Error Estimates for Approximations from Control Nets

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Abstract: This paper provides numerically accessible bounds for the error of certain simple approximations to functions defined by a control net and a partition of unity. This generalizes "flatness tests" for the results of refinement algorithms like knot insertion and subdivision. The order of the error terms may be more than quadratic with respect to the domain size. A large number of examples is given, including rational representations.

Keywords: Computer Aided Geometric Design, flatness test, control net, Bézier net, subdivision, knot insertion, error estimate, rational element.

Classifications:

1 Introduction

Let $\{B_i\,;i \in I\}$ be some collection of scalar valued functions satisfying

$$B_i(x) \geq 0,\ \sum_{i\in I} B_i(x) = 1,\ x \in \Omega \tag{1}$$

on some fixed parameter domain $\Omega \subseteq \mathbb{R}^s$. If I is infinite, we assume that

$$|I(x)| < \infty,\ I(x) := \{i \in I \mid B_i(x) \neq 0\},\ x \in \Omega.$$

Using *control nets* $b := \{b_i\}_{i\in I} \in B := R^I$ of vectors from some normed linear space R we consider the space

$$L(B) := \left\{ L(b) := \sum_{i\in I} b_i B_i \ \mid\ b = \{b_i\}_{i\in I} \in B \right\}$$

of R–valued functions on Ω. A standard technique in CAGD is the replacement of $L(b)$ by the control net b itself or by a simple function which in turn depends on b, e.g. a piecewise linear interpolant of b. Starting with a coarse control net over a large domain, refinement techniques like subdivision or knot insertion are applied until the refined control nets $b^h := \{b_i^h \mid i \in I^h\}$ over domains Ω^h satisfy certain "flatness tests" and can be used as approximations to $L(b)$. Error estimates for this process were mainly given in the form

$$\|b_i^h - L(b)(\xi_i^h)\| \leq C(b)h^2,\ i \in I^h,\ h \to 0 \tag{2}$$

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indicating quadratic convergence of the refined control net points b_i^h towards the function $L(b)$ defined by the coarse control net, evaluated at certain points $\xi_i^h \in \Omega$ (see results of Prautzsch [6], Dahmen, Dyn, and Levin [4], and Dahmen [3]). Within this approach, the order 2 has been proven by Dahmen [3] to be optimal. Higher orders were proven by Prautzsch [6] and Dahmen, Dyn, and Levin [4] to hold locally for the box spline case. Equation (2) follows from reproduction of linear polynomial curves or surfaces. In general, $\mathcal{O}(h^r)$ error bounds usually result for schemes of polynomial precision $r-1$. Such error bounds go back at least to the theory of Bramble and Hilbert [2] and to the practice of Barnhill and Gregory [1], using Sard's work, with others having contributed since.

This paper uses a very simple method to derive some rather general error estimates for approximations to $L(b)$ depending on the control net b. As a consequence, it turns out that error estimates like (2) have nothing to do with refinement processes; they simply follow from the "linear precision" [5] of the representation process $b \mapsto L(b)$. Furthermore, our error estimates

- cover NURBS and are
- uniform rather than pointwise,
- numerically accessible, and
- of more that second order for specific applications.

The following preliminary examples serve to give the reader an idea of the applications of the results of this paper. They easily generalize to surfaces, but are given in terms of curves, for simplicity.

Example 1.1 For polynomial curves of degree n in Bernstein–Bézier form on $\Omega := [\alpha, \beta] \in I\!R$ we have $I := \{0, 1, \ldots, n\}$ and

$$L(b)(x) = \sum_{i=0}^{n} b_i B_i^{(n)}(x) = \sum_{i=0}^{n} b_i \left(\begin{array}{c} n \\ i \end{array} \right) \left(\frac{\beta - x}{\beta - \alpha} \right)^{n-i} \left(\frac{x - \alpha}{\beta - \alpha} \right)^{i}. \tag{3}$$

If Ω is small enough, replacement of $L(b)$ with the polygon defined by the control net $b = \{b_0, \ldots, b_n\}$ produces an error of $\mathcal{O}(\beta - \alpha)^2$, usually proven in the discrete form

$$\|b_i - L(b)(\alpha + \frac{i}{n}(\beta - \alpha))\| \le C(\beta - \alpha)^2.$$

Our approach yields the uniform and computable bound

$$\|L(b)(x) - b_0 - \frac{x - \alpha}{\beta - \alpha}(b_n - b_0)\| \le \max_{1 \le i \le n-1} \|b_i - b_0 - \frac{i}{n}(b_n - b_0)\| \tag{4}$$

for all $x \in [\alpha, \beta]$, and the right–hand side is proven to be of the order $(\beta - \alpha)^2$ when $L(b)$ represents a fixed polynomial over varying domains $\Omega = [\alpha, \beta]$. Note that the intermediate control points $b_1, \ldots, b_{n-1}$ are unnecessary within the approximation on the left–hand side of (4); the simple linear interpolant between the control points b_0 and b_n produces sufficient accuracy. Equation (4) resembles a "flatness test" (or, more precisely, a "linearity test").

Example 1.2 We now assume $n \geq 3$ and replace the linear interpolant of the previous example by a simple cubic $H_b(x)$ defined via Hermite interpolation at α and β. For $\tau := (x-\alpha)/(\beta-\alpha)$ we construct H_b canonically as

$$\begin{aligned} H_b(x) &= b_0(1-\tau)^3 + (b_0 + \tfrac{n}{3}(b_1 - b_0)) \cdot 3(1-\tau)^2\tau \\ &\quad + (b_n - \tfrac{n}{3}(b_n - b_{n-1})) \cdot 3\tau^2(1-\tau) + b_n\tau^3 \end{aligned}$$

and get the explicit error estimate

$$\|L(b)(x) - H_b(x)\| \leq \max_{2 \leq i \leq n-2} \|b_i - \widetilde{b_i}\|, \tag{5}$$

where

$$\begin{aligned} \widetilde{b_i} := & \Big(b_0(n-i)(n-i-1)(n-i-2) + b_n i(i-1)(i-2) \\ & + 3(b_0 + \tfrac{n}{3}(b_1 - b_0))i(n-i)(n-i-1) \\ & + 3(b_n - \tfrac{n}{3}(b_n - b_{n-1}))i(i-1)(n-i)\Big) / (n(n-1)(n-2)) \end{aligned}$$

are the control points obtained when raising the degree of H_b from 3 to n. The error estimate is of *fourth* order with respect to the interval size, if a fixed polynomial is represented over various domains.

If there is an efficient method for evaluating cubic polynomials (e.g.: in case of special hardware or an assembler routine) subdivision need be carried out only until the right–hand side of (5) is small enough, and this occurs when $(\beta - \alpha)^4$ is sufficiently small.

2 Approximations from control nets

The properties (1) of the functions B_i imply the simple but very useful error estimate

$$\|L(b)(x) - L(c)(x)\| \leq \max_{i \in I(x)} \|b_i - c_i\| \text{ for } b,\ c \in B,\quad x \in \Omega. \tag{6}$$

Now we choose certain control nets $c = \{c_i \mid i \in I\} \in B$, constructed from the given control net b, such that

- $L(c)$ is much easier to evaluate than $L(b)$,
- the error estimate (6) is numerically accessible and
- of sufficiently high order.

To get an idea of the scope of (6) one can imagine that $c = c(b)$ might be constructed such that the left hand side of (6) is zero for polynomials $L(b)$ of degree r. Then one can expect the left hand side of (6) to be bounded by the $(r+1)$–th power of the diameter of Ω. If the functions B_i satisfy a "uniform stability property" in the sense of Dahmen [3], such a bound carries over to the right–hand side. This is how "linear precision" of $L(b)$ implies quadratic convergence of refinement algorithms. Furthermore, in this setting there need not be a saturation bound like $r \leq 1$ as given in [3], and uniform stability implies that the error estimates of type (6) are sharp up to constants.

3 Linear precision

We now assume "linear precision" in the sense of Farin [5] of the functions (1) in the form

$$x = \sum_{i \in I(x)} \xi_i B_i(x), \ x \in \Omega, \tag{7}$$

for a control net $\xi := \{\xi_i\}_{i \in I}$ of points from $\mathbb{R}^s \supseteq \Omega$. Then, for any affine map $T : \mathbb{R}^s \supseteq \Omega \to R$ we can use $c_i := T(\xi_i)$ in (6) and get

$$\|L(b)(x) - T(x)\| \leq \max_{i \in I(x)} \|b_i - T(\xi_i)\|, \ \ x \in \Omega. \tag{8}$$

Thus $L(b)$ can be approximated by an affine function T as good as the control net b can be approximated by the control net $c = T(\xi)$. Now T could be optimally chosen to minimize the right–hand side of (8); however, some handy suboptimal choices of T will be sufficient in most cases.

The properties of the partition (1) of unity immediately imply another useful error bound, which does not require (7), but involves a change of parametrization within the approximation:

Theorem 3.1 *If T is any affine map $\mathbb{R}^s \supset \Omega \to R$, and if points $\eta_i \in \Omega$, $0 \leq i \leq n$ are chosen arbitrarily, then there is a transformation $\phi : \Omega \to \Omega$ such that*

$$\|L(b)(x) - T(\phi(x))\| \leq \max_{i \in I(x)} \|b_i - T(\eta_i)\|, \ \ x \in \Omega. \tag{9}$$

Proof: Because of (1),

$$\phi(x) := \sum_{i \in I(x)} \eta_i B_i(x), \ x \in \Omega,$$

maps Ω into itself. Then

$$T(\phi(x)) = \sum_{i \in I(x)} T(\eta_i) B_i(x), \ x \in \Omega,$$

and the assertion follows.□

Example 3.1 To cover the situation of Example 1.1 we use

$$\xi_i = \alpha + \frac{i}{n}(\beta - \alpha) \ \text{ and } \ T(x) = b_0 + \frac{x - \alpha}{\beta - \alpha}(b_n - b_0). \tag{10}$$

Theorem 3.2 *If a fixed vector–valued polynomial p is given in Bernstein–Bézier representation* (3) *over a domain $\Omega = [\alpha, \beta]$, the error term in* (4) *is bounded by $C(p)(\beta - \alpha)^2$ with a constant depending only on p, not on the representation.*

Proof: Let $M_p(x_1, \ldots, x_n)$ be the unique symmetric multiaffine function corresponding to p, i.e.

$$p(x) = M_p(n\#x), \ x \in [\alpha, \beta],$$

where the symbol $n\#x$ means "n repetitions of x". Then (see e.g. Seidel [7]) we have

$$b_i = M_p((n - i)\#\alpha, i\#\beta)$$

and expansion around $n\#\alpha$ yields

$$b_i = b_0 + \left.\frac{\partial M_p}{\partial x_1}\right|_{n\#\alpha} \cdot i(\beta-\alpha) + \mathcal{O}((\beta-\alpha)^2),$$

proving the assertion.□

Clearly, (10) is not the best possible choice. It can be improved by adding $\frac{1}{n+1}\sum b_i - \frac{1}{2}(b_0+b_n)$, gaining a factor of circa $\frac{1}{2}$ in the error estimate. In theory, this modification is optimal with respect to minimization of

$$\sum_{i=0}^{n} \|b_i - c - \xi_i d\|_2^2$$

over all $c,\ d \in R$. Such modifications are possible in most of the examples of this paper.

Furthermore, the restriction to the ξ_i is unnecessary because of Theorem 3.1. If an affine map T represents a suitably parametrized line, the restriction of the η_i in Theorem 3.1 to Ω is no problem. Therefore the error in (9) can be bounded by the generalized distance of the whole control net from a line.

Example 3.2 Following the lines of the last example, it is fairly easy to get an error estimate for tensor product polynomial surfaces over rectangles, given in Bernstein–Bézier representation. The four "corners" $b_{00}, b_{m0}, b_{0n}, b_{mn}$ of the Bézier net $b = \{b_{ij} \mid 0 \le i \le m,\ 0 \le j \le n\}$ define a bilinear approximation of the surface, and the corresponding error is explicitly bounded by

$$\max\left\{\|\epsilon_{ij}\| \ \mid\ 1 \le i \le m-1,\ 1 \le j \le n-1\right\},$$

$$\epsilon_{ij} := b_{ij} - b_{00} - \frac{i}{m}(b_{m0} - b_{00}) - \frac{j}{n}(b_{0n} - b_{00}) - \frac{i}{m}\frac{j}{n}(b_{mn} - b_{m0} - b_{0n} + b_{00}).$$

The error bound has the behavior $\mathcal{O}(\mathrm{diam}^2(\Omega))$ when a fixed polynomial is represented over small domains Ω.

Example 3.3 Univariate splines can be written as

$$s(x) = \sum_{i=0}^{m} b_i N_i^{(n)}(x)$$

in the B–spline basis $N_i^{(n)}$ of degree n with a knot sequence

$$t_0 = \ldots = t_n < t_{n+1} \le \ldots \le t_m < t_{m+1} = \ldots = t_{n+m+1}$$

satisfying $t_i < t_{i+n+1}$ for $i = 0, \ldots, m$ (notation as in [7]). They enjoy linear precision (7) with the Greville abscissae $\xi_i = t_i^* = \frac{1}{n}(t_{i+1} + \ldots + t_{i+n}),\ 0 \le i \le m$, and we get the explicit error estimate

$$\begin{aligned} & \|s(x) - b_{j-n} - \frac{x - \xi_{j-n}}{\xi_j - \xi_{j-n}}(b_j - b_{j-n})\| \\ \le\ & \max_{j-n+1 \le i \le j-1} \|b_i - b_{j-n} - \frac{\xi_i - \xi_{j-n}}{\xi_j - \xi_{j-n}}(b_j - b_{j-n})\| \end{aligned} \qquad (11)$$

for $x \in [t_j, t_{j+1})$, where $I(x) = \{i \mid j-n \le i \le j\}$.

Theorem 3.3 *If a fixed polynomial piece p of degree n is represented over $[t_j, t_{j+1})$, the error term in* (11) *can be bounded by*

$$C \cdot \left(\max_{j-n+1 \leq i \leq j+n} |t_i - \xi_{j-n}| \right)^2 \text{ or } C \cdot (t_{j+n} - t_{j-n+1})^2,$$

where C depends on p only.

Proof: From [7] we use the multiaffine representation

$$b_i = M_p(t_{i+1}, \ldots, t_{i+n}) \text{ for } j - n \leq i \leq j \tag{12}$$

and expand around $n\#\xi_{j-n}$ to get

$$b_i \approx p(\xi_{j-n}) + \left. \frac{\partial M_p}{\partial x_1} \right|_{n\#\xi_{j-n}} \cdot n(\xi_i - \xi_{j-n})$$

up to terms of the asserted order. □

When refinement is done by knot insertion into $[t_j, t_{j+1})$, the polynomial p does not change and we get quadratic convergence of the refinement.

A similar analysis is possible for tensor product B–spline surfaces.

Example 3.4 For an n–th degree polynomial surface in Bernstein–Bézier representation in barycentric coordinates

$$x = (u, v, w) \in [0,1]^3, \ u + v + w = 1$$

over a triangle, we have a control net

$$\{b_{ijk} \mid 0 \leq i, j, k \leq n, \ i + j + k = n\}$$

and get an error estimate

$$\begin{aligned} & \|L(b)(x) - T(x)\| \\ = & \ \| \sum_{\substack{i,j,k=0 \\ i+j+k=n}}^{n} b_{ijk} \frac{n!}{i!j!k!} u^i v^j w^k - (b_{n00}u + b_{0n0}v + b_{00n}w) \| \\ \leq & \ \max \left\{ \|b_{ijk} - (b_{n00}\tfrac{i}{n} + b_{0n0}\tfrac{j}{n} + b_{00n}\tfrac{k}{n})\| \mid 0 \leq i, j, k < n, \ i + j + k = n \right\}. \end{aligned}$$

4 Higher–order approximations

A general way to exploit (6) is to take only part of the information in the control net b to construct a simple approximation $T_b : \Omega \to R$ to $L(b)$. Then T_b is written in the basis B_i by use of a standard technique (degree elevation or knot insertion), and a representation

$$T_b(x) = \sum_{i \in I(x)} B_i(x) c_i(b) = L(c(b)) \tag{13}$$

is the result, giving

$$\|L(b)(x) - T_b(x)\| \le \max_{i\in I(x)} \|b_i - c_i(b)\|. \tag{14}$$

Therefore (14) can also be seen as an error estimate for degree reduction or knot elimination.

Whenever the basis $B_i(x)$ satisfies a uniform stability condition

$$\tilde{c}\cdot \max_{i\in I} \|b_i\| \le \sup_{x\in\Omega} \|L(b)(x)\| \le \max_{i\in I} \|b_i\|,\ b\in B,$$

where $\tilde{c}$ does not depend on domain scalings, the estimate (14) is asymptotically sharp for domain sizes tending to zero. This gives a number of easy applications: just take a simple approximation of well–known order (e.g. the Hermite cubic interpolant of Example 1.2), embed it into the full space $L(B)$ by degree elevation or knot insertion, and use the resulting formulae for the new control points $c_i(b)$ for error estimation.

Example 4.1 For the univariate B–spline setting of Example 3.3 we can generate higher–order methods by specializing (14) to

$$\|L(b)(x) - \sum_{s=0}^{r} a_s x^s\| \le \max_{j-n\le i\le j} \|b_i - \sum_{s=0}^{r} a_s \begin{pmatrix} n \\ s \end{pmatrix}^{-1} S_s^{(n)}(t_{i+1},\ldots,t_{i+n})\|$$

for $x\in [t_j, t_{j+1})$, $r\le n$, where $S_s^{(n)}$ is the s–th degree elementary symmetric function of n variables. This follows easily from (12) when applied to the representation of a polynomial as a linear combination of B–splines. Good choices of the a_s will compensate the first $r+1$ terms of the expansion of

$$b_i = M_p(t_{i+1},\ldots,t_{i+n}) = \sum_{s=0}^{n} d_s \begin{pmatrix} n \\ s \end{pmatrix}^{-1} S_s^{(n)}(t_{i+1},\ldots,t_{i+n}) \tag{15}$$

around a fixed argument, e.g. ξ_{j-n}, resulting in approximations of order $r+1$. Example 3.3 covers the case $r=1$, and higher–order cases can easily be calculated explicitly from (15).

We treat the case $r=2$ by introducing the generalized Greville abscissae

$$\xi_i^{(s)} := \begin{pmatrix} n \\ s \end{pmatrix}^{-1} S_s^{(n)}(t_{i+1},\ldots,t_{i+n}),\ \ \xi_i := \xi_i^{(1)},$$

for a fixed degree n which is suppressed in the notation. Expressing the coefficients in (15) approximately in terms of the control net we find a third–order approximation

$$\begin{aligned} T_b(x) = \quad & b_{j-n} + (x-\xi_{j-n})[j,j-n]b \\ & + \ \left(x^2 - \xi_{j-n}^{(2)} - (x-\xi_{j-n})[j,j-n]\xi^{(2)}\right)\cdot \tilde{d}_2, \\ \tilde{d}_2 := \quad & \frac{[j,j-n]b - [k,j-n]b}{[j,j-n]\xi^{(2)} - [k,j-n]\xi^{(2)}}, \end{aligned}$$

where we use the divided difference operators $[i,j]u := (u_i - u_j)/(\xi_i - \xi_j)$ with respect to the standard Greville abscissae and took some k midway between $j-n$ and j. The control points $c_i(b)$ to be checked are

$$\begin{aligned} c_i(b) = \quad & b_{j-n} + (\xi_i - \xi_{j-n})[j, j-n]b \\ + \quad & \left(\xi_i^{(2)} - \xi_{j-n}^{(2)} - (\xi_i - \xi_{j-n})[j, j-n]\xi^{(2)}\right) \cdot \tilde{d}_2. \end{aligned}$$

5 Minimal error bounds in L_2

Consider an approximation $L(c)$ with c from a subspace $C \subset B$ of Bézier nets, where we have in mind that the net c should be easily computable from b and $L(c)$ should be a simple approximation of $L(b)$. Then (14) implies that one should best–approximate the control net b by $c \in C$. Replacing (14) by an estimate in L_2 we get

$$\begin{aligned} \|L(b)(x) - L(c)(x)\| &\leq \|\textstyle\sum_{i\in I(x)} B_i(x)(b_i - c_i)\| \\ &\leq \textstyle\sum_{i\in I(x)} |B_i(x)| \|b_i - c_i\| \\ &\leq \left(\textstyle\sum_{i\in I(x)} |B_i(x)|^2\right)^{1/2} \left(\textstyle\sum_{i\in I(x)} \|b_i - c_i\|^2\right)^{1/2} \\ &\leq 1 \cdot \left(\textstyle\sum_{i\in I(x)} \|b_i - c_i\|^2\right)^{1/2} =: \|b - c\|_{2,I(x)} \end{aligned}$$

because of (1). Now one can minimize $\|b-c\|_{2,I(x)}$ or $\|b-c\|_{2,I}$ over all $c \in C$, and the result can be written in the form of a linear operator $C(x)$ or $C : B \to B$, which can be calculated beforehand and implemented as a simple matrix operation.

Specific examples of this type can easily be generated from formulae for degree elevation or knot insertion. The nets c then are linear functions of very coarse control nets d which are refined to the basis (1) to yield the nets c. Then the best approximation of b by such c can be described by a linear transformation $D : d \mapsto c$, which can be explicitly calculated by classical L_2 minimization techniques.

Example 5.1 A linear curve with control points d_0, d_1, when raised to degree n, has control points $c_i = d_0 + \frac{i}{n}(d_1 - d_0)$, $1 \leq i \leq n$. The best L_2 approximation of a control net b_i by such a simple control net in the sense defined above is given by

$$\begin{aligned} d_0 \ &:= \ \bar{b} - \tilde{b}/2 \ := \ \tfrac{1}{n+1}\textstyle\sum_{i=0}^n b_i - \tfrac{1}{2}\tilde{b}, \\ d_1 - d_0 \ &:= \qquad \tilde{b} \quad\ := \ \tfrac{12n}{n+2}\left(\tfrac{1}{n(n+1)}\textstyle\sum_{i=0}^n i b_i - \tfrac{1}{2}\bar{b}\right). \end{aligned}$$

6 Rational curves and surfaces

Any set (1) of functions yields a projectively invariant family of rational functions

$$\rho_i^{(\gamma)}(x) := B_i(x)\gamma_i / \sum_{j\in I} B_j(x)\gamma_j, \ i \in I(x), \ x \in \Omega,$$

where $\gamma = \{\gamma_i\}_{i\in I}$ is a vector of positive weights. We assume linear precision of the B_i in the form (7) and get

$$x\left(\sum_{i\in I(x)} B_i(x)\gamma_i\right)^{-1} = \sum_{i\in I(x)} \frac{\xi_i}{\gamma_i}\rho_i^{(\gamma)}(x)$$

for the rational counterpart. Now let $c \in R$ be a vector and $D : I\!R^s \to R$ be a linear transformation. Then the rational map

$$T(x) = c + Dx \cdot \left(\sum_{i\in I(x)} B_i(x)\gamma_i\right)^{-1}$$

is a generalization of the affine maps of section 3, and we get

$$\| \sum_{i\in I(x)} b_i\rho_i^{(\gamma)}(x) - T(x)\| \le \max_{i\in I(x)} \|b_i - c - D\xi_i \cdot \gamma_i^{-1}\|.$$

The rational representation can therefore be approximated by the rational mapping T as well as the control net can be approximated by an affine map of the points $\xi_i\gamma_i^{-1}$. Examples for rational curves and surfaces, including splines, can easily be constructed using the techniques of the previous examples. Note that this case is also covered by Theorem 3.1.

Another approach to rational representations uses homogeneous coordinates in $R \times I\!R$ and applies former results. Clearly, linear precision (10) implies

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \sum_{i\in I(x)} B_i(x)\begin{pmatrix} \xi_i \\ 1 \end{pmatrix},$$

and a control net $\begin{pmatrix} b_i\gamma_i \\ \gamma_i \end{pmatrix} \approx \begin{pmatrix} b_i \\ 1 \end{pmatrix}$ yields the estimate

$$\left\| \sum_{i\in I(x)} B_i(x)\begin{pmatrix} b_i\gamma_i \\ \gamma_i \end{pmatrix} - \begin{pmatrix} T_1(x) \\ T_2(x) \end{pmatrix}\right\|$$

$$\le \max_{i\in I(x)} \left\|\begin{pmatrix} b_i\gamma_i \\ \gamma_i \end{pmatrix} - \begin{pmatrix} T_1(\xi_i) \\ T_2(\xi_i) \end{pmatrix}\right\|$$

for affine maps $T_1 : I\!R^s \to R$, $T_2 : I\!R^s \to I\!R$. If both transformations are chosen to give quadratic order, the resulting order for the rational representation is quadratic, as long as both the γ_i and the $T_2(\xi_i)$ are bounded from below by a positive constant.

Example 6.1 For rational curves of degree n over $[\alpha, \beta]$ the results of Example 1.1 carry over when we use

$$\begin{aligned} T_2(x) &= \gamma_0 &+& \frac{x-\alpha}{\beta-\alpha}(\gamma_n - \gamma_0), \\ T_1(x) &= b_0\gamma_0 &+& \frac{x-\alpha}{\beta-\alpha}(b_n\gamma_n - b_0\gamma_0), \\ \xi_i &= \alpha &+& \frac{i}{n}(\beta-\alpha). \end{aligned}$$

If $\gamma_i \geq \gamma > 0$, we have $T_2(\xi_i) \geq \gamma$, and the rational representation has the error bound

$$\left\| \frac{\sum_{i=0}^{n} B_i(x) b_i \gamma_i}{\sum_{i=0}^{n} B_i(x) \gamma_i} - \frac{T_1(x)}{T_2(x)} \right\|$$

$$\leq \quad \frac{1}{\gamma} \max_{1 \leq i \leq n-1} \| b_i \gamma_i - b_0 \gamma_0 - \tfrac{i}{n}(b_n \gamma_n - b_0 \gamma_0) \|$$

$$+ \quad \frac{1}{\gamma} \left\| \frac{T_1(x)}{T_2(x)} \right\| \max_{1 \leq i \leq n-1} \| \gamma_i - \gamma_0 - \tfrac{i}{n}(\gamma_n - \gamma_0) \|,$$

where we applied

$$\frac{a}{b} - \frac{r}{s} = \frac{a-r}{b} + \frac{r}{s} \cdot \frac{s-b}{b}.$$

This technique can easily be adapted to more general rational representations (surfaces and splines).

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