H^2–CONVERGENCE OF LEAST-SQUARES KERNEL COLLOCATION METHODS

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Abstract. The strong-form asymmetric kernel-based collocation method, commonly referred to as the Kansa method, is easy to implement and hence is widely used for solving engineering problems and partial differential equations despite the lack of theoretical support. The simple least-squares (LS) formulation, on the other hand, makes the study of its solvability and convergence rather nontrivial. In this paper, we focus on general second order linear elliptic differential equations in $\Omega \subset \mathbb{R}^d$ under Dirichlet boundary conditions. With kernels that reproduce $H^m(\Omega)$ and some smoothness assumptions on the solution, we provide conditions for a constrained least-squares method and a class of weighted least-squares algorithms to be convergent. Theoretically, for $\max(2, \lceil (d+1)/2 \rceil) \le \nu \le m$, we identify some $H^\nu(\Omega)$ convergent LS formulations that have an optimal error behavior like $h^{m-\nu}$. For $d \le 3$, the proposed methods are optimal in $H^2(\Omega)$. We demonstrate the effects of various collocation settings on the respective convergence rates.

Key words. Meshfree method, radial basis function, Kansa method, overdetermined collocation.

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1. Introduction.

Mathematical models or differential equations are meaningful only if they can somehow mirror the overly complicated real world. Similarly, numerical methods are useful only if they can produce approximations guaranteed to converge to the outcome that the mathematical model predicts. It could take tens of years for some good numerical strategies to mature and become a well-established class of numerical methods with a complete and rigid theoretical framework. Take the finite element method as an example. It waited for a quarter of a century [5] to get its rigorous mathematical foundation. This paper aims to continue our theoretical contributions to the unsymmetric radial basis function (RBF) collocation method, which is also known as the Kansa method in the community and we shall use this name throughout this paper.

To quickly overview the development of the Kansa method and its connection to the radial basis function scattered data interpolation problem, let us look at some of its cornerstones [7, 8, 37]. An RBF is a smooth scalar function $\phi : \mathbb{R}^+ \to \mathbb{R}$, which usually is induced from a kernel function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ in today's applications, such that the interpolant of an interpolation problem is given as a linear combination

$$u = \sum_{j=1}^{n_Z} \lambda_j \phi(\| \cdot - z_j \|_2) = \sum_{j=1}^{n_Z} \lambda_j \Phi(\cdot, z_j), \tag{1.1}$$

of shifted RBFs in which the set $Z = \{z_1, \dots, z_{n_Z}\}$ contains *trial centers* that specify the shifts of the kernel function in the expansion. Dealing with scaling has been another huge topic in Kansa methods [12, 18, 35] for a decade, but we will ignore this point for the sake of brevity.

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Impressed by the meshfree nature, simplicity to program, dimension independence, and arbitrarily high convergence rates in interpolations, E.J. Kansa [16, 17] proposed to modify the RBF interpolation method to solve partial differential equations (PDEs) in the early 90s. Using the same RBF expansion (1.1), Kansa imposed strong-form collocation conditions instead of interpolation conditions for identifying the unknown coefficients. Consider a PDE given by $\mathcal{L}u = f$ in Ω and $\mathcal{B}u = g$ on $\Gamma = \partial\Omega$. The Kansa method collocates the PDE at the trial centers Z to yield exactly n_Z conditions:

$$\begin{array}{rcll} \mathcal{L}u(z_i) & = & \sum \lambda_j \mathcal{L}\phi(\|z_i - z_j\|), & \text{for } z_i \in Z \cap \Omega, \\ \mathcal{B}u(z_i) & = & \sum \lambda_j \mathcal{B}\phi(\|z_i - z_j\|), & \text{for } z_i \in Z \cap \Gamma, \end{array} \tag{1.2}$$

for identifying the unknown λ_j or equivalently, a numerical approximation to u from the *trial space*

$$\mathcal{U}_Z = \mathcal{U}_{Z,\Omega,\Phi} := \text{span}\{\Phi(\cdot\,, z_j) : \, z_j \in Z\}. \tag{1.3}$$

This approach requires no re-formulation of the PDE and no triangularization. As long as one knows how to program for an interpolation problem, it only takes minutes to understand and code up something for the Kansa method. Since invented, the Kansa method has been widely used in vast numbers of applications in physics and engineering [3, 19, 22, 31].

Since the differential and boundary operators of a PDE are independently applied to yield different rows of the final linear system of equations, it is easy to see why any Kansa system matrix is unsymmetric. While this has some implications for the choice of linear solvers, the unsymmetric matrix places the Kansa method far away from the approximation theories from which RBFs interpolation theories were built. Though the technique introduced by Kansa is very successful in a large variety of applications in Engineerings and Science, there were no proven results about it for over 10 years. After many unsuccessful attempts to establish such a foundation, Hon and Schaback [14] showed in 2001 that there are extremely rare cases where the original approach can fail because the underlying linear system can be singular. This puts an end to all attempts to prove stability of the Kansa method in general. One workaround is to apply *symmetric collocation* [6, 9] that mimics scattered Hermite interpolation. While the Kansa trial space basis in (1.1) is independent of the collocation, the symmetric method takes a basis that is itself dependent on the collocation. This approach yields positive definite symmetric system matrices [39] at the expense of higher smoothness requirements and less stability. On the positive side, symmetric collocation can be proven [33] to be error-optimal, because it is a pointwise optimal recovery of the solution from discrete input data.

The lack of theory for the Kansa method remained the same until 2006, when we provided the first solvability results for an extended Kansa method. In order to ensure solvability, *overtesting* is applied. Keeping the trial space (1.3) based on a set Z of *trial centers*, the standard Kansa system (1.2) is modified by taking another, but usually larger discrete set X of collocation points that is sufficiently fine relative to the set Z of trial centers. Readers are referred to the original articles [26] and an extension [32] to the corresponding weak problems for details. In 2008, we had a partial answer to the convergence of an overdetermined Kansa formulation [27]. Our analysis was carried out based on the continuous and discrete maximum norms. We showed that the ℓ_∞-minimizer of a residual functional converges to the exact solution at the optimal speed, i.e. with the same convergence rate as the interpolant converges to the exact solution. From then on, we attempted to extend the theories to the least-squares (LS) minimizer [20] and numerically verified in extended precision arithmetic that the LS-minimizer also converges at the optimal rate [21]. Recently, in [34], we gave an L^∞ convergence rate of $m-2-d/2$ for an overdetermined Kansa method in H^m for $m > 2+d/2$. In this study, we continue to work on the overdetermined Kansa method and concentrate on the popular LS solution. In Section 2, we will provide all the necessary assumptions and prove error estimates for a constrained least-squares (CLS) and a class of *weighted least-squares* (WLS) formulations. The convergence for the CLS formulation will then be given in Section 3. In Sections 4 and 5, the theory for WLS formulations in two trial spaces will be given. Lastly, we will numerically verify the accuracy and convergence rates of some proven convergent formulations in Section 6.

2. Notations, assumptions and main theorems. Throughout the paper, the notation C will be reserved for generic constants whose subscripts indicate the dependencies of the constant.

We consider a general second order elliptic differential equation in some bounded domain $\Omega \subset \mathbb{R}^d$ subject to the Dirichlet boundary condition on $\Gamma = \partial\Omega$:

$$\mathcal{L}u = f \text{ in } \Omega \qquad \text{and} \qquad u = g \text{ on } \Gamma, \tag{2.1}$$

where

$$\begin{aligned}\mathcal{L}u(x) \;:=\; & \sum_{i,j=1}^{d} \frac{\partial}{\partial x^j}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u(x)\right) + \sum_{j=1}^{d}\frac{\partial}{\partial x^j}\left(b^j(x)u(x)\right) \\ & + \sum_{i=1}^{d} c^i(x)\frac{\partial}{\partial x^i}u(x) + d(x)u(x).\end{aligned} \tag{2.2}$$

The Sobolev regularity of the true solution will be denoted by m, and we will work in Hilbert spaces $H^k(\Omega)$ and $H^{k-1/2}(\Gamma)$ with standard Sobolev norms $\|u\|^2_{k,\Omega} = \sum_{|\alpha|\le k}\|D^\alpha u\|^2_{L^2(\Omega)}$ and $\|u\|_{k-1/2,\Gamma}$ defined via atlas [13] respectively, for $k \le m$.

Assumption 2.1 (Smoothness of domain). *We assume that the bounded domain Ω has a piecewise C^m–boundary Γ so that Ω is Lipschitz continuous and satisfies an interior cone condition.* □

Now the trace theorem [38] can be applied and we can define a trace operator:

$$\mathcal{T} : H^m(\Omega) \to H^{m-1/2}(\Gamma) \text{ such that } \mathcal{T}u = u_{|\Gamma} \text{ for all } u \in C^m(\bar{\Omega}),$$

for $m > 1/2$, with a continuous right-inverse linear extension operator $\mathcal{E}$ such that

$$\mathcal{T} \circ \mathcal{E}g = g \text{ for all } g \in H^{m-1/2}(\Gamma).$$

The smoothness assumption also allows a partition of unity of the boundary [38], each part of which can be mapped to the unit ball in $\mathbb{R}^{d-1}$ by a C^m–diffeomorphism. This allows us to define Sobolev norms on Γ and apply some Sobolev inequalities (i.e., kernel independent ones).

Assumption 2.2 (Differential operator and solution). *Assume that $\mathcal{L}$ as in (2.2) is a strongly elliptic operator with coefficients belonging to $W^m_\infty(\Omega)$. Also, we assume that the functions f and g are smooth enough to admit a* classical *solution*

$u^* \in H^m(\Omega)$. □

By results in [11], $\mathcal{L}$ is a bounded operator from $H^m(\Omega)$ to $H^{m-2}(\Omega)$ with

$$\|\mathcal{L}u\|_{m-k-2,\Omega} \le C_{\Omega,\mathcal{L}} \|u\|_{m-k,\Omega},\ 0 \le k \le m-2,\ k \in \mathbb{N}, \tag{2.3}$$

for all $u \in H^m(\Omega)$. Moreover, the following boundary regularity estimate [15] holds:

$$\|u\|_{k+2,\Omega} \le C_{\Omega,\mathcal{L},k} \left(\|\mathcal{L}u\|_{k,\Omega} + \|u\|_{k+1+1/2,\Gamma} \right),\ 0 \le k \le m-2, \tag{2.4}$$

for all $u \in H^m(\Omega)$ with $C_{\Omega,\mathcal{L},k}$ depending on Ω, the ellipticity constant of $\mathcal{L}$, and $k \ge 0$.

Assumption 2.3 (Kernel). *Assume Φ_m is a reproducing kernel of $H^m(\Omega)$ for some integer $m > \max(2, \lceil (d+1)/2 \rceil) + d/2$. More precisely, we use a symmetric positive definite kernel Φ_m on $\mathbb{R}^d$ with smoothness m that satisfies*

$$c_{\Phi_m} (1 + \|\omega\|_2^2)^{-m} \le \widehat{\Phi_m}(\omega) \le C_{\Phi_m} (1 + \|\omega\|_2^2)^{-m} \quad \textit{for all } \omega \in \mathbb{R}^d, \tag{2.5}$$

for two constants $0 < c_{\Phi_m} \le C_{\Phi_m}$. □

For any $m > d/2$, its native space $\mathcal{N}_{\Omega,\Phi_m}$ on $\mathbb{R}^d$ [2, 37] is norm-equivalent to $H^m(\mathbb{R}^d)$. This includes the standard *Whittle-Matérn-Sobolev* kernel with exact Fourier transform $(1 + \|\omega\|_2^2)^{-m}$ that takes the form

$$\Phi_m(x) := \|x\|_2^{m-d/2} \mathcal{K}_{m-d/2}(\|x\|_2) \quad \text{for all } x \in \mathbb{R}^d, \tag{2.6}$$

where $\mathcal{K}_\nu$ is the Bessel functions of the second kind. The compactly supported piecewise polynomial Wendland functions [36] are another examples of kernels satisfying (2.5).

Let $\mathcal{S}$ be any discrete set of $n_{\mathcal{S}}$ points in Ω. For describing the denseness of $\mathcal{S} \subset \Omega$, its *fill distance* for fixed Ω and *separation distance* are defined as

$$h_{\mathcal{S}} := \sup_{\zeta \in \Omega} \min_{z \in \mathcal{S}} \|z - \zeta\|_{\ell_2(\mathbb{R}^d)} \quad \text{and} \quad q_{\mathcal{S}} := \frac{1}{2} \min_{\substack{z_i, z_j \in \mathcal{S} \\ z_i \ne z_j}} \|z_i - z_j\|_{\ell_2(\mathbb{R}^d)},$$

respectively, and the quantity $h_{\mathcal{S}}/q_{\mathcal{S}} =: \rho_{\mathcal{S}}$ is commonly referred as its *mesh ratio*. For any $u \in H^m(\Omega)$, we define discrete norms on $\mathcal{S}$ by $\|u\|_{\mathcal{S}} = \|u\|_{0,\mathcal{S}} = \|u\|_{\ell_2(\mathcal{S})}$, for $0 \le k < m - d/2$. The same notations will also be used to denote discrete norms on boundary for any discrete set $\mathcal{S} \subset \Gamma$.

Assumption 2.4 (Data points and trial space). *Let $Z = \{z_1, \dots, z_{n_Z}\}$ be a discrete set of* trial centers *in Ω. Let $X = \{x_1, \dots, x_{n_X}\}$ be a discrete set of* PDE collocation points *in Ω and $Y = \{y_1, \dots, y_{n_Y}\}$ be a set of* boundary collocation points *on Γ. We assume the set Z of trial centers is sufficiently dense with respect to Ω, Φ, and $\mathcal{L}$ but independent of the solution. In analogy to* (1.3)*, but now with translation-invariance, we define the finite-dimensional trial space as*

$$\mathcal{U}_Z = \mathcal{U}_{Z,\Omega,\Phi_m} := \operatorname{span}\{\Phi_m(\cdot - z_j) : z_j \in Z\} \subset \mathcal{N}_{\Omega,\Phi_m}.$$

For any u in the native space $\mathcal{N}_{\Omega,\Phi_m}$ of Φ_m, we denote $I_Z u = I_{Z,\Phi_m} u$ to be the unique interpolant of u on Z from the trial space $\mathcal{U}_Z$. We assume all point sets are quasi-uniform *and maintain* linear ratios of oversampling *as they were refined. That is, there exist constants $\rho_{\mathcal{S}} \ge 1$ such that*

$$q_{\mathcal{S}} \le h_{\mathcal{S}} \le \rho_{\mathcal{S}} q_{\mathcal{S}} \quad \textit{for } \mathcal{S} \in \{X, Y, Z\} \tag{2.7}$$

and ratios of oversampling $\gamma_{\mathcal{S}} \geq 1$ *such that*

$$h_Z \leq \gamma_{\mathcal{S}} h_{\mathcal{S}} \quad \text{for } \mathcal{S} \in \{X, Y\} \tag{2.8}$$

holds for all admissible sets of data points. □

Note that the sets X and Y of collocation points together have to be as dense as the trial centers in Z to ensure stability. This paper will provide rigid sufficient conditions for this.

Imposing strong testing on (2.1) at collocation points in X and Y yields $n_X+n_Y > n_Z$ conditions, from which one can hopefully identify a numerical approximation from some trial spaces. The following theorems summarize our convergence results for three possible least-squares alternatives. The first concerns the case where we enlarge the set Z of trial points by adding the set Y of boundary collocation points to it. Then, we can keep the numerical solution to be exact on Y, and we add this as a constraint.

THEOREM 2.5 (Constrained least squares (CLS)). *Suppose the Assumptions 2.1 to 2.4 hold. Let* $u^* \in H^m(\Omega)$ *denote the exact solution of the elliptic PDE* (2.1), *and let an integer* ν *with* $\max(2, \lceil (d+1)/2 \rceil) \leq \nu < m - d/2$ *be given. In addition, suppose the sets* X, Y *and* $Z \cup Y$ *satisfy conditions* (2.8) *with* $Z \cup Y$ *instead of* Z *and* (3.2). *Let* $u^{CLS}_{X,Y} \in \mathcal{U}_{Z\cup Y}$ *be the CLS solution defined as*

$$u^{CLS}_{X,Y} := \underset{u \in \mathcal{U}_{Z\cup Y}}{\arg\inf} \|\mathcal{L}u - f\|^2_X \quad \text{subject to } u_{|Y} = g_{|Y}. \tag{2.9}$$

Then, the error estimates

$$\|u^{CLS}_{X,Y} - u^*\|_{\nu,\Omega} \leq C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X,\gamma_X} h^{m-d/2-\nu}_{Z\cup Y} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + \tfrac{d}{2},$$

and

$$\|u^{CLS}_{X,Y} - u^*\|_{\nu,\Omega} \leq C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X,\gamma_X} h^{m-\nu}_{Z\cup Y} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + 1 + \tfrac{d}{2},$$

hold for some constants that depend only on Ω, Φ_m, $\mathcal{L}$, ν, *mesh ratio* ρ_X *and ratio of oversampling* γ_X.

The next case does not require exactness on Y but still keeps $Z \cup Y$ as the set of trial centers.

THEOREM 2.6 (Weighted least squares (WLS)). *Suppose all the assumptions in Theorem 2.5 hold. We further assume that* $h_X \leq h_Y < 1$. *For any* $\theta \geq 0$ *and integer* ν *with* $\max(2, \lceil (d+1)/2 \rceil) \leq \nu < m - d/2$, *let* $u^{WLS,\theta}_{X,Y,Z\cup Y} \in \mathcal{U}_{Z\cup Y}$ *be the WLS solution defined as*

$$u^{WLS,\theta}_{X,Y,Z\cup Y} := \underset{u \in \mathcal{U}_{Z\cup Y}}{\arg\inf} \|\mathcal{L}u - f\|^2_X + \left(\frac{h_Y}{h_X}\right)^{(d/2+2-\nu)\theta} h_Y^{-2\theta} \|u - g\|^2_Y. \tag{2.10}$$

Then, the error estimates

$$\|u^{WLS,\theta}_{X,Y,Z\cup Y} - u^*\|_{\nu,\Omega} \leq C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X,\gamma_X,\gamma_Y} h^{m-d/2-\nu-(2-\theta)_+}_{Z\cup Y} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + \tfrac{d}{2},$$

and

$$\|u^{WLS,\theta}_{X,Y,Z\cup Y} - u^*\|_{\nu,\Omega} \leq C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X,\gamma_X,\gamma_Y} h^{m-\nu-(2-\theta)_+}_{Z\cup Y} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + 1 + \tfrac{d}{2},$$

hold for some constants that depend only on Ω*,* Φ_m*,* $\mathcal{L}$*,* ν*,* ρ_X*, and ratios of oversampling* γ_X *and* γ_Y*.*

Due to the presence of h_Y and $q_{Z\cup Y}$, condition (3.2) in Theorems 2.5 and 2.6 is satisfiable only when the generic constant there, which depends on the domain, differential operator, kernel and the value of ν, is small. Since there is very little known about the magnitude of such a constant, it is not straightforward to verify if condition (3.2) is being met. If it is unsatisfied, the theoretical error bounds of both CLS and WLS will stagnate at a constant. In Section 6, we will numerically confirm that both formulations converge. To be more general, we go back to the case where Z is the set of trial nodes, independent of X and Y. Instead of (3.2), this allows us to use another condition (3.3), whose satisfiability is trivial by using sufficiently small h_X and h_Y with respect to q_Z.

Theorem 2.7 (WLS in a smaller trial space). *Suppose the Assumptions 2.1 to 2.4 hold. Moreover, the sets* X*,* Y *and* Z *satisfy conditions* (2.8) *and* (3.3)*. For any* $0 \le \theta \le 2$*, suppose the trial space of the weighted least-squares approximation* (2.10) *is restricted to* $u^{WLS,\theta}_{X,Y,Z} \in \mathcal{U}_Z$ *instead of* $\mathcal{U}_{Z\cup Y}$*. Then, the error estimates in Theorem 2.6 with* $h_{Z\cup Y}$ *replaced by* h_Z*, i.e.,*

$$\|u^{WLS,\theta}_{X,Y,Z} - u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi,\mathcal{L},\nu,\rho_X,\gamma_X,\gamma_Y} h_Z^{m-d/2-\nu-(2-\theta)_+} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + \tfrac{d}{2},$$

and

$$\|u^{WLS,\theta}_{X,Y,Z} - u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi,\mathcal{L},\nu,\rho_X,\rho_Y,\rho_Z,\gamma_X,\gamma_Y} h_Z^{m-\nu-(2-\theta)_+} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + 1 + \tfrac{d}{2},$$

hold for some constants that also depend on the mesh ratios ρ_Y *and* ρ_Z*.*

3. Optimal convergence rates for CLS. We first prove some necessary inequalities essential to our proofs.

Lemma 3.1 (Sampling Inequality of fractional order). *Suppose* $\Omega \subset \mathbb{R}^d$ *is a bounded Lipschitz domain with a piecewise* C^m*–boundary. Then, there exists some constant that depends only on* Ω*,* m *and* s *such that the followings hold:*

$$\|u\|_{s,\Omega} \le C_{\Omega,m,s} \left(h_X^{m-s} \|u\|_{m,\Omega} + h_X^{d/2-s} \|u\|_X \right) \quad \text{for } 0 \le s \le m,$$

and

$$\|u\|_{s-\frac{1}{2},\Gamma} \le C_{\Omega,m,s} \left(h_Y^{m-s} \|u\|_{m,\Omega} + h_Y^{d/2-s} \|u\|_Y \right) \quad \text{for } \tfrac{1}{2} \le s \le m,$$

for any $u \in H^m(\Omega)$ *with* $m > d/2$ *and any discrete sets* $X \subset \Omega$ *and* $Y \subset \Gamma$ *with sufficiently small mesh norm* h_X *and* h_Y*.*

Proof. The interior sampling inequality for $X \subset \Omega$, which only requires Ω be a bounded Lipschitz domain, is a special case of a sampling inequality in [1]. Applying the interior sampling inequality to the union of unit balls in $\mathbb{R}^{d-1}$, which are images of the partition of unity of Γ under the C^m–diffeomorphism in Assumption 2.1, yields

$$\|u\|_{s-1/2,\Gamma} \le C \left(h_Y^{(m-1/2)-(s-1/2)} \|u\|_{m-1/2,\Gamma} + h_Y^{(d-1)/2-(s-1/2)} \|u\|_Y \right),$$

for all $1/2 \le s \le m$. Finally, by applying the trace theorem, the desired boundary sampling inequality is obtained. □

Lemma 3.2 (Inverse Inequality). *Let a kernel $\Phi_m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying* (2.5) *with smoothness $m > d/2$ be given. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain satisfying an interior cone condition. Assume $d/2 < \nu \le m$. Then there is a constant depending only on Ω, Φ_m, and ν such that*

$$\|u\|_{m,\Omega} \le C_{\Omega,\Phi_m,\nu}\, q_Z^{-m+\nu} \|u\|_{\nu,\Omega} \quad \text{for all } \ u \in \mathcal{U}_Z \tag{3.1}$$

holds in the trial space of Φ_m and all finite sets $Z \subset \Omega$ with separation distance q_Z.

Proof. Let $u_\nu = I_{\Phi_\nu,Z} u_m$ be the interpolant of $u_m \in \mathcal{U}_{Z,\Omega,\Phi_m}$ on Z in the trial space $\mathcal{U}_{Z,\Omega,\Phi_\nu}$. By [30, Cor.3.5], there exists a Paley-Wiener band-limited function

$$f_\sigma \in \mathcal{B}_\sigma = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp}\hat{f} \subseteq B(0,\sigma)\}$$

for some $\sigma = C_{d,\nu} q_Z^{-1}$ such that $f_\sigma|_Z = u_\nu|_Z = u_m|_Z$ and $\|f_\sigma\|_{\nu,\mathbb{R}^d} \le C_{\Omega,\nu}\|u_\nu\|_{\nu,\Omega}$. Since f_σ is analytic, we have $f_\sigma \in H^m(\mathbb{R}^d)$ and hence

$$\|u_m\|_{m,\Omega} \le \|u_m - f_\sigma\|_{m,\Omega} + \|f_\sigma\|_{m,\Omega} \le C_{\Phi_m,\Omega}\|f_\sigma\|_{m,\Omega} + \|f_\sigma\|_{m,\Omega}.$$

The last inequality is based on the fact that $f_\sigma|_Z = u_m|_Z$ and therefore $u_m = I_{\Phi_m,Z} f_\sigma$ is the unique interpolant in $\mathcal{U}_{Z,\Omega,\Phi_m}$ of f_σ. By [37, Lem.10.24], we have

$$(I_{\Phi_m,Z} f_\sigma - f_\sigma,\ I_{\Phi_m,Z} f_\sigma)_{\mathcal{N}_{\Phi_m}(\Omega)} = 0,$$

which proves that the functions $u_{m,f} - f_\sigma$ and $u_{m,f}$ are mutually orthogonal:

$$\|u_{m,f} - f_\sigma\|^2_{\mathcal{N}_{\Phi_m}(\Omega)} + \|u_{m,f}\|^2_{\mathcal{N}_{\Phi_m}(\Omega)} = \|f_\sigma\|^2_{\mathcal{N}_{\Phi_m}(\Omega)}.$$

Hence, $\|u_m - f_\sigma\|_{\mathcal{N}_{\Phi_m}(\Omega)} \le \|f_\sigma\|_{\mathcal{N}_{\Phi_m}(\Omega)}$. By the norm equivalence between $\mathcal{N}_{\Phi_m}(\Omega)$ and $H^m(\Omega)$, we have

$$\|u_m - f_\sigma\|_{m,\Omega} \le C_{\Phi_m,\Omega}\|f_\sigma\|_{m,\Omega}.$$

Using the Bernstein inequality for band-limited functions, e.g.,

$$\|D^\alpha f\|_{L^2(\mathbb{R}^d)} \le \sigma^{|\alpha|}\|f\|_{L^2(\mathbb{R}^d)} \ \text{ for all } \ f \in \mathcal{B}_\sigma,$$

we obtain

$$\begin{aligned}
\|u_m\|_{m,\Omega} &\le (C_{\Phi_m,\Omega} + 1)\|f_\sigma\|_{m,\Omega} \\
&\le (C_{\Phi_m,\Omega} + 1)\|f_\sigma\|_{m,\mathbb{R}^d} \\
&\le (C_{\Phi_m,\Omega} + 1) C_{d,\nu} q_Z^{-m+\nu}\|f_\sigma\|_{\nu,\mathbb{R}^d} \\
&\le (C_{\Phi_m,\Omega} + 1) C_{d,\nu} C_{\Omega,\nu} q_Z^{-m+\nu}\|u_\nu\|_{\nu,\Omega}.
\end{aligned}$$

Finally, $\|u_\nu\|_{\nu,\Omega} \le C_{\Phi_\nu,\Omega}\|u_m\|_{\nu,\Omega}$ holds because u_ν interpolates u_m. □

Lemma 3.3 (Stability). *Let ν be an integer $\nu \ge 2$ and $\nu > d/2$. Let a kernel Φ_m as in* (2.5) *with $m \ge \nu$ be given. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain satisfying an interior cone condition. If the elliptic operator $\mathcal{L}$ satisfies all assumptions to allow regularity* (2.4)*, then there exists a constant depending only on Ω, Φ_m, $\mathcal{L}$ and ν such that*

$$\|u\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu}\left(h_X^{d/2+2-\nu}\|\mathcal{L}u\|_X + h_Y^{d/2-\nu}\|u\|_Y\right)$$

holds in two circumstances:

- *for all $u \in \mathcal{U}_{Z\cup Y}$ under the condition*

$$C_{\Omega,\Phi_m,\mathcal{L},\nu}(h_X^{m-\nu}+h_Y^{m-\nu})q_{Z\cup Y}^{-m+\nu}<\frac{1}{2}, \tag{3.2}$$

- *or for all $u \in \mathcal{U}_Z$ under the condition*

$$C_{\Omega,\Phi_m,\mathcal{L},\nu}(h_X^{m-\nu}+h_Y^{m-\nu})q_{Z}^{-m+\nu}<\frac{1}{2}, \tag{3.3}$$

for any finite sets $X \subset \Omega$ and $Y \subset \Gamma$.

Proof. For $0 \le k \le m-2$, we apply the first inequality of Lemma 3.1 (for $u = \mathcal{L}u$, $s = k$, $m = m-2$) to get

$$\|\mathcal{L}u\|_{k,\Omega} \le C_{\Omega,m-2,k}\left(h_X^{m-2-k}\|\mathcal{L}u\|_{m-2,\Omega}+h_X^{d/2-k}\|\mathcal{L}u\|_X\right) \text{ for all } u\in H^m(\Omega)$$

and, by (2.3),

$$\|\mathcal{L}u\|_{k,\Omega} \le C_{\Omega,m-2,k}\left(h_X^{m-2-k}\|u\|_{m,\Omega}+h_X^{d/2-k}\|\mathcal{L}u\|_X\right) \text{ for all } u\in H^m(\Omega).$$

Using the second inequality of Lemma 3.1 (for $s = k+2, m$), we get

$$\|u\|_{k+1+1/2,\Gamma} \le C_{\Omega,m,2}\left(h_Y^{m-2-k}\|u\|_{m,\Omega}+h_Y^{d/2-2-k}\|u\|_Y\right) \text{ for all } u\in H^m(\Omega)$$

and this combines with the boundary regularity estimate (2.4) into

$$\|u\|_{k+2,\Omega} \le C_{\Omega,\mathcal{L},m}\left((h_X^{m-2-k}+h_Y^{m-2-k})\|u\|_{m,\Omega}+h_X^{d/2-k}\|\mathcal{L}u\|_X+h_Y^{d/2-2-k}\|u\|_Y\right).$$

Up to here, we are still in full Sobolev space. Now we use the inverse inequality in Lemma 3.2, whatever the trial space is. If we only take Z nodes like in the lemma, then

$$\begin{aligned}\|u\|_{k+2,\Omega} \le C_{\Omega,\Phi,\mathcal{L},\nu}\Big((h_X^{m-2-k}+h_Y^{m-2-k})q_Z^{-m+\nu}\|u\|_{\nu,\Omega}\\ +h_X^{d/2-k}\|\mathcal{L}u\|_X+h_Y^{d/2-2-k}\|u\|_Y\Big)\end{aligned}$$

for all $u \in \mathcal{U}_Z$. Setting $k = \nu-2$ with $2 \le \nu \le m$, the stability is

$$\|u\|_{\nu,\Omega} \le C_{\Omega,\mathcal{L},m,\nu}\left(h_X^{d/2+2-\nu}\|\mathcal{L}u\|_X+h_Y^{d/2-\nu}\|u\|_Y\right)$$

for all $u \in \mathcal{U}_Z$ under the condition $C_{\Omega,\Phi,\mathcal{L},\nu}(h_X^{m-\nu}+h_Y^{m-\nu})q_Z^{-m+\nu} < 1/2$. If we now take $Z\cup Y$ nodes, then the same stability holds for all $u \in \mathcal{U}_{Z\cup Y}$ under the condition $C_{\Omega,\Phi,\mathcal{L},\nu}(h_X^{m-\nu}+h_Y^{m-\nu})q_{Z\cup Y}^{-m+\nu} < 1/2$. □

Lemma 3.4 (Consistency). *If the elliptic operator $\mathcal{L}$ satisfies Assumption 2.2, the kernel Φ_m as in* (2.5) *has smoothness order $m > 2+d/2$, and X is quasi-uniform, then there exist some constants depending only on Ω, Φ_m, $\mathcal{L}$ and ρ_X such that*

$$\min_{\substack{v\in\mathcal{U}_{Z\cup Y}\\ v_{|Y}=u^*{}_{|Y}}}\|\mathcal{L}v-\mathcal{L}u^*\|_X \le C_{\Omega,\Phi_m,\mathcal{L},\rho_X}h_X^{-d/2}h_{Z\cup Y}^{m-2-d/2}\|u^*\|_{m,\Omega} \quad \textit{for } m>2+\tfrac{d}{2},$$

$$\text{and} \quad \min_{\substack{v \in \mathcal{U}_{Z\cup Y} \\ v_{|Y} = u^*{}_{|Y}}} \|\mathcal{L}v - \mathcal{L}u^*\|_X \le C_{\Omega,\Phi_m,\mathcal{L},\rho_X} h_X^{-d/2} h_{Z\cup Y}^{m-2} \|u^*\|_{m,\Omega} \quad \text{for } m > 3 + \tfrac{d}{2},$$

hold for any $u^* \in H^m(\Omega)$.

Proof. By comparing the minimizer $v^* \in \mathcal{U}_{Z\cup Y}$ of the optimization problem with the interpolant $I_{Z\cup Y}u^* \in \mathcal{U}_{Z\cup Y}$ that also satisfies the constraints at Y, we turn the problem into an error estimate for radial basis function interpolation:

$$\|\mathcal{L}v^* - \mathcal{L}u^*\|_X \le \|\mathcal{L}I_{Z\cup Y}u^* - \mathcal{L}u^*\|_X.$$

The first error estimate can be derived based on native space error estimates [37, Thm.11.9] and upper bounds of power functions [7, Sec.15.1.2]. For $m > 2 + d/2$, we have

$$\begin{aligned}
\|\mathcal{L}I_{Z\cup Y}u^* - \mathcal{L}u^*\|_X &\le n_X^{1/2} \|\mathcal{L}I_{Z\cup Y}u^* - \mathcal{L}u^*\|_{L^\infty(\Omega)} \\
&\le C_{\Omega,\mathcal{L}} n_X^{1/2} \max_{|\alpha|\le 2} |D^\alpha I_{Z\cup Y}u^* - D^\alpha u^*| \\
&\le C_{\Omega,\mathcal{L}} n_X^{1/2} h_{Z\cup Y}^{m-d/2-2} \|u^*\|_{m,\Omega}.
\end{aligned}$$

Noting that $n_X \le C_\Omega q_X^{-d} \le C_{\Omega,\rho_X} h_X^{-d}$ yields the first estimate. If we employ kernels with a higher smoothness parameter $m > 3 + d/2$, we can use the estimates for functions with scattered zeros. Applying [29, Prop.3.3] to our Hilbert space setting and taking care of the definitions of discrete norms yield the desired error bound. □

To prove Theorem 2.5, suppose Assumptions 2.1–2.4 hold so that all lemmas in this section can be applied. Let $u_{X,Y}^{CLS} \in \mathcal{U}_{Z\cup Y}$ be the CLS approximation of (2.1), defined as in (2.9). Moreover, let $I_{Z\cup Y}u^*$ denote the unique interpolant of the exact solution $u^* \in H^m(\Omega)$ from the trial space $\mathcal{U}_{Z\cup Y} \subset \mathcal{N}_{\Omega,\Phi_m} = H^m(\Omega)$. Assume the condition (3.2) holds; we shall show that the CLS solution converges to the interpolant in $\mathcal{U}_{Z\cup Y}$. Since the stability result in Lemma 3.3 only applies to functions in the trial space, the inequality

$$\|u_{X,Y}^{CLS} - u^*\|_{\nu,\Omega} \le \|u_{X,Y}^{CLS} - I_{Z\cup Y}u^*\|_{\nu,\Omega} + \|I_{Z\cup Y}u^* - u^*\|_{\nu,\Omega},$$

suggests that we can focus on the difference $u_{X,Y}^{CLS} - I_{Z\cup Y}u^* \in \mathcal{U}_{Z\cup Y}$, which has zeros at nodes Y. Using Lemma 3.3, we have

$$\|u_{X,Y}^{CLS} - I_{Z\cup Y}u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu}\big(h_X^{d/2+2-\nu} \|\mathcal{L}u_{X,Y}^{CLS} - \mathcal{L}I_{Z\cup Y}u^*\|_X + 0\big).$$

Applying Lemma 3.4 yields

$$\|u_{X,Y}^{CLS} - u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X}(h_X^{2-\nu} + h_{Z\cup Y}^{2-\nu}) h_{Z\cup Y}^{m-2-d/2} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + d/2,$$

and

$$\|u_{X,Y}^{CLS} - u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X}(h_X^{2-\nu} + h_{Z\cup Y}^{2-\nu}) h_{Z\cup Y}^{m-2} \|u^*\|_{m,\Omega} \quad \text{for } m > \nu + 1 + d/2.$$

For any $\nu > 2$, we use (2.8), i.e., $h_X^{2-\nu} \le \gamma_X^{2-\nu} h_{Z\cup Y}^{2-\nu}$, to complete the proof.

4. Convergence for WLS. Instead of a specific weight, we will consider a class of weighted least-squares formulations by a simple inequality.

LEMMA 4.1. *Let $a, b > 0$, $0 < \epsilon < 1$, and $0 \le \theta \le 2$. Then the inequalities $(\epsilon a + b)^2 \le 2\big(\epsilon^\theta\, a^2 + b^2\big)$ holds.*

PROOF. Consider $0 \le \theta/2 \le 1$. From $(\epsilon a + b)^2 \le 2(\epsilon^2 a^2 + b^2)$ and $\epsilon \le \epsilon^{\theta/2}$, we have $(\epsilon a + b)^2 \le 2(\epsilon^\theta a^2 + b^2)$. □

LEMMA 4.2 (Stability). *Suppose the assumptions in Lemma 3.3 hold under the condition* (3.2) *and we further assume $h_X \le h_Y < 1$. Let*

$$W_\theta := (h_Y/h_X)^{(d/2+2-\nu)\theta} h_Y^{-2\theta}.$$

Then there exists a constant depending only on Ω, Φ_m, $\mathcal{L}$ and ν such that

$$\|u\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu} h_Y^{d/2-\nu} W_\theta^{-\frac{1}{2}} \Big(\|\mathcal{L}u\|_X^2 + W_\theta \|u\|_Y^2\Big)^{\frac{1}{2}} \quad \text{for all } 0 \le \theta \le 2,\ u \in \mathcal{U}_{Z\cup Y}$$

holds for any finite sets $X \subset \Omega$ and $Y \subset \Gamma$.

PROOF. The CLS stability in Lemma 3.3 has to be further modified to suit the need of WLS. With the denseness requirement (3.2), let us start with

$$\|u\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu}(h_X^{d/2+2-\nu}\|\mathcal{L}u\|_X + h_Y^{d/2-\nu}\|u\|_Y) \tag{4.1}$$

for some $\max(2, \lceil (d+1)/2 \rceil) \le \nu < m - d/2$ and all $u \in \mathcal{U}_{Z\cup Y}$. We want to obtain a stability estimate with discrete sum of squares. Rewrite (4.1) as

$$\|u\|_{\nu,\Omega}^2 \le C_{\Omega,\Phi_m,\mathcal{L},\nu} h_Y^{d-2\nu}\big(\epsilon\|\mathcal{L}u\|_X + \|u\|_Y\big)^2 \quad \text{with } \epsilon = (h_X/h_Y)^{d/2+2-\nu} h_Y^2.$$

Note that having $\epsilon < 1$ is a very mild requirement, for example $h_X \le h_Y < 1$, and will not be an obstacle between theories and practice. By Lemma 4.1, we have

$$\begin{aligned}\|u\|_{\nu,\Omega} &\le \Big(C_{\Omega,\Phi_m,\mathcal{L}} h_Y^{d-2\nu}\big(\epsilon^\theta\|\mathcal{L}u\|_X^2 + \|u\|_Y^2\big)\Big)^{1/2} \\ &\le C'_{\Omega,\Phi_m,\mathcal{L}} h_Y^{d/2-\nu}\epsilon^{\theta/2}\big(\|\mathcal{L}u\|_X^2 + \epsilon^{-\theta}\|u\|_Y^2\big)^{1/2},\end{aligned}$$

for any $0 \le \theta \le 2$. Substituting ϵ back yields and we obtain the desired WLS stability after simplification. □

LEMMA 4.3 (Consistency). *For any $W > 0$, define a functional $J_W : H^m(\Omega) \to \mathbb{R}$ by $J_W(u) := \big(\|\mathcal{L}u\|_X^2 + W\|u\|_Y^2\big)^{1/2}$. Suppose the assumptions in Lemma 3.4 hold. Then, the error estimates in Lemma 3.4 also hold if the left-handed sides are replaced by $\min\limits_{v\in\mathcal{U}_{Z\cup Y}} J_W(v - u^*)$.*

PROOF. Again, we compare the minimizer v^* with the interpolant $I_{Z\cup Y}u^*$ in $\mathcal{U}_{Z\cup Y}$:

$$J_W^2(v^* - u^*) \le J_W^2(I_{Z\cup Y}u^* - u^*) = \|\mathcal{L}I_{Z\cup Y}u^* - \mathcal{L}u^*\|_X^2 + W\|I_{Z\cup Y}u^* - u^*\|_Y^2,$$

where the last term vanishes due to the zeros of $I_{Z\cup Y}u^* - u^*$ at Y. □

With both consistency and stability results, we can now prove the convergence of a class of WLS solutions defined by (2.10). By similar arguments used in Section 3, we only need to show that the WLS solution converges to the interpolant $I_{Z\cup Y}u^*$ of the exact solution u^* from the trial space $\mathcal{U}_{Z\cup Y}$. Consider the functional

$$J_{W_\theta}(u) := \Big(\|\mathcal{L}u\|_X^2 + W_\theta\|u\|_Y^2\Big)^{1/2} \quad \text{for } 0 \le \theta \le 2, \tag{4.2}$$

where W_θ is as defined in Lemma 4.2. Applying the results of Lemmas 4.2 and 4.3, we have the WLS solution convergence within the trial space; for simplicity, let τ be $d/2$ if $m > \nu + d/2$ and zero if $m > \nu + 1 + d/2$. Then,

$$\begin{aligned}
\|u^{WLS,\theta}_{X,Y,Z\cup Y} - I_{Z\cup Y}u^*\|_{\nu,\Omega} &\le C_{\Omega,\Phi_m,\mathcal{L},\nu} h_Y^{d/2-\nu} W_\theta^{-1/2} J_{W_\theta}\Big(u^{WLS,\theta}_{X,Y,Z\cup Y} - I_{Z\cup Y}u^*\Big) \\
&\le C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X} h_X^{-(1-\theta/2)d/2-(\nu-2)\theta/2} h_Y^{-(1-\theta/2)(\nu-d/2)} h_{Z\cup Y}^{m-2-\tau}\|u^*\|_{m,\Omega} \\
&\le C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X} \max(\gamma_X,\gamma_Y)^{\theta-\nu} h_{Z\cup Y}^{m-\nu-(2-\theta)-\tau}\|u^*\|_{m,\Omega}.
\end{aligned} \tag{4.3}$$

The last holds under the assumption (2.8) and $\max(\gamma_X,\gamma_Y)^{\theta-\nu} \le C_{\nu,\gamma_X,\gamma_Y}$ for all $0 \le \theta \le 2$. We can compare the difference between the WLS solution and the exact solution to obtain error estimates for any $0 \le \theta \le 2$. Now, we consider $\theta = 2$

$$\|u^{WLS,2}_{X,Y,Z\cup Y} - u^*\|_{\nu,\Omega} \le C_{\Omega,\Phi_m,\mathcal{L},\nu,\rho_X,\gamma_X,\gamma_Y} h_{Z\cup Y}^{m-\nu-\tau}\|u^*\|_{m,\Omega}.$$

The CLS and the above optimal WLS formulation share convergence estimates of the same form. They both match the convergence estimate of the interpolant exactly, that in turn confirms their optimality. To complete proving Theorem 2.6 for $\theta > 2$, we consider the stability for $\theta = 2$ and Lemma 4.2 gives

$$\begin{aligned}
\|u\|_{\nu,\Omega} &\le C_{\Omega,\Phi_m,\mathcal{L}} h_X^{d/2+2-\nu}\Big(\|\mathcal{L}u\|_X^2 + (h_Y/h_X)^{d+4-2\nu} h_Y^{-4}\|u\|_Y^2\Big)^{1/2} \\
&\le C_{\Omega,\Phi_m,\mathcal{L}} h_X^{d/2+2-\nu}(\|\mathcal{L}u\|_X^2 + (h_Y/h_X)^{d+4-2\nu} h_Y^{-2\theta}\|u\|_Y^2)^{1/2},
\end{aligned}$$

for any $\theta \ge 2$ as long as $h_Y < 1$. We extend the definition of functional J_{W_θ} in (4.2) to $\theta \ge 2$ by $W_\theta = (h_Y/h_X)^{d+4-2\nu} h_Y^{-2\theta}$. Then, for any $u \in H^m(\Omega)$, we have

$$J_{\theta_1}(u) \le J_{\theta_2}(u) \quad \text{for } 2 \le \theta_1 \le \theta_2 \le \infty.$$

Since the CLS formulation is equivalent to the WLS with $\theta = \infty$, we have

$$\min_{v\in\mathcal{U}_{Z\cup Y}} J_{W_\theta}(v-u^*) \le J_{W_\theta}(u^{CLS}_{X,Y} - u^*) \le J_{W_\infty}(u^{CLS}_{X,Y} - u^*) \quad \text{for } \theta \ge 2,$$

where the last term is minimal by the definition of CLS solution. Theorem 2.6 can now be concluded based on Theorem 2.5. We have to take (3.2) into account in both theorems.

5. WLS in a smaller trial space. In this section, we will prove Theorem 2.7. To begin, let us return to the proof for WLS consistency (Lemma 4.3) but restrict the approximation in the smaller trial space $\mathcal{U}_Z$, within which the stability result in Lemma 4.2 remains valid. However, we can only compare the minimizer $v^* \in \mathcal{U}_Z$ with the interpolant $I_Z u^* \in \mathcal{U}_Z$ to the exact solution $u^* \in H^m(\Omega)$:

$$\min_{v\in\mathcal{U}_Z} J^2_{W_\theta}(v-u^*) \le J^2_{W_\theta}(I_Z u^* - u^*) = \|\mathcal{L}I_Z u^* - \mathcal{L}u^*\|_X^2 + W_\theta\|I_Z u^* - u^*\|_Y^2.$$

The PDE residual on X is exactly the same as that in the previous section. Without Y in the trial centers to annihilate the boundary collocation, we simply need to identify the extra terms associated to boundary error on Y. Following the ideas in the proof of Lemma 3.4, for $m > \nu + d/2$, we can bound the boundary term by

$$\begin{aligned}\|I_Z u^* - u^*\|_Y &\le n_Y^{1/2}\|I_Z u^* - u^*\|_{L^\infty(\Gamma)} \le n_Y^{1/2}\|I_Z u^* - u^*\|_{L^\infty(\Omega)}\\ &\le C_{\Omega,\Phi_m,\mathcal{L},\rho_Y} h_Y^{(d-1)/2} h_Z^{m-d/2}\|u^*\|_{m,\Omega}.\end{aligned}$$

Hence, the error estimate for WLS on $\mathcal{U}_Z$ contains a non-dominating extra term

$$\begin{aligned}&h_Y^{d/2-2} {W_\theta}^{-1/2}\left({W_\theta}^{1/2} C_{\Omega,\Phi_m,\mathcal{L},\rho_Y} h_Y^{(d-1)/2} h_Z^{m-d/2}\|u^*\|_{m,\Omega}\right)\\ &\quad\le C_{\Omega,\Phi_m,\mathcal{L},\rho_Y} h_Y^{d/2-2} h_Y^{-(d-1)/2} h_Z^{m-d/2}\|u^*\|_{m,\Omega}\\ &\quad\le C_{\Omega,\Phi_m,\mathcal{L},\rho_Y,\gamma_Y} h_Z^{m-d/2-3/2}\|u^*\|_{m,\Omega}. \qquad (5.1)\end{aligned}$$

For the other case when $m > \nu + 1 + d/2$, we want to bound the $\ell_2(Y)$ norm on the boundary by some $\ell_2(\widetilde{Z}\cup Y)$ norm in the domain (like the trace theorem does). For any subset $\widetilde{Z} \subseteq Z$, we have

$$\|I_Z u^* - u^*\|_Y = \|I_Z u^* - u^*\|_{\widetilde{Z}\cup Y} \le C_{\Omega,\Phi_m} n_{\widetilde{Z}\cup Y}^{1/2} \rho_{\widetilde{Z}\cup Y}^{d/2} h_Z^m \|u^*\|_{m,\Omega}.$$

We want to select $\widetilde{Z}$ so that $\rho_{\widetilde{Z}\cup Y}$ can be bounded by the denseness measures of Y and Z. We already assumed Y and Z are quasi-uniform in Assumption 2.4. We also assume Y is sufficiently dense with respect to Ω so that $q_{Y,\Gamma} < q_{Y,\Omega}$ (see [10, Thm.6]).

Consider the following subset that excludes all points in Z that are within distance h_Z to the boundary: $\widetilde{Z} := \Big\{z \in Z \cap \{\zeta \in \Omega \,:\, \operatorname{dist}(\zeta - \Gamma) > h_Z\}\Big\} \subseteq Z$. Then, we have

$$\begin{aligned}\min(q_Z, q_Y) \le q_{\widetilde{Z}\cup Y} \le h_{\widetilde{Z}\cup Y} &\le \Big(\sup_{\zeta\in\Omega_{h_Z}} + \sup_{\zeta\in\Omega\setminus\Omega_{h_Z}}\Big) \min_{z\in\widetilde{Z}\cup Y} \|z-\zeta\|_{\ell_2(\mathbb{R}^d)}\\ &\le h_Z + (h_Z + h_Y).\end{aligned}$$

It is now clear that the set $\widetilde{Z}\cup Y$ is also quasi-uniform with respect to some parameter $\rho_{\widetilde{Z}\cup Y}$ that depends on ρ_Z and ρ_Y. Hence, we can bound $\rho_{\widetilde{Z}\cup Y}$ by some generic constant C_{ρ_Y,ρ_Z}. To control the term $n_{\widetilde{Z}\cup Y}$, consider

$$n_{\widetilde{Z}\cup Y} \le n_Z + n_Y \le C_{\Omega,\rho_Z} h_Z^{-d} + C_{\Omega,\rho_Y} h_Y^{-(d-1)} \le C_{\Omega,\rho_Y,\rho_Z}\big(h_Y^{-d} + h_Y^{-d+1}\big).$$

Since we assumed $h_Y < 1$, we have $n_{\widetilde{Z}\cup Y} \le C_{\Omega,\rho_Y,\rho_Z} h_Y^{-d}$. Together, we have

$$\|I_Z u^* - u^*\|_Y \le C_{\Omega,\Phi_m,\rho_Y,\rho_Z,\gamma_Y} h_Z^{-d/2} h_Z^m \|u^*\|_{m,\Omega}$$

and the extra term associated with boundary error on Y is

$$h_Y^{d/2-2} W_\theta^{-1/2}\left(W_\theta^{1/2} C_{\Omega,\Phi_m,\rho_Y,\rho_Z} h_Z^{-d/2} h_Z^m \|u^*\|_{m,\Omega}\right) \le C_{\Omega,\Phi_m,\mathcal{L},\rho_Y,\rho_Z,\gamma_Y} h_Z^{m-2}\|u^*\|_{m,\Omega}.$$

Boundary errors in both cases do not affect the convergence rates.

6. Numerical demonstrations. We test the proposed formulations in $\Omega = [-1,1]^2$, Discretization is done by using regular Z with $n_Z = 11^2, 16^2, \ldots, 36^2$ and collocation points X are either regular or scattered, see Figure 6.1. For the regular cases, collocation points X (strictly in the interior) and Y are constructed similarly with $h_X = \gamma_X h_Z$ and $h_Y = \gamma_Y h_Z$ with $\gamma_X = 1, 1/2, 1/3$ and $\gamma_Y = 1, 1/2$ such that $Z \subseteq X$ and $(Z \cap \Gamma) \subseteq Y$ respectively. All reported (absolute) errors are approximated by using a fixed set of 100^2 regular points, which is denser than the collocation $X \cup Y$ sets in all tests.

In matrix form, collocation conditions for the PDE and boundary condition can be written as $K_{\mathcal{L},X}\lambda = f_{|X}$ and $K_{\mathcal{B},Y}\lambda = g_{|Y}$, respectively, with entries $[K_{\mathcal{L},X}]_{ij} = \mathcal{L}\Phi(x_i - z_j)$ and $[K_{\mathcal{B},Y}]_{ij} = \Phi(y_i - z_j)$ for $x_i \in X$ and $y_i \in Y$. Both resultant matrices have $n_Z + n_Y$ (and n_Z) columns for trial space $\mathcal{U}_{Z\cup Y}$ (and $\mathcal{U}_Z$) corresponding to each z_j from the trial space. In the CLS approach (2.9), the constraints at Y are enforced using the null space matrix of the boundary collocation matrix, denoted by $\mathcal{N}_{\mathcal{B},Y} := \text{null}(K_{\mathcal{B},Y})$, as in [25], so that the unknown coefficient is expressed in the form $\lambda = \mathcal{N}_{\mathcal{B},Y}\widetilde{\lambda} + K^{\dagger}_{\mathcal{B},Y} g_{|Y}$, for some new unknown $\widetilde{\lambda}$, which can be found by solving $K_{\mathcal{L},X}\mathcal{N}_{\mathcal{B},Y}\widetilde{\lambda} = f_{|X} - K_{\mathcal{L},X}K^{\dagger}_{\mathcal{B},Y} g_{|Y}$.

In all WLS formulations, with weighting specified by W_θ in Theorem 2.6, the unknown coefficient λ is obtained by solving the following overdetermined system

$$\begin{bmatrix} K_{\mathcal{L},X} \\ W_\theta K_{\mathcal{B},Y} \end{bmatrix} \lambda = \begin{bmatrix} f_{|X} \\ W_\theta g_{|Y} \end{bmatrix}$$

with the Matlab function `mldivide` in the least-squares sense. For all computations in this section, we did not employ any technique to deal with the problem of ill-conditioning. To deal with the numerical instability, readers are referred to our trial subspace selection techniques [23, 24, 28] that determine the trial set Z adaptively to circumvent the problem of ill-conditioning.

Example 6.1. ***How dense is dense?*** First, we consider a Poisson problem with Dirichlet boundary conditions generated from three different exact solutions $u^* = \sin(\pi x/2)\cos(\pi y/2)$, peaks$(3x, 3y)$, and franke$(2x-1, 2y-1)$ by the corresponding functions in MATLAB. We cast the CLS formulation (2.9) using Whittle-Matérn-Sobolev kernels that reproduce $H^m(\Omega)$ with $m = 3, \dots, 6$. Note that our proven H^2–convergence theories require $m \geq 4$ for $\Omega \subset \mathbb{R}^2$. To see the effect of "oversampling", all sets in this example are regular and we tested $h_X = \{h_Z, h_Z/2, h_Z/3\}$ and $h_Y = \{h_Z, h_Z/2\}$. Figure 6.2 compactly shows all convergence profiles in $H^2(\Omega)$ with respect to h_Z (instead of $h_{Z\cup Y}$ for ease of comparison to the results in the next example).

To begin, let us focus on the $H^2(\Omega)$ errors for $u^* = \sin(\pi x/2)\cos(\pi y/2)$ in Figure 6.2. Generally speaking, all collocation settings demonstrate an $m-2$ convergence rate for all tested smoothness m; this also includes the original Kansa formulation with $Z = X \cup Y$. It is obvious that the error profiles for each tested m are split into two groups. The least accurate groups (i.e., the group above) correspond to $h_X = h_Z$. Without over-testing the PDE, this setting would probably fail the denseness requirement (3.2) but yet allow convergence at the optimal rate. All errors reduce at a rather constant rate, except that we can see two unstable profiles in the cases of $m = 6$. These numerical instabilities correspond to the two cases with large numbers of boundary collocations; $(h_X, h_Y) = (h_Z/2, h_Z/2)$ and $(h_X, h_Y) = (h_Z/3, h_Z/2)$. In comparison, the other two tested solutions, $u^* = $ peaks$(3x, 3y)$ and franke$(2x-1, 2y-1)$, are more oscillatory. We can see the CLS convergence rates slow down and approach an optimal $m-2$ order.

We omit the $L^2(\Omega)$ error profiles, which show exactly two extra orders as one would expect and achieve an order-m convergence before numerical instability kicks in.

$\mathcal{U}_{Z\cup Y}$		$\mathcal{U}_Z$	
n_Y	rank$(K_{\mathcal{B},Y})$	n_Y	rank$(K_{\mathcal{B},Y})$
80	80	80	80
160	104	160	96
244	108	244	96
328	100	328	96

TABLE 6.1
Numerical ranks of boundary collocation matrices resulting from different n_Y.

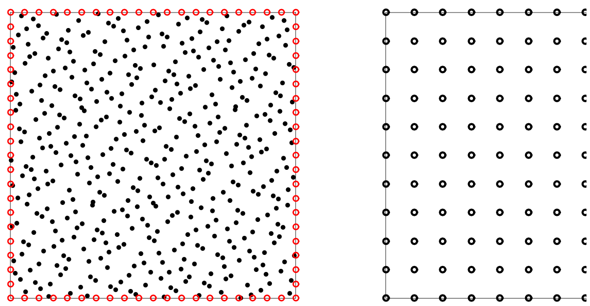

FIG. 6.1. *Schematic point sets, collocations X and Y (left), and trial Z (right), used to solve various PDEs.*

EXAMPLE 6.2. ***CLS convergence in trial space $\mathcal{U}_Z$.*** Putting the theoretical requirements aside, we are interested in the numerical performance of casting the CLS in the smaller and more practical trial spaces $\mathcal{U}_Z$. Elementary linear algebra says that if $n_Z < n_Y$, then we may not be able to find nontrivial functions from $\mathcal{U}_Z$ with zeros at Y. However, one can observe numerically that the CLS formulation hardly runs into trouble when it is cast in this smaller trial space. Numerically, as $h_Y \to 0$, the rank of the boundary collocation matrix is bounded; for example, for $n_Z = 21^2$ with finer and finer Y, we can see from Table 6.1 that the numerical ranks of the boundary matrices are bounded.

Figure 6.3 shows the $H^2(\Omega)$ error profiles for the CLS performance in $\mathcal{U}_Z$ with all other settings identical to those in Example 6.1. Comparing the CLS convergence rates in the two trial spaces, we observe that optimal convergence is also possible in $\mathcal{U}_Z$ but only for small enough h_Z. The CLS accuracy can "catch up" when the numerical rank of $K_{\mathcal{B},Y}$ is relatively insignificant compared to n_Z. Therefore, the larger the rank$(K_{\mathcal{B},Y})$ the longer CLS takes to achieve optimal convergence. By using a smaller trial space, we not only gain computational efficiency but suffer less ill-conditioning. In all tested cases and parameters, we see no accuracy drop due to ill-conditioning.

EXAMPLE 6.3. ***Numerically optimal weight for WLS.*** Now we consider the WLS formulation in (2.10) with $\theta \in \{\infty, 0, 0.5, 1, 2\}$. We begin with the same set up as in Example 6.1 and set $h_X = h_Y = h_Z/2$ to solve the Poisson problem in $\Omega = [-1, 1]^2$. The WLS weighting, as in (4.2), in this test are $W_\theta \sim 1$, n_Z, n_Z^2 and n_Z^4, and using W_∞ is equivalent to the CLS. Figures 6.4–6.5 show the $H^2(\Omega)$ error resulting from various W_θ associated with $u^* = \sin(\pi x/2, \pi y/2)$ and $u^* = \text{peaks}(3x, 3y)$ respectively. The estimated convergence rates shown in the legends are obtained from least-squares fitting to all data; if the convergence profile is not "straight" enough, the corresponding estimate is not trustworthy.

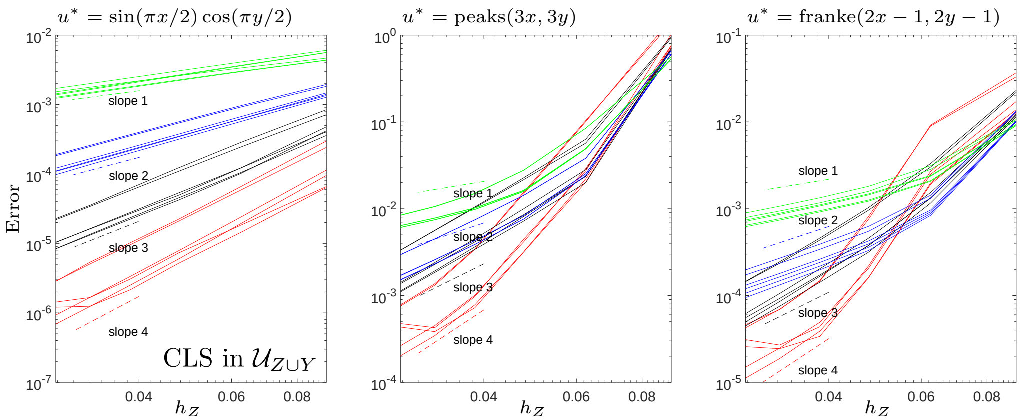

FIG. 6.2. *Example 6.1: $H^2(\Omega)$ error profiles for casting the CLS formulation in $\mathcal{U}_{Z\cup Y}$ with Whittle-Matérn-Sobolev kernels of order $m = 3, \ldots, 6$ (colored green, blue, black, and red with the associated reference slopes $1, \ldots, 4$ respectively) to solve $\Delta u = f$ with different exact solution u^*.*

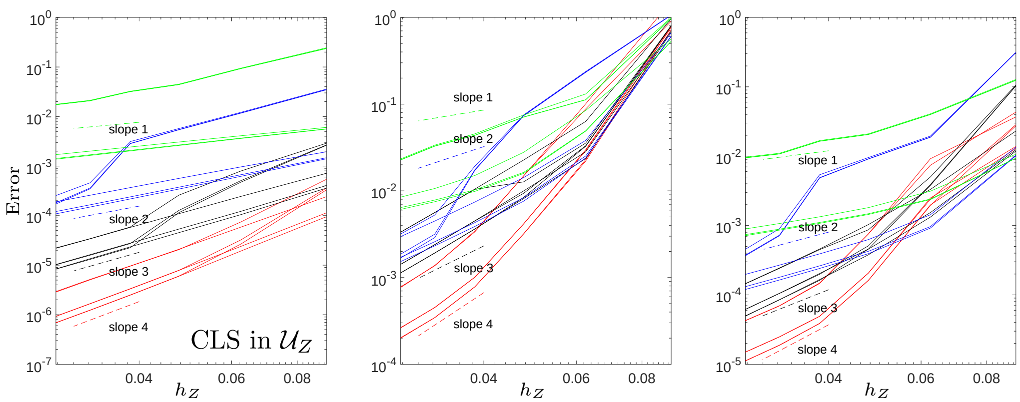

FIG. 6.3. *Example 6.2: $H^2(\Omega)$ error profiles for casting the CLS formulation in $\mathcal{U}_Z$ to the same settings as in Figure 6.2.*

From these figures, we immediately see that there is no benefit at all (in terms of both efficiency and accuracy) to go for the unweighted W_0 formulation. Unlike the CLS in $\mathcal{U}_Z$, all tested W_θ with $\theta > 0$ do not have a lag in convergence rate but may suffer ill-conditioning for large θ. By comparing all tested cases, we see that $\theta = 1$ allows good accuracy and an optimal convergence rate in both trial spaces.

To further verify these observations, we now use sets of n_X scattered collocation points (generated by the Halton sequence; see Figure 6.1) to solve three different PDEs. Boundary collocation points remain regular with $h_Y = h_Z/2$. We present the numerical result for $m = 4$ in Figure 6.6. All PDEs have

$$u^*(x, y) = \max(0, x)^4 - \max(0, y)^4,$$

whose biharmonic is discontinuous, as the exact solution. These results are compared to the results of $m = 4$ in Figure 6.5 and the convergence patterns are very similar. More numerical examples in 3D can be found in [4].

7. Conclusion. We have proven some error estimates for a constraint least-squares and a class of weighted least-squares strong-form RBF collocation formula-

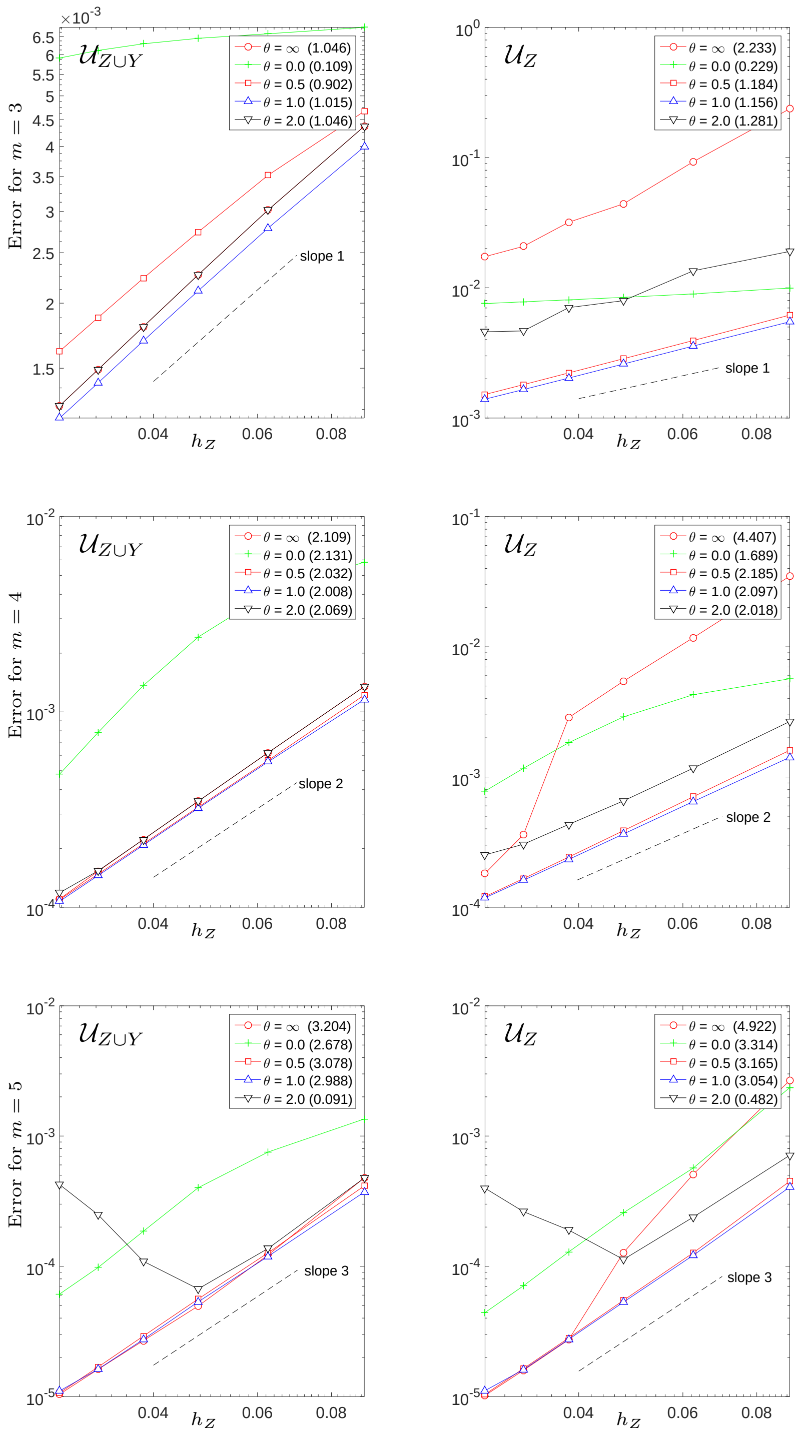

FIG. 6.4. *Example 6.3: $H^2(\Omega)$ error profiles and estimated convergence rates for casting the WLS formulation in $\mathcal{U}_{Z\cup Y}$ and $\mathcal{U}_Z$ with Whittle-Matérn-Sobolev kernels of order $m = 3, 4, 5$ (with estimated slopes in the legend) to solve $\Delta u = f$ with exact solution $u^* = \sin(\pi x/2, \pi y/2)$.*

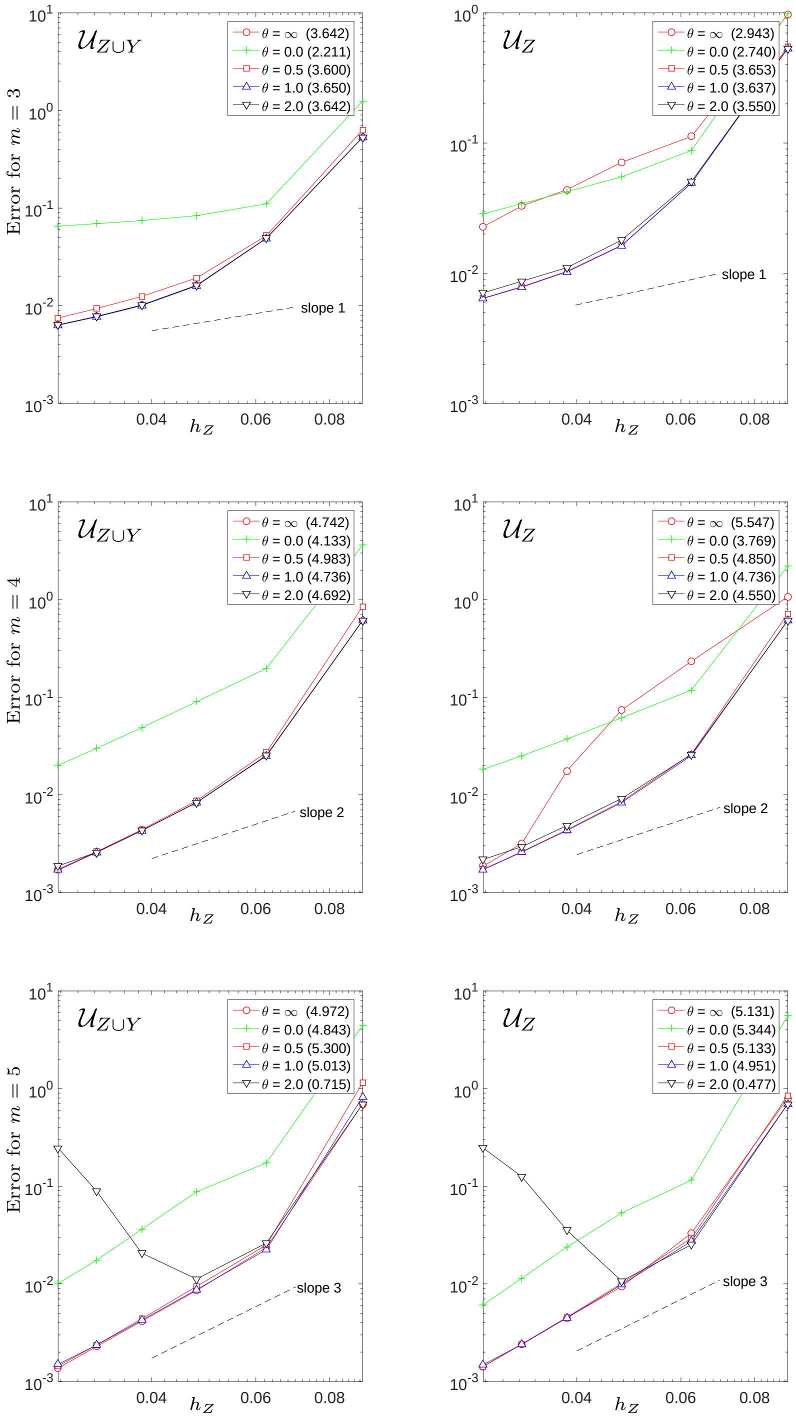

FIG. 6.5. *Example 6.3: $H^2(\Omega)$ error profiles and estimated convergence rates for casting the WLS formulation in $\mathcal{U}_{Z\cup Y}$ and $\mathcal{U}_Z$ with Whittle-Matérn-Sobolev kernels of order $m = 3, 4, 5$ (with estimated slopes in the legend) to solve $\Delta u = f$ with exact solution $u^* = \mathrm{peaks}(3x, 3y)$.*

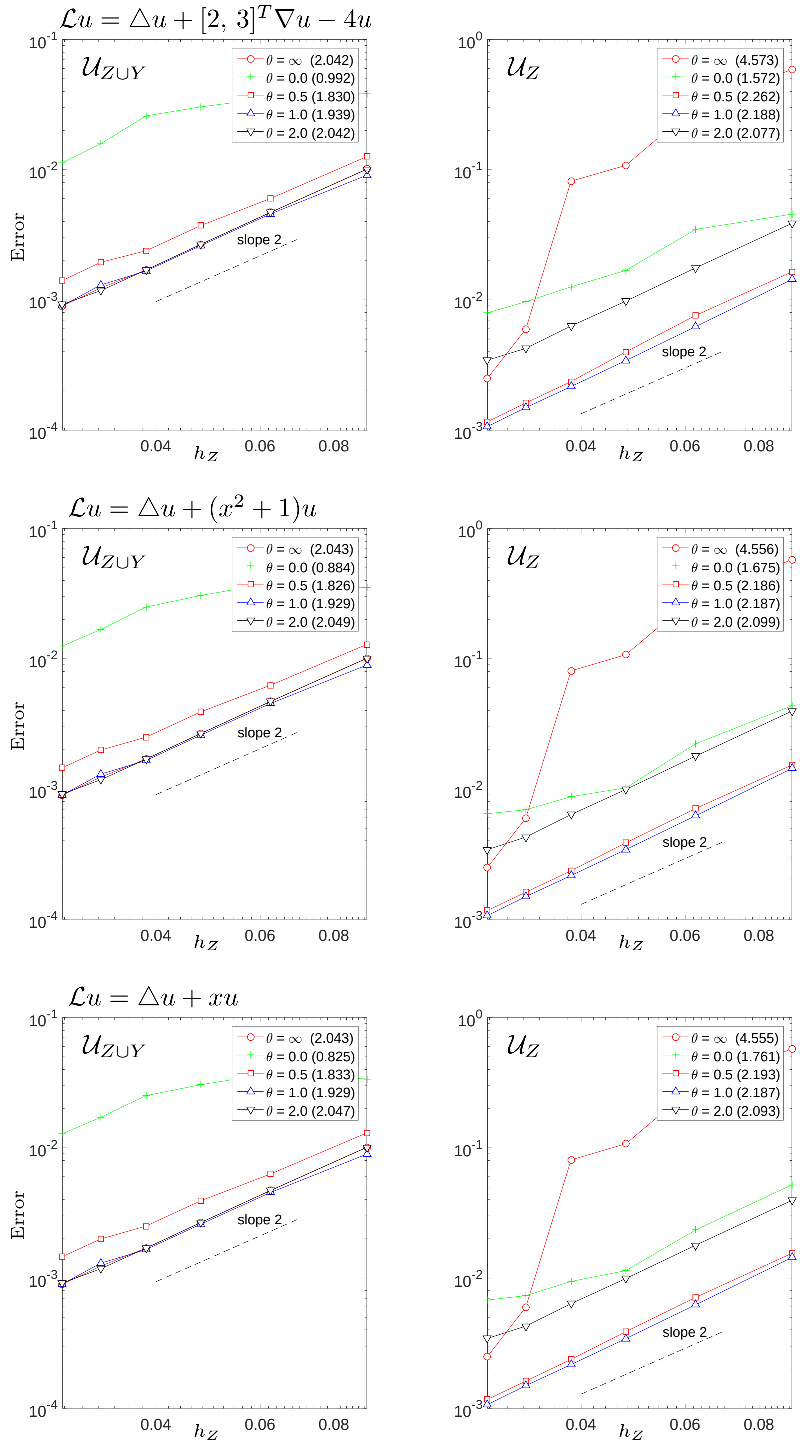

FIG. 6.6. *Example 6.3: $H^2(\Omega)$ error profiles and estimated convergence rates for casting the WLS formulation in $\mathcal{U}_{Z\cup Y}$ and $\mathcal{U}_Z$ with Whittle-Matérn-Sobolev kernels of order $m = 4$ (with estimated slopes in the legend) to solve various PDEs with exact solution $u^*(x,y) = \max(0,x)^4 - \max(0,y)^4$.*

tions for solving general second order elliptic problem with nonhomogenous Dirichlet boundary condition. All analysis is carried out in Hilbert spaces so that both PDE and RBF theories apply. We show that the CLS and WLS formulations using kernels, which reproduce $H^m(\Omega)$, with sufficient smoothness can converge at the optimal $m-2$ rate in $H^2(\Omega)$ for $d \le 3$ with respect to the fill distance of the trial centers. Besides some standard smoothness assumptions for high order convergence, the convergence theories require some linear ratios of oversampling on both sets of PDE and boundary collocation points to hold. The mesh ratios of the collocation points will only affect the constant in the convergence estimates, which allows adaptive collocation when necessary. The techniques used in this paper cannot be further generalized to obtain optimal estimates in weaker norms. In particular, the consistency of PDE residuals measured in $\ell_2(X)$ is capped at a rate of $m-2$, which is a bottleneck for getting an optimal order-m (or $m-1$) convergence estimates in $L^2(\Omega)$ (or $H^1(\Omega)$). We leave this problem open for further studies.

We verify by numerical examples that there are many convergent formulations for $\Omega \subset \mathbb{R}^2$ that enjoy the optimal convergence rate. We thoroughly study the numerical performance of Whittle-Matérn-Sobolev kernels in two trial spaces. The larger space that includes all boundary collocation points as trial centers is more theoretically sound (in the sense of the range of optimal weighting), whereas the small one is computationally more efficient. Taking both accuracy and efficiency into consideration, casting WLS in the small trial space with a moderate weight consistently yields competitive accuracy and numerical stability.

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