. \tag{3.13}$$

*This equation is the dual of equation (2.9). For $\Lambda = \delta_X \circ L$ we get the equation*

$$\inf_{s \in \operatorname{span} \overline{\Lambda}^y \Phi(\cdot,y)} \|u - s\|_{\mathcal{F}_\Phi} \;=\; \inf_{s \in \operatorname{span} \delta_X^y \Psi_L(\cdot,y)} \|(Lu) - s\|_{\mathcal{F}_{\Psi_L}} \;. \tag{3.14}$$

*The right hand term is the 'classical' approximation error of $Lu$. If $\mathcal{F}_\Phi \subseteq \mathcal{F}_{\Psi_L}$ and $L : \mathcal{F}_{\Psi_L} \to \mathcal{F}_{\Psi_L}$ is bounded by $\|L\|_{\Psi_L} < \infty$, we find moreover*

$$\inf_{s \in \operatorname{span} \overline{\Lambda}^y \Phi(\cdot,y)} \|u - s\|_{\mathcal{F}_\Phi} \;\le\; \inf_{s \in \operatorname{span} \overline{\Lambda}^y \Phi(\cdot,y)} \|u - s\|_{\mathcal{F}_{\Psi_L}} \cdot \|L\|_{\Psi_L} \;. \tag{3.15}$$

**Proof.** With the equations (3.8) and (3.10), we calculate for $f \in \mathcal{F}_\Phi$:

$$\begin{aligned} L(f)(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot \operatorname{sym}_{\delta_0 \circ L}(\omega) \cdot e^{i\,\omega^{\mathrm{tr}} x}\, d\omega \\ &= \mathrm{FT}^{-1}\left(\widehat{f} \cdot \operatorname{sym}_{\delta_0 \circ L}\right)(x) \;. \end{aligned}$$

The equations (3.11) and (3.12) yield

$$\begin{aligned} \|f\|_{\mathcal{F}_\Phi}^2 &= (2\pi)^{-d} \int_{\mathbb{R}^d} \left|\widehat{f}(\omega)\right|^2 / \varphi(\omega)\, d\omega \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \left|\widehat{f}(\omega) \cdot \operatorname{sym}_{\delta_0 \circ L}(\omega)\right|^2 / \psi_L(\omega)\, d\omega \\ &= \left\|\mathrm{FT}^{-1}\left(\widehat{f} \cdot \operatorname{sym}_{\delta_0 \circ L}\right)\right\|_{\mathcal{F}_{\Psi_L}}^2 \;. \end{aligned}$$

The combination of these results prove (3.13). To prove the second assertion, we use (3.13) to calculate

$$\begin{aligned} \inf_{s \in \operatorname{span} \overline{\Lambda}^y \Phi(\cdot,y)} \|u - s\|_{\mathcal{F}_\Phi} &= \inf_{s \in \operatorname{span} \overline{\Lambda}^y \Phi(\cdot,y)} \|Lu - Ls\|_{\mathcal{F}_{\Psi_L}} \\ &= \inf_{s' \in \operatorname{span} L \circ \overline{\Lambda}^y \Phi(\cdot,y)} \|Lu - s'\|_{\mathcal{F}_{\Psi_L}} \\ &= \inf_{s \in \operatorname{span} \delta_X^y \Psi_L(\cdot,y)} \|Lu - s\|_{\mathcal{F}_{\Psi_L}} \;. \end{aligned}$$

The last assertion follows from the first line of this equation. ∎

We give two simple, but typical examples for translation invariant linear operators $L$:

**Example 3.6** *Let $L := p(D)$ be a partial differential operator with constant coefficients, i.e. $p$ is a polynomial on $i \cdot \mathbb{R}^d$ and $D := (\partial/\partial x_j)_{j=1,\dots,d}$. Then we find*

$$\Psi_L(x,y) \;=\; p(D)^x\, \overline{p(D)}^y\, \Phi(x,y)$$

*and*

$$\mathrm{sym}_{\delta_x \circ L}(\omega) \;=\; p(i\omega)\cdot e^{i\,\omega^{\mathrm{tr}} x} \tag{3.16}$$

*for every $x \in \Omega$. With equation (3.7), this implies*

$$\|\delta_x\|^2_{\mathcal{F}^*_{\Psi_L}} \;=\; \|\delta_x \circ L\|^2_{\mathcal{F}^*_{\Phi}} \;=\; (2\pi)^{-d}\int_{\mathbb{R}^d} |p(i\omega)|^2 \cdot \varphi(\omega)\, d\omega\ .$$

**Example 3.7** *Let*

$$L(f)(x) \;:=\; \int_{\mathbb{R}^d} K(x-x')\, f(x')\, dx' \;=\; (K * f)(x)$$

*be an integral operator of convolution type. We calculate*

$$\Psi_L(x,y) \;=\; \int_{\mathbb{R}^d} K(x-x') \int_{\mathbb{R}^d} \overline{K(y-y')}\ \Phi(x',y')\ dy'\, dx'$$

*and*

$$\mathrm{sym}_{\delta_x \circ L}(\omega) \;=\; \mathrm{FT}(K)(\omega)\cdot e^{i\,\omega^{\mathrm{tr}} x} \tag{3.17}$$

*for every $x \in \Omega$. With equation (3.7), we get*

$$\|\delta_x\|^2_{\mathcal{F}^*_{\Psi_L}} \;=\; (2\pi)^{-d}\int_{\mathbb{R}^d} |\mathrm{FT}(K)(\omega)|^2 \cdot \varphi(\omega)\, d\omega\ .$$

**Remark 3.8** *The paper [6] shows a general technique to apply the above abstract results to general problems involving partial differential equations. However, [6] does not produce explicit convergence orders, and thus we add a section to demonstrate how the technique works for elliptic boundary value problems.*

# 4 Application to Mixed Linear Problems

In the following example, we need the classical theory of partial differential equations, which uses *Sobolev spaces.* There are two common versions of such spaces, defined by

$$\mathsf{H}^l(\mathbb{R}^d) \;:=\left\{ \lambda \in \left(\mathcal{S}(\mathbb{R}^d)\right)' :\ \int_{\mathbb{R}^d} |\mathrm{FT}(\lambda)(\omega)|^2 \left(1+\|\omega\|_2^2\right)^l d\omega < \infty \right\}$$

and

$$\mathsf{W}_p^l(\Omega) \ := \Big\{ \, f \in \mathsf{L}_p(\Omega) : \ D^\alpha f \text{ exists for all } \alpha \in \mathbb{N}_0^d, |\alpha| \le l \, \Big\} \ .$$

Since we need domains $\Omega$ with sufficient smooth boundaries, for example *Lipschitz-boundaries*, Theorem 5.3 of [18] implies $\mathsf{W}_2^l(\Omega) \cong \mathsf{H}^l(\Omega)$ for all $l \ge 0$. We shall use the notation $\mathsf{H}^l(\Omega)$. (The sign $\cong$ denotes *norm isomorphy.*)

We treat the problem

$$\begin{array}{rclcll} L_1 u & := & p(D)\, u & = & f_1 & \text{on} \quad \Omega_1 \\ L_2 u & := & K * u & = & f_2 & \text{on} \quad \Omega_2 \\ L_3 u & := & u & = & f_3 & \text{on} \quad \Omega_3 = \partial\Omega_1 \end{array} \tag{4.1}$$

as an example, where $\overline{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \mathbb{R}^d$ is bounded, $p$ is a polynomial of order $m$ and $K \in \mathsf{L}_2(\mathbb{R}^d)$. In addition, we want the polynomial $p$ to have no zeros in $i \cdot \mathbb{R}^d$, for example $p(x) = -x^{\mathrm{tr}} x + 1$. We assume there is a solution $u$ in the Sobolev space $\mathsf{H}^\rho(\mathbb{R}^d)$. Moreover, we need $u \in C^m(\mathbb{R}^d)$, since the first condition of (4.1) shall be satisfied pointwise. Due to the *Sobolev imbedding theorems* we have to assume $\rho > m + d/2$ to ensure $\mathsf{H}^\rho(\mathbb{R}^d) \subseteq C^m(\mathbb{R}^d)$, cf. [18], Corollary 6.1. To satisfy equation (2.2), we need

$$\mathcal{L} \ := \ \operatorname{span} \Big( \{ \ \delta_x \circ L_1 \ \}_{x \in \Omega_1} \cup \ \{ \ \delta_x \circ L_2 \ \}_{x \in \Omega_2} \cup \ \{ \ \delta_x \ \}_{x \in \Omega_3} \Big) \ \subseteq \ \mathcal{F}_\Phi^* \ .$$

We first have to choose a feasible basis function $\Phi(x, y)$ which allows the application and commutation of any pair of the above functionals with respect to $x$ and $y$. The following theorem shows that it is possible to choose $\Phi$ with $\mathcal{F}_\Phi \cong \mathsf{H}^\sigma(\mathbb{R}^d)$, where $\sigma \le \rho$. Since $u \in \mathsf{H}^\sigma(\mathbb{R}^d)$, we then know $u \in \mathcal{F}_\Phi$. To ensure the above-mentioned commutation property, it suffices to increase $\sigma$ as needed.

**Theorem 4.1** *We assume $\partial\Omega$ to be Lipschitz continuous. If a function $\Phi_0 \in \mathsf{L}_1(\mathbb{R}^d)$ has a Fourier transform $\varphi$ satisfying*

$$c_1 \left(1 + \|\omega\|_2\right)^{-2\sigma} \ \le \ \varphi(\omega) \ \le \ c_2 \left(1 + \|\omega\|_2\right)^{-2\sigma} \quad \textit{for all } \omega \in \mathbb{R}^d \tag{4.2}$$

*with certain constants $0 < c_1 \le c_2$ and $2\,\sigma \in \mathbb{N}$, $\sigma \ge 2$, its corresponding function*

$$\Phi(x, y) = \Phi_0(x - y) = \mathrm{FT}^{-1}(\varphi)(x - y)$$

*is a feasible basis function with respect to $\mathcal{L}_\delta := \operatorname{span}\{\,\delta_x\}_{x \in \mathbb{R}^d}$. The native space $\mathcal{F}_\Phi$ is norm isomorphic to the Sobolev space $\mathsf{H}^\sigma(\mathbb{R}^d)$, i.e.: $\mathcal{F}_\Phi \cong \mathsf{H}^\sigma(\mathbb{R}^d)$.*

This is a re-formulation of [16], Theorem 2.1, which is based on [20]. For example, Wendland's compactly supported radial basis functions satisfy equation (4.2), cf. [16], Theorem 3.6.

Now, we examine the Properties of $\mathcal{F}_{\Psi_{L_1}}$. Due to example 3.6, we know that if $\Phi_0 \in C^{2m}(\mathbb{R}^d)$ induces a feasible basis function with respect to $\mathcal{L}_\delta$, then

$$\Psi_{L_1}(x,y) \;=\; {L_1}^x\, \overline{L_1}^y\, \Phi(x,y) \;=\; \Psi_{L_1,0}(x-y)$$

is feasible with respect to $\delta_x$-functionals, too. Therefore $\Phi$ is feasible with respect to $\delta_x$- *and* $\delta_x \circ L_1$-functionals. With the equations (3.12) and (3.16), we find

$$c_1\,(1+\|\omega\|_2)^{-2\sigma}\;|p(i\omega)|^2 \;\le\; \psi_{L_1}(\omega) \;\le\; c_2\,(1+\|\omega\|_2)^{-2\sigma}\;|p(i\omega)|^2$$

for $\psi_{L_1} = \mathrm{FT}(\Psi_{L_1,0})$ and for all $\omega \in \mathbb{R}^d$. Since $p$ is a polynomial of order $m$ and does not vanish anywhere, we get with new constants $c_2 \ge c_1 > 0$ the inequalities

$$c_1\,(1+\|\omega\|_2)^{-2(\sigma-m)} \;\le\; \psi_{L_1}(\omega) \;\le\; c_2\,(1+\|\omega\|_2)^{-2(\sigma-m)} \quad \text{for all } \omega \in \mathbb{R}^d\,, \tag{4.3}$$

which imply $\mathcal{F}_{\Psi_{L_1}} \cong \mathsf{H}^{\sigma-m}(\mathbb{R}^d)$ by use of equation (4.2). We need $\Psi_{L_1,0} \in C^0(\mathbb{R}^d)$, therefore we have to assume $\sigma \ge m + d/2$.

Since $K \in \mathsf{L}_2(\mathbb{R}^d)$, we find $L_2 : \mathcal{F}_\Phi \to \mathcal{F}_\Phi$ to be bounded by the norm of $K$, i.e. $\|L_2\|_\Phi \le \|K\|_{\mathsf{L}_2(\mathbb{R}^d)}$. We can set $\mathcal{F}_{\Psi_{L_2}} :\cong \mathcal{F}_\Phi$, but we have to obey equation (3.13), which says $\|f\|_{\mathcal{F}_\Phi} = \|L_2 f\|_{\mathcal{F}_{\Psi_{L_2}}}$. This $\mathcal{F}_\Phi$ may not be the maximal possible native space of $\Psi_{L_2}$, but we do not need the maximal space here.

We choose finite sets $X_\nu \subset \Omega_\nu$, $\nu = 1,\ldots,3$ of centers and construct $s_{u,\Lambda,\Phi}$ according to the equations (1.2.6) and (1.2.7). The centers are to be distributed nicely, i.e. there exists an $h_0 > 0$ with

$$h_{X,\Omega} \;:=\; \sup_{x\in\Omega}\,\min_{x'\in X}\,\|x-x'\|_2 \;<\; h_0\,, \tag{4.4}$$

and $\Lambda$ from (1.2.5) has to be linearly independent.

To proceed towards error bounds, we have to use the uniform ellipticity of the partial differential operator. The following theorem requires $m = 2$, but there are similar and slightly more complicated theorems for differential operators of higher orders, cf. [18], Theorem 12.12 and Theorem 13.1.

**Theorem 4.2** (cf. [7], Theorem 3.7) *Let the polynomial according to $L_1$ have the form $p(x) = \sum_{i,j=1}^d a_{i,j}\,x_i\,x_j + b^{\mathrm{tr}}x + c$, where $A := (a_{i,j})_{i,j=1,\ldots,d}$, $b$ and*

*$c \le 0$ are real valued and constant. The operator $L_1$ shall be* elliptic *in the sense of $A = A^{\mathrm{tr}}$ and*

$$\gamma \cdot x^{\mathrm{tr}} x \;\le\; x^{\mathrm{tr}} A\, x \;\le\; M \cdot x^{\mathrm{tr}} x$$

*for every $x \in \mathbb{R}^d$ with constants $M \ge \gamma > 0$. The domain $\Omega_1$ shall be bounded. If the functions $\widetilde{u} \in C^2(\Omega_1) \cap C^0(\overline{\Omega_1})$ and $\widetilde{f} \in C^0(\overline{\Omega_1})$ satisfy $L_1 \widetilde{u} = \widetilde{f}$ on $\Omega_1$, then*

$$\sup_{x\in\Omega_1} |\widetilde{u}(x)| \;\le\; \sup_{x\in\partial\Omega_1} |\widetilde{u}(x)| \;+\; \frac{c}{\gamma} \sup_{x\in\Omega_1} \left|\widetilde{f}(x)\right| \tag{4.5}$$

*holds, where the constant $c$ depends only on* diam $\Omega$ *and $\|b\|_2/\gamma$.*

We apply this theorem to $\widetilde{u} := u - s_{u,\Lambda,\Phi}$ and $\widetilde{f} := L_1(\widetilde{u})$ to get

$$\begin{aligned} \sup_{x\in\Omega_1} |u(x) - s_{u,\Lambda,\Phi}(x)| \;&\le\; \sup_{x\in\Omega_3} |\delta_x(u - s_{u,\Lambda,\Phi})| \\ &\quad + \frac{c}{\gamma} \sup_{x\in\Omega_1} |\delta_x \circ L_1(u - s_{u,\Lambda,\Phi})| \ . \end{aligned} \tag{4.6}$$

We can use the theorem, since $\rho \ge \sigma > m + d/2$, $u \in \mathsf{H}^\rho(\mathbb{R}^d)$ and $s_{u,\Lambda,\Phi} \in \mathcal{F}_\Phi \cong \mathsf{H}^\sigma(\mathbb{R}^d)$ implies $\widetilde{u} \in \mathsf{H}^\sigma(\mathbb{R}^d) \subset C^2(\mathbb{R}^d)$ and $\widetilde{f} \in C^0(\mathbb{R}^d)$. Due to Theorem 2.5, we know

$$\begin{aligned} |\delta_x(u - s_{u,\Lambda,\Phi})| \;&\le\; P_{\Phi,\Lambda}(\delta_x) \cdot \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} && \text{for all } x \in \Omega_3 \ , \\ |\delta_x \circ L_1(u - s_{u,\Lambda,\Phi})| \;&\le\; P_{\Phi,\Lambda}(\delta_x \circ L_1) \cdot \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} && \text{for all } x \in \Omega_1 \ , \\ |\delta_x \circ L_2(u - s_{u,\Lambda,\Phi})| \;&\le\; P_{\Phi,\Lambda}(\delta_x \circ L_2) \cdot \|u - s_{u,\Lambda,\Phi}\|_{\mathcal{F}_\Phi} && \text{for all } x \in \Omega_2 \ . \end{aligned} \tag{4.7}$$

Combining this with the Splitting Theorem 2.6, we find

$$\begin{aligned} |\delta_x(u - s_{u,\Lambda,\Phi})| \;&\le\; P_{\Phi,\delta_{X_3}}(\delta_x) \cdot \left\|u - s_{u,\delta_{X_3},\Phi}\right\|_{\mathcal{F}_\Phi} \text{for all } x \in \Omega_3 \ , \\ |\delta_x \circ L_\nu(u - s_{u,\Lambda,\Phi})| \;&\le\; P_{\Phi,\delta_{X_\nu}\circ L_\nu}(\delta_x \circ L_\nu) \cdot \left\|u - s_{u,\delta_{X_\nu}\circ L_\nu,\Phi}\right\|_{\mathcal{F}_\Phi} \end{aligned} \tag{4.8}$$

for all $x \in \Omega_\nu$, $\nu = 1, 2$. The first line allows the application of the 'classical' theory of interpolation with radial basis function, while the second line still needs some work. We use the Transformation Theorem 2.7 and see

$$P_{\Phi,\delta_{X_\nu}\circ L_\nu}(\delta_x \circ L_\nu) \;=\; P_{\Psi_{L_\nu},\delta_{X_\nu}}(\delta_x) \quad \text{for all } x \in \Omega_\nu;\ \nu = 1, 2 \ . \tag{4.9}$$

The Primal Transformation Theorem 3.5 yields

$$\begin{aligned} \left\|u - s_{u,\delta_{X_\nu}\circ L_\nu,\Phi}\right\|_{\mathcal{F}_\Phi} \;&=\; \left\|L_\nu u - L_\nu s_{u,\delta_{X_\nu}\circ L_\nu,\Phi}\right\|_{\mathcal{F}_{\Psi_{L_\nu}}} \\ &=\; \left\|L_\nu u - s_{L_\nu u,\delta_{X_\nu},\Psi_{L_\nu}}\right\|_{\mathcal{F}_{\Psi_{L_\nu}}} \quad \text{for } \nu = 1, 2 \ , \end{aligned} \tag{4.10}$$

where we know that its hypotheses (3.3) and (3.5) are satisfied, since $\tilde{u} \in \mathsf{H}^\rho(\mathbb{R}^d)$. The second equality is due to Theorem 2.8. It contains the approximation error of $L_\nu u$ instead of $u$'s. Note that $L_\nu u = f_\nu$ on $\Omega_\nu$.

Let us collect our error bounds now. We found

$$|u(x) - s_{u,\Lambda,\Phi}(x)| \leq P_{\Phi,\delta_{X_3}}(\delta_x) \cdot \left\| u - s_{u,\delta_{X_3},\Phi} \right\|_{\mathcal{F}_\Phi} \tag{4.11}$$

for all $x \in \Omega_3$ from (4.8),

$$|u(x) - s_{u,\Lambda,\Phi}(x)| \leq \sup_{x\in\Omega_3} P_{\Phi,\delta_{X_3}}(\delta_x) \cdot \left\| u - s_{u,\delta_{X_3},\Phi} \right\|_{\mathcal{F}_\Phi} + \frac{c}{\gamma} \sup_{x\in\Omega_1} P_{\Psi_{L_1},\delta_{X_1}}(\delta_x) \cdot \left\| f_1 - s_{f_1,\delta_{X_1},\Psi_{L_1}} \right\|_{\mathcal{F}_{\Psi_{L_1}}} \tag{4.12}$$

for all $x \in \Omega_1$ using (4.6), (4.8), (4.9) and (4.10), and finally we saw

$$|L_2(u - s_{u,\Lambda,\Phi})(x)| \leq P_{\Phi,\delta_{X_2}\circ L_2}(\delta_x \circ L_2) \cdot \left\| f_2 - s_{f_2,\delta_{X_2},\Psi_{L_2}} \right\|_{\mathcal{F}_{\Psi_{L_2}}}$$

for all $x \in \Omega_2$ from (4.8), and (4.10). Applying the equation (2.11), we can bound the last item by

$$|L_2(u - s_{u,\Lambda,\Phi})(x)| \leq P_{\Phi,\delta_{X_2}}(\delta_x) \, \|L_2\|_\Phi \cdot \left\| f_2 - s_{f_2,\delta_{X_2},\Psi_{L_2}} \right\|_{\mathcal{F}_{\Psi_{L_2}}} \tag{4.13}$$

for all $x \in \Omega_2$.

If $\Lambda = \delta_X$, then a finer distribution $X' \supseteq X$ of centers implies a decrease of $P_{\Phi,\Lambda}(\delta_x)$ by means of the Splitting Theorem 2.6. This effect shall be used now to establish convergence orders in terms of powers of $h_{X,\Omega}$. We recall Theorem 5 of [20]. It says that for $\Phi$ and $\sigma$ satisfying condition (4.2) there exist constants $h_0, C > 0$ with

$$P_{\Phi,\delta_X}(x) \leq C\, h_{X,\Omega}^{\sigma-d/2} \tag{4.14}$$

for any distribution $X \subset \Omega$ of centers with $h_{X,\Omega} < h_0$ and any $x \in \Omega$.

**Corollary 4.3** *We recall $\mathcal{F}_{\Psi_{L_2}} :\cong \mathcal{F}_\Phi \cong \mathsf{H}^\sigma(\mathbb{R}^d)$ and $\mathcal{F}_{\Psi_{L_1}} \cong \mathsf{H}^{\sigma-m}(\mathbb{R}^d)$. We assumed $u \in \mathsf{H}^\rho(\Omega)$ with $\rho \geq \sigma > m+d/2$. Below, $C$ denotes a generic constant. Using equation (4.11), we find*

$$|u(x) - s_{u,\Lambda,\Phi}(x)| \leq C \left\| u - s_{u,\delta_{X_3},\Phi} \right\|_{\mathsf{H}^\sigma(\mathbb{R}^d)} \cdot h_{X_3,\Omega_3}^{\sigma-d/2}$$

*for all* $x \in \Omega_3$*. From equation (4.12) and using* $f_1 \in \mathsf{H}^{\rho-m}(\mathbb{R}^d)$*, we get*

$$\begin{aligned} |u(x) - s_{u,\Lambda,\Phi}(x)| \quad &\leq \quad C \, \left\| u - s_{u,\delta_{X_3},\Phi} \right\|_{\mathsf{H}^\sigma(\mathbb{R}^d)} \cdot h_{X_3,\Omega_3}^{\sigma-d/2} \\ &\quad + C \, \frac{c}{\gamma} \, \left\| f_1 - s_{f_1,\delta_{X_1},\Psi_{L_1}} \right\|_{\mathsf{H}^{\sigma-m}(\mathbb{R}^d)} \cdot h_{X_1,\Omega_1}^{\sigma-m-d/2} \end{aligned}$$

*for all* $x \in \Omega_1$*. Finally, we take equation (4.13) to calculate*

$$|L_2(u - s_{u,\Lambda,\Phi})(x)| \quad \leq \quad C \, \left\| f_2 - s_{f_2,\delta_{X_2},\Psi_{L_2}} \right\|_{\mathsf{H}^\sigma(\mathbb{R}^d)} \cdot h_{X_2,\Omega_2}^{\sigma-d/2}$$

*for all* $x \in \Omega_2$*. Here, we assumed* $f_2 \in \mathsf{H}^\sigma(\Omega)$*.*

**Remark 4.4** *The* $\mathsf{L}_\infty$*-norm can be replaced by the* $\mathsf{L}_2$*-norm to gain an additional factor* $h_X^{d/2}$*, cf. [17], Theorem 5. Since every* $s_{f,\delta_{X_\nu},\Psi_{L_\nu}}$ *is the result of a minimization, the norms on the right hand side can be bounded by* $\|u\|_{\mathsf{H}^\sigma(\mathbb{R}^d)}$*,* $\|f_1\|_{\mathsf{H}^{\sigma-m}(\mathbb{R}^d)}$ *and* $\|f_2\|_{\mathsf{H}^\sigma(\mathbb{R}^d)}$*, respectively.*

# 5 Conclusion

The theory of finite element methods (FEM) yields the following bound of approximation error.

**Theorem 5.1** (cf. [9], Satz 4.2) *Let the domain* $\Omega \subset \mathbb{R}^d$ *have a polyhedral boundary and a quasi-uniform decomposition* $\mathcal{T}$ *into finite elements, which are affine images of a common reference element. Let the maximal diameter of all finite elements be* $2h$*.*

*The order* $m$ *of the given differential operator* $L$ *shall be even. Let* $V$ *be a subspace of* $\mathsf{H}^{m/2}(\Omega)$*. Partial integration is used to define the continuous and* $V$*-elliptical bilinear form* $a(u,v) := \langle Lu \,|\, v \rangle_V$ *on* $V \times V$*. Let* $f$ *be a continuous linear form on* $V$*. We assume there is a solution* $u \in V \cap \mathsf{H}^\rho(\Omega)$ *of the problem* $a(u,v) = f(v)$ *for all* $v \in V$ *with the higher regularity* $\rho = q_{\mathrm{FEM}} + 1 > m/2$*.*

*Let the span* $V_h \subseteq V$ *of basis and test functions contain the space of on* $\mathcal{T}$ *piecewise polynomial functions of degree at most* $q_{\mathrm{FEM}}$*.*

*Then the conforming finite element problem* $a(u_h, v_h) = f(v_h)$ *for all* $v_h \in V_h$ *has a unique solution* $u_h$ *which satisfies*

$$\|u - u_h\|_{\mathsf{H}^{m/2}(\Omega)} \;\leq\; C \; |u|_{\mathsf{H}^\rho(\Omega)} \; h^{\rho-m/2} \; . \tag{5.1}$$

Comparing the error bound (5.1) of the FEM with the error bounds of Corollary 4.3 for collocation, we note several points:

- The collocation method requires a very regular solution $u \in \mathsf{H}^\rho(\Omega)$ with $\rho > m + d/2$. It constructs an approximation of smoothness order $\sigma > m + d/2$ and approximation order $\sigma - m - d/2$. The FEM needs only $\rho \geq m/2$. Its approximating function has smoothness order $\rho - 1 \geq m/2 - 1$ and approximation order $\rho - m/2$.

  But our method yields an $\mathsf{L}_\infty$-error bound, while FEM yields a $\mathsf{H}^{m/2}$ one. There is an additional $h^{d/2}$ convergence factor, if our estimate is rewritten to an $\mathsf{L}_2$-norm.

  The additional regularity required by our method clearly limits its direct applicability. However, current research along the directions of e.g. [3] shows that there are promising techniques to handle cases of low regularity in such a way that the core solution method has to deal only with the regular part of the solution. Combined with such techniques, the regularity requirements are much less serious.

- Collocation as a meshless method needs no geometric information. Thus the main impact of our approach will be towards high-dimensional problems with high regularity.

- We recall that Wendland's functions produce sparse systems due to their compact support. Therefore the complexity of the collocation method can possibly be reduced to $\mathcal{O}(\#centers)$, cf. [14]. However, the underlying theory is difficult and still incomplete. In this direction, multilevel techniques are currently under investigation, cf. [5, 12].

- The FEM can treat operators $L$ which are not translation invariant. We expect that the collocation method can be extended to such problems, too.

- The smoothness of the boundary $\partial\Omega$ does not influence our method, except that we need the existence of a solution of sufficient high regularity.

Altogether we see: The collocation method is feasible for problems with very regular solutions in high space dimensions or with many different operators. This roughly complements the set of problems where the FEM has proven to be an extremely effective tool.

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