Interpolation with piecewise quadratic visually $C^2$ Bézier polynomials

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Abstract. For data satisfying certain generalized convexity conditions, the existence and uniqueness of a piecewise quadratic curvature continuous parametric spline interpolant in $\mathbb{R}^2$ is proved. Convexity is preserved by the interpolation, and the numerical construction of the interpolant can be carried out efficiently. The interpolation is 4-th order accurate, and a number of examples shows both the applicability and the limitations of this interpolation scheme.

1 Introduction

For given data points

$$f_1, \ldots, f_n, n \in \mathbb{R}^2, \ n \geq 2, \ f_i \neq f_{i+1} \text{ for } 1 \leq i \leq n - 1$$

we want to construct a continuous interpolating curve

$$s : I \rightarrow \mathbb{R}^2, \ I \subset \mathbb{R} \text{ a closed interval}$$

whose pieces are of class $C^2$ and whose tangent direction and curvature vary continuously. This property is called “visual” or “geometric” $C^2$ continuity and is abbreviated by $GC^2$ (see e.g. [Boehm et. al., '84] and [Farin, '82]). The interpolant should be convexity preserving in order to avoid unwanted oscillations. Since nonparametric strictly convex or concave $C^2$ functions have no inflection points and nonvanishing curvature, we introduce a natural common generalization of convexity/concavity to the parametric situation:

Definition 1.1 A $GC^2$ curve $s$ in $\mathbb{R}^2$ is (strictly) noninflexing, if the curvature of $s$ nowhere vanishes.

If the cross-product for planar vectors is defined as

$$x \times y = -x_1y_2 + x_2y_1 \text{ for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

the (signed) curvature of a planar $GC^2$ curve $f$ is $f'(t) \times f''(t)/\|f'(t)\|^3$ at $f(t)$. The sign of the curvature indicates whether the osculating circle lies to the “left” or “right” of the curve in the sense of a moving observer looking into the direction of the tangential vector. Reversing the parametrization will change the sign of this form of curvature.

The following theorem shows that parametric quadratic splines are appropriate tools for constructing noninflexing curves:
Theorem 1.1 A visual $C^2$ parametric quadratic spline curve is strictly noninfecting or a straight line.

Proof. It is easy to verify that a quadratic polynomial in Bézier form either has identically zero curvature (in case it is a straight line) or has nonvanishing curvature.

The sign of the curvature of a quadratic polynomial $s(t)$ in Bézier form with control points $b_0, b_1, b_2$ in $\mathbb{R}^2$ and

$$s(t) = b_0(1-t)^2 + 2b_1t(1-t) + b_2t^2, \quad t \in [0,1]$$

is determined by the position of $b_1$ relative to $b_0$ and $b_2$. If $b_1$ is collinear to $b_0$ and $b_2$, the curve is linear. Otherwise, $b_1$ lies in the “left” or “right” half-space defined by the oriented line $L(b_0, b_2)$, thus determining the sign of the curvature.

2 Solvability conditions

In the following we shall construct the interpolating curve by patching together $n-1$ quadratic polynomials in Bézier form with control points $f_i = b_{2i}; b_{2i+1}, b_{2i+2} = f_{i+1}$ for $1 \leq i \leq n-1$. The interior control points $b_{2i+1}$ will be determined by continuity requirements only. As a result of the above discussion, we can assume for the rest of this paper that the curve is always “turning to the right”. For “left-turning” data the indexing can be reversed to get “right-turning” data (positive curvature). Then the points $b_{2i+1}$ have to be constructed on the “left” side of the oriented lines $L(b_{2i}; b_{2i+2}) = L(f_i, f_{i+1})$, except for the trivial case that all data points $f_i$ are collinear, which we exclude from now on.

Since each quadratic piece has two degrees of freedom and visual $C^2$ continuity imposes two conditions on each breakpoint $f_2, f_3, \ldots, f_{n-1}$, we have to fix two additional parameters via boundary conditions. This can be done by adding two points $b_0 \neq b_2$ and $b_{2n+2} \neq b_{2n}$ defining tangent directions at $b_2$ and $b_{2n}$, respectively. The complete interpolation problem then will be as follows:

Construct a visually $C^2$ curve consisting piecewise of quadratic polynomials in Bézier form with control points $f_i = b_{2i}; b_{2i+1}, b_{2i+2} = f_{i+1}$ $1 \leq i \leq n-1$, and satisfying boundary conditions

$$b_{3} - b_{2} = \rho_1 (b_{2} - b_{0}), \quad \rho_1 > 0 \text{ arbitrary},$$

$$b_{2n} - b_{2n-1} = \rho_n (b_{2n+2} - b_{2n}), \quad \rho_n > 0 \text{ arbitrary},$$

by finding suitable control points $b_{2i+1}, 1 \leq i \leq n-1$. Since curvature continuity is a nonlinear condition and no additional degrees of freedom are available, the problem is equivalent to a nonlinear system of equations (see (2)). It is not always solvable (see Figure 1), even if the data form a piecewise linear curve that is a limit of noninfecting curves. We therefore must look for a weak sufficient condition for solvability. Furthermore, we shall see that unicity of the interpolant requires an even stronger condition than the one that will be used for the existence proof.
To discuss solvability conditions we introduce tangents $T_i$ at interpolation points $f_i = b_{2i}, 1 \leq i \leq n$, where the boundary conditions force $T_1$ and $T_n$ to coincide with the lines $L(b_0, b_4)$ and $L(b_{2n}, b_{2n+1})$, respectively. Then $b_{2i+1}$ is the intercept of $T_i$ with $T_{i+1}$. The problem posed in Figure 1 is unsolvable, because $T_2$ has to intersect both $T_1$ and $T_3$ in finite points $b_3$ and $b_5$ which must lie on the “left” of the lines $L(b_2, b_4)$ and $L(b_4, b_6)$, respectively.

Another example is given by Figure 2. This problem is unsolvable for $\sqrt{x} + \sqrt{y} < 2\sqrt{10}$. The piecewise linear interpolant of $b_0, b_2, \ldots, b_{10}$ takes two successive 90° turns and fails to satisfy the second part of
Definition 2.1 A data set \( b_0, b_2, \ldots, b_{2n+2} \) is called noninfecting, if all angles
\[
\gamma_i := \angle (b_{2i+2} - b_{2i}; b_{2i} - b_{2i-2}), \quad (1 \leq i \leq n),
\]
measured in counterclockwise direction, are in \([0, 180^\circ]\) or in \((180, 360^\circ)\) (i.e. the data are always “turning right” or always “turning left”). If the sum of two consecutive angles \( \gamma_i \) always is in \((0, 180^\circ)\) or in \((180, 360^\circ)\), the data set is called nonreflecting.

Note that any nonreflecting and noninfecting data set admits \( n - 1 \) “accompanying triangles” formed by \( b_{2i}, b_{2i+2} \) and the intercept \( b_{2i+1} \) of the lines \( L(b_{2i-2}, b_{2i}) \) and \( L(b_{2i+2}, b_{2i+4}) \) for \( 1 \leq i \leq n - 1 \) (see Figure 4). Each accompanying triangle must contain a control point \( b_{2i+1} \) of the solution, if the latter exists. The main existence result of this paper will be

Theorem 2.1 For any noninfecting and nonreflecting data set there is at least one visually \( C^2 \) interpolant by a parametric quadratic spline.

The proof is contained in the next two sections.

Note that Figure 2 can be made into a solvable problem by small perturbations of \( b_2 \) and \( b_8 \). However, as Figure 3 shows, a perturbed problem with noninfecting and nonreflecting data may have multiple solutions. To get uniqueness, we therefore need additional conditions.

Definition 2.2 A nonreflecting and noninfecting data set is called nondegenerate, if all accompanying triangles have obtuse angles at the points \( b_{2i+1} \).

An equivalent form of this condition is

Either all \( \gamma_i \) satisfy \( \gamma_i \in (0, 90^\circ) \) and \( \gamma_i + \gamma_{i+1} < 90^\circ \)

or all \( \gamma_i \) satisfy \( \gamma_i \in (270^\circ, 360^\circ) \) and \( 720^\circ - \gamma_i - \gamma_{i+1} < 90^\circ \),

and in short: A nondegenerate data set

- always turns to the left or always turns to the right,
- the direction changes only by acute angles,
- two successive direction changes sum up to an acute angle.

We remark that this form of nondegeneracy is a natural notion:

Theorem 2.2 An ordered and sufficiently dense sample of sufficiently many points from a noninfecting \( C^2 \) curve will form a nondegenerate data set.

Our main result concerning uniqueness will be

Theorem 2.3 A nondegenerate data set has a unique visually \( C^2 \) piecewise quadratic interpolant.
3 Principles of construction

Consider part of a noninfecting and nonreflecting data set as in Figure 5. We introduce tangents $T_i$ at $b_{2i}$ forming angles $\alpha_i$ with $b_{2i+2} - b_{2i}$. The intercept $b_{2i+1}$ of $T_i$ and $T_{i+1}$ must lie in the accompanying triangle $b_{2i}; b_{2i+2}; b_{i+1}$ to avoid unwanted inflection points. Curvature at $b_{2i}$ is (up to a constant factor) equal to

$$\kappa_i^- = \frac{\text{dist} (b_{2i-2}, T_i)}{||b_{2i-1} - b_{2i}||^2}$$

from the left and

$$\kappa_i^+ = \frac{\text{dist} (b_{2i+2}, T_i)}{||b_{2i+1} - b_{2i}||^2}$$

from the right

(see e.g. [Boehm et. al., '84, p. 11]).

Using the obvious identities

$$\frac{\text{dist} (b_{2i+2}, T_i)}{h_i} = \sin \alpha_i$$

$$\frac{\text{dist} (b_{2i-2}, T_i)}{h_{i-1}} = \sin (\gamma_i - \alpha_i)$$
Figure 4: Accompanying triangles

\[
\frac{\text{dist} (b_{2i-2}, T_i)}{\|b_{2i-2} - b_{2i-1}\|_2} = \sin \delta_{i-1} = \sin (\gamma_i - \alpha_i + \alpha_{i-1})
\]

\[
\frac{\text{dist} (b_{2i-1}, T_{i-1})}{\|b_{2i-1} - b_{2i}\|_2} = \sin \delta_{i-1}
\]

we can express curvature continuity at \( b_{2i}, \ 2 \leq i \leq n - 1, \) by the equation

\[
\frac{1}{h_i} \sin \alpha_i \frac{\sin^2 (\gamma_{i+1} - \alpha_{i+1} + \alpha_i)}{\sin^2 (\gamma_{i+1} - \alpha_{i+1})} = \frac{1}{h_{i-1}} \sin (\gamma_i - \alpha_i) \frac{\sin^2 (\gamma_i - \alpha_i + \alpha_{i-1})}{\sin^2 \alpha_{i-1}}
\]  (2)

involving only constants \( h_j, \gamma_j \) and the variables \( \alpha_j. \) This gives \( n - 1 \) nonlinear equations for the unknowns \( \alpha_2, \ldots, \alpha_{n-1}; \) we use \( \alpha_1 = \gamma_1 \) and \( \alpha_n = 0 \) to satisfy the boundary conditions.

The approach via the system (2) is used later for efficient numerical treatment of the problem.

It does not yield much insight for the proof of existence and uniqueness of a solution.

For the latter we theoretically investigate a numerically hazardous “shooting strategy”:

**Start.** Choose an arbitrary point \( b_3 \) between \( b_2 \) and \( \widetilde{b_3} \) on \( T_1. \)

**Iteration.** For \( 2 \leq i \leq n - 1, \) if \( b_{2i-1} \) and \( T_i \) are given, construct \( b_{2i+1} \) on \( T_i \) by using curvature continuity at \( b_{2i} \) in the form

\[
\|b_{2i+1} - b_{2i}\|^2 = \frac{\text{dist} (b_{2i+2}, T_i)}{\text{dist} (b_{2i-2}, T_i)} \|b_{2i+1} - b_{2i-1}\|^2.
\]

If this quantity comes out to be too large, \( b_{2i+1} \) moves out of the accompanying triangle formed by \( b_{2i}, b_{2i+2}, \widetilde{b_{i+1}}, \) and the process cannot be repeated. Otherwise the tangent \( T_{i+1} \) is fixed by \( b_{2i+1} \) and \( b_{2i+2}, \) and the next value of \( i \) can be used.
This process tries to get through to the triangle \( b_{2n-2} b_{2n} b_{2n-1} \) and to hit the line \( L(b_{2n}, b_{2n-1}) \) to produce a curvature continuous solution. It is comparable to the "single shooting" method for solving a boundary value problem in ordinary differential equations. Its (theoretical) success is equivalent to the solvability of the problem, but experience shows that it is numerically extremely sensitive. There are simple nondegenerate data sets for which 32-bit precision does not suffice to find some \( b_3 \) to shoot through to the last triangle. Replacement by a "multiple shooting" method is equivalent to solving the system (2), which is recommended for practical purposes (see section 6).

4 Existence proof

We now proceed to study the topological properties of a single step of the shooting method defined in the last section.

Let two lines \( L_1 \) and \( L_2 \) intersect in some point \( S \) with an angle \( \alpha \) of less than 180 degrees. This angle forms two open cones \( C_1 \) and \( C_2 \): let \( A_1 \neq S \) and \( A_2 \neq S \) be points on \( L_1 \) and \( L_2 \), respectively, but on different sides of \( S \) (see Figure 6).

Definition 4.1 A continuous curve \( z : [0, 1] \to \mathbb{R}^2 \) is "transversal in \( C_1 \)", if (see Figure 6)
Figure 6: Transversal curve

\[ a) \ z((0,1)) \subset C_1, \]
\[ b) \ z(0) = A_1, \]
\[ c) \ z(1) \in L_2, \]
\[ d) \ z(1) \neq S. \]

**Definition 4.2** On the configuration of Figure 6 we define a mapping \( F : C_1 \to C_2 \) by

\[ a) \ F(x) \in C_2 \text{ lies on the line } L(x) \text{ through } x \text{ and } S \]
\[ b) \ \frac{\text{dist} (A_1, L(x))}{\|x - S\|_2} = \frac{\text{dist} (A_2, L(x))}{\|F(x) - S\|_2} \]

This definition makes sure that the piecewise quadratic polynomial defined in Bézier form by control points \( A_1, x, S \) and \( S, F(x), A_2 \) is curvature continuous at \( S \). Therefore \( F \) corresponds to a single step of the shooting method.

If we introduce polar coordinates as in Figure 7, we get for the image \( F(r, \varphi) = (R(r, \varphi), \varphi) \) of \((r, \varphi)\) the expression

\[
R^2(r, \varphi) = \frac{h_2}{h_1} \frac{\sin \varphi}{\sin (\alpha - \varphi)} r^2 \tag{3}
\]

which shows that \( F \) is a smooth function.

This follows from

\[
\frac{h_1}{r^2} = \frac{h_2}{R^2}, \quad h_1 = h_1 \sin (\alpha - \varphi), \quad h_2 = h_2 \sin \varphi.
\]

**Lemma 4.1** Let \( z \) be a transversal curve in \( C_1 \). Then:
Figure 7: Construction of $F(x)$ in polar coordinates

a) $u(t) := F(z(t))$ is a continuous curve on $(0, 1)$ in $C_2$.

b) $u(1) := S$ makes $u$ continuous on $(0, 1]$.

c) $\lim_{t \to 0} \|u(t)\|_2 = \infty$.

d) $\lim_{t \to 0} \text{dist}(u(t), L_1) = 0$.

**Proof.** Everything follows easily from elementary continuity arguments using the properties of $z$ and $F$ described in the polar coordinate system of (3) and Figure 7.

We now observe that in the configuration of Figure 8 two instances of Figure 7 are intertwined. A transversal curve $z_{i-1}$ in the cone $C_i^-$ will map by Lemma 4.1 into a curve $z_i$ in the cone $C_i^+$. This curve starts in $b_{2i}$ and has the line $L(b_{2i-2}, b_{2i})$ as an asymptote. Now the intersection of this curve with the accompanying triangle $b_{2i} \ b_{2i+2} \ b_{2i+1}$ is transversal in the cone $C_{i+1}^-$. Applying Lemma 4.1 again we get a curve $z_{i+1}$ in the cone $C_{i+1}^+$ and so on. The shooting process starts with $z_1$ being part of $T_1$.

To prove Theorem 2.1 we now use induction on the statement

"There is an open nonvoid interval on the line $L(b_2, \tilde{b}_3)$ that maps uniquely onto a curve $z_i$ in $C_i^+$ that starts from $b_{2i}$ and has the line $L(b_{2i-2}, b_{2i})$ as an asymptote."

This statement follows from the previous lemma for $i = 2$, and the discussion illustrated by Figure 8 gives the induction step.

The resulting curve $z_{n-1}$ in $C_n^+$ necessarily intersects the line $T_n$ given by the boundary condition at $b_{2n}$ at least once. This proves existence of a solution.

Uniqueness is not guaranteed, because $z_{n-1}$ might intersect $T_n$ in several points. This can be actually observed by plotting the curves $z_i$, and this is how the solutions of Figure 3 were found.
Figure 8: Shooting method

5 Uniqueness

Definition 5.1 A transversal curve \( z \) in \( C_1 \) is strictly monotonic, if in the polar coordinates of Figure 7, the radius \( r(\varphi) \) is strictly monotonic.

Lemma 5.1

a) For the image \((R(r(\varphi), \varphi), \varphi)\) of a monotonic transversal curve \( z = (r(\varphi), \varphi) \) in \( C_1 \) the radius \( R(r(\varphi), \varphi) \) is strictly monotonic.

b) If \( B_2 \neq S \) is a point on the line \( L_1 \) and the boundary of \( C_2 \) such that the angle \( \gamma = \angle (A_2B_2S) \) is obtuse, the image curve \( F(z) \) meets the line \( L(B_2, A_2) \) in exactly one point (see Figure 9).

Proof. The first assertion is a direct consequence of (3). If \( F(z) \) meets \( B_2A_2 \) in two points \( P_1, P_2 \) with \( \varphi_1 < \varphi_2 \) (see Figure 10), then \( R(r(\varphi), \varphi) \) is no monotonic function of \( \varphi \).

For a nondegenerate data set we can apply Lemma 5.1 in each step of the shooting method, because all angles at \( b_{2i+1} \) are obtuse. Then the curves constructed in the previous section have unique intersections with the sides \( b_{2i+1} \) of the triangles \( b_{2i} b_{2i+1} b_{2i+2} \). This also holds for \( i = n - 1 \), proving uniqueness.
Figure 9: Monotonicity

Figure 10: Uniqueness

6 Convergence

Let $f : [a,b] \to \mathbb{R}^2$ be a smooth planar curve with positive curvature, parametrized by arclength. This section analyzes the convergence of the interpolation process for data from $f$, i.e., for data density tending to zero. We can follow [deBoor et. al., '87] to assume a local two-dimensional representation of $f$ around a single point by

$$f(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f'(t) = \begin{pmatrix} \cos \theta(t) \\ \sin \theta(t) \end{pmatrix}$$

$$\theta(t) = \theta_1 t + \theta_2 t^2 + \mathcal{O}(t^3)$$

without loss of generality. Curvature at $f(t)$ is

$$\kappa(t) = \theta'(t) = \theta_1 + \frac{1}{2} \theta_2 t + \mathcal{O}(t^2).$$
The angle $\alpha(t)$ between the tangent at $f(0)$ and the chord

$$f(t) - f(0) = \int_0^t f'(\sigma) d\sigma = \frac{1}{6} \left( \frac{6t - \theta_1^3 t^3}{3\theta_1 t^2 + 2\theta_2 t^3} \right) + O(t^4)$$

(4)
can be expanded via

$$\tan \alpha(t) = \frac{3\theta_1 t^2 + 2\theta_2 t^3 + O(t^4)}{6t - \theta_1^3 t^3 + O(t^4)}$$
as

$$\alpha(t) = \tan \alpha(t) + O(t^3) = \frac{1}{2} \theta_1 t + \frac{1}{3} \theta_2 t^2 + O(t^3)$$

(5)
for $t \to 0$. The chord length

$$h(t) = \|f(t) - f(0)\|_2 = t - \frac{1}{24} \theta_1^3 t^3 + O(t^4)$$
can replace arc length $t$ up to terms of order $O(t^3)$. This allows expansions like (5) to be written in terms of chord length $h$ instead of arc length:

**Lemma 6.1** The angle $\alpha = \alpha(f(t_0, t_1))$ between the tangent to a smooth planar curve $f$ at $f(t_0)$ and a chord $f(t_1) - f(t_0)$ of length $h = \|f(t_1) - f(t_0)\|$ is

$$\alpha = \frac{1}{2} \kappa(t_0) h + \frac{1}{3} \kappa'(t_0) h^2 + O(h^3)$$

(6)
for $t_1 > t_0, t_1 \to t_0$, where $\kappa$ is the curvature of the curve.

The expansion (6) is independent of the parametrization except for the orientation hidden in the correspondence of $\kappa'$ and $t_1 > t_0$. For $t_1 < t_0$, the analogous expansion is

$$\alpha = -\frac{1}{2} \kappa(t_0) h + \frac{1}{3} \kappa'(t_0) h^2 + O(h^3),$$

(7)
obtained from Lemma 6.1 by reversing the parametrization and the sign of curvature. Note that in both cases the angle has to be taken as oriented from the tangent to the chord.

Similarly, one easily verifies expansions of angles between successive chords in terms of the curvature at the central or a non-central point:

**Lemma 6.2** The angle $\gamma = \gamma(f(t_0, t_1, t_2))$ between two chords $f(t_{i+1}) - f(t_i), i = 0, 1$, of a planar curve $f$ with chord lengths $h_i = \|f(t_{i+1}) - f(t_i)\|$ is

$$\gamma = \frac{1}{2} \kappa(t_1)(h_0 + h_1) + \frac{1}{3} \kappa'(t_1)(h_1^2 - h_0^2) + O(\max(h_0^3, h_1^3))$$
$$= \frac{1}{2} \kappa(t_0)(h_0 + h_1) + \frac{1}{3} \kappa'(t_0)(2h_0^2 + 3h_0h_1 + h_1^2) + O(\max(h_0^3, h_1^3))$$

(8)
for $t_2 > t_1 > t_0, t_2 - t_0 \to 0$, where $\kappa$ is the curvature of the curve. The angle between the tangent at $f(t_1)$ and the chord $f(t_1) - f(t_0)$ is

$$\alpha = \frac{1}{2} \kappa(t_0) h_0 + \frac{2}{3} \kappa'(t_0) h_0^2 + O(h_0^3)$$

(9)
in terms of the curvature at $t_0$. 

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**Proof:** Expansion (9) simply follows from (8) for \( t_2 = t_1, h_1 = 0 \), and the first representation of \( \gamma \) in (6) can be written as the difference between (6) and (7). To obtain the non-central representation, the angle between the chord \( f(t_2) - f(t_1) \) and the tangent at \( f(t_0) \) is evaluated as

\[
\frac{1}{2} \theta_1(h_0 + h_2) + \frac{1}{3} \theta_2(h_0^3 + h_0h_1 + h_1^2) + \mathcal{O}(h_2^3)
\]

via an expansion of the chord like (4), using \( h_2 = \|f(t_2) - f(t_0)\| \). The rest follows easily, using \( h_2 = h_0 + h_1 + \mathcal{O}(\max(h_0^3, h_2^3)) \). ■

We now go over to the interpolation of \( f \) at points \( f_i = b_{2i} = f(t_i) \) for \( 1 \leq i \leq n \) with boundary conditions using the directions of \( f'(t_1) \) and \( f'(t_n) \). The density of knots is measured by

\[
h := \max_{1 \leq i \leq n-1} h_i, \\
h_i := \|f(t_{i+1}) - f(t_i)\|, 1 \leq i \leq n - 1, \quad h_0 := h_n := 0,
\]

and we consider the asymptotic situation \( h \to 0, n \to \infty \). The data-defined chord angles \( \gamma_i \) and the unknown exact tangent angles \( \alpha_i^* \) are representable as

\[
\gamma_i := \frac{1}{2} \kappa_i(h_i + h_{i-1}) + \frac{1}{3} \kappa_i'(h_i^3 - h_{i-1}^3) + \mathcal{O}(\max(h_{i-1}^3, h_i^3)), \\
\alpha_i^* := \frac{1}{2} \kappa_i h_i + \frac{1}{2} \kappa_i' h_i^2 + \mathcal{O}(h_i^3),
\]

using Lemmas 6.1 and 6.2 and setting \( \kappa_i := \kappa(t_i), \ \kappa_i' := \kappa'(t_i) \). As a first application we eliminate the unknown curvature \( \kappa_i \) and get numerically available tangent angles

\[
\hat{\alpha}_i := \gamma_i h_i/(h_{i-1} + h_i) \\
= \frac{1}{2} \kappa_i h_i + \frac{1}{3} \kappa_i' h_i^2/(h_i - h_{i-1}) + \mathcal{O}(h_i^3)
\]

as good approximations of \( \alpha_i^* \). We therefore recommend (10) for local methods and for starting Newton’s method to solve the system (2).

The goal of this section is to prove

**Theorem 6.1** Global \( GC^2 \) piecewise quadratic interpolation of data from a smooth planar curve with nonvanishing curvature has an error of \( \mathcal{O}(h^4) \) for data density \( h \to 0 \).

The proof will be reduced to an application of

**Theorem 6.2** If local Hermite interpolation of a smooth planar curve near a point with nonvanishing curvature is done by quadratic polynomials using exact positions and approximate tangent directions with errors \( \mathcal{O}(h^k) \) for \( k = 2 \) or 3, the error will be \( \mathcal{O}(h^{k+1}) \) in terms of the distance \( h \) between interpolation points.

The proof of this result can be omitted because it is just a slight variation of the argument given by deBoor, Höllig, and Sabin in [deBoor et. al., ’87] to prove their \( \mathcal{O}(h^6) \) result for \( GC^2 \) cubics.

Convergence of order \( \mathcal{O}(h^3) \) for the convexity preserving local Hermite interpolation by \( GC^1 \) piecewise quadratics using approximate tangent angles \( \hat{\alpha}_i \) defined in (10) follows immediately from Theorem 6.2. To prove \( \mathcal{O}(h^4) \) convergence for the global \( GC^2 \) method of this paper, we have to show

\[
\tilde{\alpha}_i = \alpha_i^* + \mathcal{O}(h^3), \quad h \to 0
\]

(11)
uniformly with respect to $i$ for the solution $\tilde{\alpha}_i$ of the system (2). This will be done via Kantorovitch’s convergence theorem for Newton’s method (see e.g. [Ortega–Rheinboldt, ’70, p. 421]) applied to a scaled version of the system (2).

To evaluate (2) at the exact tangent angles $\alpha^*$ we fix an index $i$ and apply lemmas 6.1 and 6.2 to get the expansions

\[
\begin{align*}
\alpha_{i-1}^* &= \frac{1}{2}\kappa(t_i)h_{i-1} - \frac{2}{3}\kappa'(t_i)h_{i-1}^2 + O(h^3) \\
\alpha_{i+1}^* &= \frac{1}{3}\kappa(t_i)h_i - \frac{2}{3}\kappa'(t_i)h_i + O(h^3) \\
\gamma_i + 1 &= \frac{1}{3}\kappa(t_i)(h_i - h_{i+1}) + \frac{1}{3}\kappa'(t_i)(2h_i^2 + 3h_i h_{i+1} + h_{i+1}^2) + O(h^3) \\
\gamma_i - \alpha_i^* &= \frac{1}{3}\kappa(t_i)h_{i-1} - \frac{1}{3}\kappa'(t_i)h_{i-1}^2 + O(h^3) \\
\gamma_i^* - \alpha_{i+1}^* &= \frac{1}{3}\kappa(t_i)h_i + \frac{1}{3}\kappa'(t_i)h_i^2 + O(h^3) \\
\gamma_i^* + \alpha_{i+1}^* &= \frac{1}{3}\kappa(t_i)h_i + \kappa'(t_i)h_i^2 + O(h^3).
\end{align*}
\]

Putting these into (2) yields an $O(h^2)$ expression after some calculations. Similar expansions can be derived for the angles $\tilde{\alpha}_i$, replacing $\alpha_i^*$, and (2) comes out to be of order $O(h)$ at $\tilde{\alpha}$. This is not surprising, because the approximate angles defined in (10) are exact up to $O(h^3)$ for circular arcs.

To evaluate derivatives, we rescale (2) from the form $F(\alpha) = 0$ into $G(\beta) = 0$ by setting $\beta := T(\alpha)$, $G(\beta) := F(T^{-1}\beta)$ with $T$ defined by

\[
\beta_j := T_j(\alpha) := \frac{\alpha_j}{\min(h_{j-1}, h_j)}, \quad 2 \leq j \leq n - 1.
\]

Using $\hat{T} := T(\tilde{\alpha})$ and $\beta^* := T(\alpha^*)$ we have

\[
F(\tilde{\alpha}) = G(\hat{T}) = O(h), \quad F(\alpha^*) = G(\beta^*) = O(h^2)
\]

for the scaled version. Now we examine $G'$ at $\beta^*$ and get

\[
\frac{\partial G_i}{\partial \beta_{i-1}} = 2\frac{\sin^2(\gamma_i - \alpha_i) \sin(\gamma_i - \alpha_i + \alpha_{i-1}) \min(h_{i-1}, h_{i-2})}{\sin^3\alpha_{i-1} h_{i-1}} \min(h_{i-1}, h_{i-2}) = (4 + O(h)) \frac{\min(h_{i-1}, h_{i-2})}{h_{i-1}} = 4 + O(h),
\]

\[
\frac{\partial G_i}{\partial \beta_{i+1}} = 2\frac{\sin^2(\gamma_i - \alpha_i + \alpha_{i+1} \alpha_i) \min(h_{i-1}, h_i)}{\sin^3(\gamma_{i+1} - \alpha_{i+1}) h_i} \min(h_{i-1}, h_i) = (4 + O(h)) \frac{\min(h_{i-1}, h_i)}{h_i} = 4 + O(h),
\]

\[
\frac{\partial G_i}{\partial \beta_i} = 2\frac{\sin^2(\gamma_i - \alpha_i + \alpha_{i+1}) \min(h_{i-1}, h_{i-2})}{\sin^2 \alpha_{i-1} h_{i-1}} \left(\frac{\sin(\alpha_{i-1} + 3\sin(\gamma_i - \alpha_i) \cos(\gamma_i - \alpha_i + \alpha_{i-1}))}{\sin^2(\gamma_{i+1} - \alpha_i + \alpha_{i-1}) h_i} \min(h_{i-1}, h_i) \right)
\]

\[
+ 2\frac{\sin^2(\gamma_{i+1} - \alpha_{i+1}) h_{i-1}}{\sin^2(\gamma_{i+1} - \alpha_{i+1}) h_{i-1}} \left(\frac{\sin(\gamma_{i+1} - \alpha_{i+1}) \cos(\gamma_{i+1} - \alpha_{i+1} + \alpha_i)}{\sin^2(\gamma_{i+1} - \alpha_{i+1}) h_i} \right)
\]

\[
= (8 + O(h)) \frac{\min(h_{i-1}, h_{i-2})}{h_{i-1}} + (8 + O(h)) \frac{\min(h_{i-1}, h_i)}{h_i} = 16 + O(h)
\]

\[
= 2 \frac{\partial G_i}{\partial \beta_{i-1}} + 2 \frac{\partial G_i}{\partial \beta_{i+1}} + O(h).
\]
The row-sum norm of $G'$ at $\beta^*$ then is bounded by $32 + O(h)$, while the inverse has a norm bound of $\frac{1}{4} + O(h)$. A similar calculation proves that all second derivatives of $G_i$ at $\beta^*$ are bounded with respect to $h \to 0$. This is where the special form of the scaling pays off.

Kantorovich’s theorem now implies that there exist $r > 0$ and $h_0 > 0$ such that there is a unique solution $\tilde{\beta}$ of $G(\beta) = 0$ for each interpolation problem for $f$ with data density $h$ less than $h_0$, satisfying $\|\tilde{\beta} - \beta^*\| < r$ in the sup-norm. Furthermore, the solution is reached with quadratic convergence from every starting point $\beta$ with $\|\beta - \beta^*\| < r$. Because of

$$|\tilde{\beta}_i - \beta^*_i| = \frac{\tilde{\alpha}_i - \alpha^*_i}{\min(h_i, h_{i-1})} = (\frac{1}{\pi} |f'(t_i)| h_i h_{i-1} + O(\max(h_i^3, h_{i-1}^3)))/\min(h_i, h_{i-1})$$

the starting value $\tilde{\beta}$ will be successful for $h \to 0$, if the ratio

$$\max(h_i, h_{i-1})/\min(h_i, h_{i-1})$$

is uniformly bounded.

Since $G(\beta^*) = O(h^2)$, the radius of the existence domain can be (theoretically) chosen as an $O(h^2)$ function, and therefore the exact solution $\beta$ satisfies $\|\beta - \beta^*\| = O(h^2)$. But then

$$\tilde{\alpha}_i = \tilde{\beta}_i(h_i + h_{i-1}) = \beta^*_i(h_i + h_{i-1}) + O(h^3)$$

$$\tilde{\alpha}_i = \alpha^*_i + O(h^3),$$

as required for $O(h^4)$ convergence of the interpolation process.

7 Numerical treatment

The asymptotic considerations of the preceding section suggest a modified Newton–Raphson method for solving the nonlinear system (2). The Jacobians will be tridiagonal, and each iteration needs only $O(n)$ operations. If (2) is written as $F(\alpha) = 0$ with $F : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$, then a simple stepsize control is recommended to make sure that

a) $\|F(\alpha)\|_2$ decreases at each step

b) $\alpha$ stays in the admissible region (e.g. $0 < \alpha_i < \gamma_i$ for “right-turning” data or $\gamma_i < \alpha_i < 2\pi$ for “left-turning” data).

(see [Schaback, ’88] for a comparable situation). Good results are obtained for starting values (10), because these satisfy (2) up to $O(h)$. A Gauss–Seidel iteration of (2) may be employed for an existence proof via Brouwer’s fixed-point theorem, but there are examples where the unique solution of (2) is a repelling fixed point of the Gauss–Seidel iteration.

Newton’s method also works well in cases with multiple solutions and in cases of reflecting data with existing solutions. In cases of nonexistent solutions the method creeps to the boundary of the admissible domain, using stepsizes tending to zero. There are nondegenerate data sets where the matrix $F'$ is not diagonally dominant at the solution, but no nondegenerate case is known where Newton’s method failed to converge. For sufficiently dense samples of data with density $h \to 0$ and a bounded ratio (12), Newton’s method will converge quadratically to the solution when started at (10), as the convergence results of the preceding section show.
8 Examples and Concluding Remarks

Figures 11 and 12 show two examples.

![Figure 11: Logarithmic spiral, 8 points](image)

For large numbers of data points, the solutions look very nice and closely approximate functions that supplied the data. If data points are far apart, the solution looks a little “too straight” at interior knots (see Figure 11). This is due to the fact that the interpolating parabolic pieces normally attain their curvature maximum in the interior of their domain.

Existence and uniqueness results extend to the case of piecewise rational quadratic curves with prescribed weights. This modifies the system (2) by some extra factors, but leaves the geometric proof for existence and uniqueness of solutions unchanged.

In a forthcoming paper we show how to place cubic polynomial pieces at positions where the data require an inflection point or a $GC^2$ continuation into a straight line. Furthermore, the tangential boundary conditions can be replaced by “not-a-knot—” conditions.

References


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Figure 12: Letter “Ω” formed by 15 points