Kernel B–Splines and Interpolation

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Abstract

This paper applies divided differences to conditionally positive definite kernels in order to generate *kernel B–splines* that have fast decay towards infinity. Interpolation by these new kernels provides better condition of the linear system, and the kernel B–spline inherits the approximation orders from its native kernel. We proceed in two different ways: either the kernel B–spline is constructed adaptively on the data knot set X, or we use a fixed generalized divided difference scheme and shift it around; special B–splines obtained by second finite differences of multiquadrics are studied. When a fixed scheme of divided differences of order two is applied and then shifted around, the kernel B–spline so obtained is strictly positive in general. We give suggestions in order to get a consistent improvement of the condition of the interpolation matrix in applications.

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1 Introduction

The reconstruction of multivariate functions from discrete data by using the reproducing kernel of some semi–Hilbert space is an optimal [9] and increasingly popular [8, 4, 5] technique. For computational efficiency, one should look for cheaply available kernels with good decay at infinity, and for other practical reasons certain unbounded kernels like the multiquadrics or the thin–plate spline are useful. To overcome this apparent contradiction, the situation on the infinite grid [3] and certain preconditioning techniques [6] suggest to take linear combinations of unbounded kernels in order to generate new kernels with strong decay properties.

In this paper we define *kernel B–splines* via generalized divided differences of conditionally positive definite kernels, and we prove sharp results on their decay at infinity in case of multiquadrics and polyharmonic splines. If the same local difference scheme is applied twice to a translation–invariant kernel, one can write the result as a new *B–spline kernel* which turns out to be (strictly) positive definite, if the order of the divided difference scheme is large enough. This leads to strongly decaying kernels with good approximation and stability properties. In contrast to [3] on the grid, we assume no regularity of the data here.

2 Notation

Let Φ be a conditionally positive definite translation–invariant kernel of order m on $I\!R^d$ and let $X, Y, \ldots$ denote finite subsets of separated points of $I\!R^d$. We let the space dimension d be fixed and remove it from further notation. The space of d–variate polynomials up to order at most k will be denoted by $I\!P_k$, its dimension is $q(k) = \binom{k+d-1}{d}$, and a basis is denoted by $p_1, \ldots, p_{q(k)}$. If X is a point set in $I\!R^d$ consisting of $M := |X|$ elements, we define the $|X| \times q(k)$ matrix

$$P_{X,k} := (p_j(x_i))_{x_i \in X, 1 \le j \le q(k)}$$

and the space

$$V_{X,k} = \left\{ \alpha \in I\!R^{|X|} \;:\; P_{X,k}^T \alpha = 0 \right\}. \tag{1}$$

We generally assume $k \ge m,\ |X| \ge q(k) \ge q(m)$, and

$$\text{rank } P_{X,k} = q(k). \tag{2}$$

Note that for $k = m$ this is the standard additional condition on the point locations, ensuring solvability of the interpolation problem on X for a conditionally positive definite kernel of order m. For any two finite sets X and

Y we use the notation

$$A_{X,Y} := (\Phi(x_j - y_l))_{x_j \in X, y_l \in Y}$$

for the $|X| \times |Y|$ matrix of values of the kernel Φ on X and Y.

Definition 2.1 *For each $|X| \subset I\!R^d$, each k with $q(k) \leq |X|$ and each $\alpha \in V_{X,k}$ we call a function*

$$u_{X,k,\alpha}(x) := A_{\{x\},X}\alpha \tag{3}$$

a kernel B–spline based on Φ with knot set X and annihilation order k.

Note that this generalizes the standard univariate B–spline definition, but it will in general not yield a function with compact support. However, it will in many cases provide a function with strong decay towards infinity, and we shall address this question in the next section.

3 Properties

Theorem 3.1 *For multiquadrics and polyharmonic splines, increasing annihilation orders of B–splines result in improved decay of the B–spline.*

Proof: We take (3) and evaluate the Fourier transform as

$$\begin{array}{rcl} \hat{u}_{X,k,\alpha}(\omega) & = & \hat{\Phi}(\omega)\sigma_{X,k,\alpha}(\omega) \\ \sigma_{X,k,\alpha}(\omega) & = & \displaystyle\sum_{x_j \in X} \alpha_j e^{-i\omega^t x_j}. \end{array} \tag{4}$$

The condition in (1) yields

$$\sigma_{X,k,\alpha}(\omega) = \mathcal{O}(\|\omega\|_2^k) \text{ near } \omega = 0,$$

following from Taylor expansion of the exponential around zero (see e.g. [10]). Thus the second factor in the Fourier transform of the B–spline removes singularities at zero in the first factor. This applies to multiquadrics and polyharmonic splines, leading to the specified decay behavior of the B–spline. $\square$

To prove somewhat more precise results, consider the identity

$$\|x - y\|_2^2 = \|x - z\|_2^2 \cdot (1 + F(x, y, z))$$

with

$$F(x, y, z) = \frac{\|y\|_2^2 - \|z\|_2^2 + 2x^T(z - y)}{\|x - z\|_2^2}$$

for sufficiently large x and bounded y, z. We shall use the fact that F is a quadratic polynomial in y and decays like $\|x\|_2^{-1}$ for $x \to \infty$.

Consider classical multiquadrics first. Then, by expansion of $-\sqrt{1+t}$ around zero, we get

$$\begin{aligned}
& -(1+\|x-x_j\|_2^2)^{1/2} \\
= & -\|x-z\|_2\Big(\|x-z\|_2^{-2}+1+F(x,x_j,z)\Big)^{1/2} \\
= & -\|x-z\|_2\sum_{\ell=0}^{\infty} c_\ell \left((\|x-z\|_2^{-2}+F(x,x_j,z)\right)^{\ell} \\
= & -\sum_{\ell=0}^{\infty} c_\ell\|x-z\|_2^{1-2\ell}\left(1-\|z\|_2^2+\|x_j\|_2^2+2x^T(z-x_j)\right)^{\ell} \\
= & -\sum_{\ell=0}^{\infty} c_\ell\|x-z\|_2^{1-2\ell}\sum_{i_0+i_1+i_2=\ell}\binom{\ell}{i}\left(1-\|z\|_2^2\right)^{i_0}\|x_j\|_2^{2i_1}\left(2x^T(z-x_j)\right)^{i_2}.
\end{aligned}$$

Now let α be a vector that annihilates polynomials on X up to order k as in (1), and form the multiquadric kernel B–spline with these coefficients. Then only terms with $2i_1+i_2 \ge k$ are left in the sum, and we get a decay order of at least

$$1-2\ell+i_2 = 1-2i_0-2i_1-2i_2+i_2 = 1-2i_0-2i_1-i_2 \le 1-2i_0-k \le 1-k.$$

Let us now do the same trick with the polyharmonic spline. First we consider the case of d even, $\Phi(t)=\|t\|^{2s-d}log\|t\|^2$ with $s\in\mathcal{Z}_+$, and we put $2r=2s-d$. We find

$$\begin{aligned}
& \|x-x_j\|_2^{2r}\log\|x-x_j\|_2^2 \\
= & \|x-x_j\|_2^{2r}\log\left(\|x-z\|_2^2\left(1+F(x,x_j,z)\right)\right) \\
= & \|x-x_j\|_2^{2r}\left(\log\|x-z\|_2^2+\log(1+F(x,x_j,z))\right) \\
= & \left(\|x-x_j\|_2^{2r}\log\|x-z\|_2^2\right)+\left(\|x-x_j\|_2^{2r}\log(1+F(x,x_j,z))\right).
\end{aligned}$$

Again, let α be a vector that annihilates polynomials on X up to order k as in (1), and form the kernel B–spline with these coefficients. We now assume $k \ge (2r+1)$ to get rid of the first summand of the above equation, because it is a quadratic polynomial in x_j. We are left with the second one, rewrite it as

$$\begin{aligned}
& \|x-x_j\|_2^{2r}\log(1+F(x,x_j,z)) \\
= & \|x-z\|_2^{2r}(1+F(x,x_j,z))^r\log(1+F(x,x_j,z))
\end{aligned}$$

and expand $(1+t)^r \log(1+t)$ around zero to get

$$
\begin{aligned}
& \|x-z\|_2^{2r}(1+F(x,x_j,z))^r \log(1+F(x,x_j,z)) \\
= \; & \|x-z\|_2^{2r}\sum_{\ell=1}^{\infty} c_\ell F(x,x_j,z)^\ell \\
= \; & \sum_{\ell=1}^{\infty} c_\ell \|x-z\|_2^{2r-2\ell}\left(\|x_j\|_2^2-\|z\|_2^2+2x^T(z-x_j)\right)^\ell \\
= \; & \sum_{\ell=1}^{\infty} c_\ell \|x-z\|_2^{2r-2\ell} \sum_{i_0+i_1+i_2=\ell} \binom{\ell}{i} \left(-\|z\|_2^2\right)^{i_0} \|x_j\|_2^{2i_1} \left(2x^T(z-x_j)\right)^{i_2}.
\end{aligned}
$$

With the same argument as for the multiquadric, we now get a decay order of at least $2r-k$ for $k \geq 2r+1$.

In the case of d odd, the polyharmonic spline is defined as $\|t\|^{2s-d}$ with $s \in \mathcal{Z}_+$; by proceeding as before, we obtain a decay order of at least $2s-d-k$ for $k \geq 2s-d+1$.

Theorem 3.2 *If k is the annihilation order of its coefficient vector, a B–spline based on a classical multiquadric has decay order $1-k$ at infinity. A B–spline based on a classical polyharmonic spline has decay order $2s-d-k$ for $k \geq 2s-d+1$.*

Numerical results show that the stated decay rates are attained. Note that the above technique does not make specific use of radiality, and it gives decay results for any point configuration.

4 Construction and Experiments

This section deals with the numerical construction of kernel B–splines. Omitting indices k and X from (1), we have to construct vectors α with $P^T\alpha = 0$. These will not be unique, and there may be additional conditions that we can impose. In what follows, we shall ignore permutations of points (or, equivalently, columns of P and elements of α). A standard way to handle the condition $P^T\alpha = 0$ in view of the rank property (2) is to find an orthogonal basis of the nullspace of P^T, as provided by the MATLAB command $B = null(P')$. This yields a matrix B of size $|X| \times (|X| - q(k))$ with $B^TB = I$ and $P^TB = 0$. Any $\alpha = B\gamma$ with some $\gamma \in I\!R^{|X|-q(k)}$ will do, and we can impose other conditions to restrict γ.

If the standard RBF system for solving an interpolation problem with $k \geq m$ on X is written as

$$
\begin{array}{rcrcl}
A_{X,X}\alpha & + & P_{X,k}\beta & = & f_X \\
P_{X,k}^T\alpha & + & 0 & = & 0
\end{array}
$$

we can use the matrix B for solving

$$\begin{aligned} B^T A_{X,X} B\gamma &= B^T f_X \\ \alpha &= B\gamma \\ P_{X,k}\beta &= f_X - A_{X,X}\alpha \end{aligned} \tag{5}$$

instead.

We did many experiments in 2D with the multiquadric and the thin plate spline, respectively, using the matrix B as above and looking at the condition of $B^T A_{X,X} B$. The experiments show that when using a radial basis function $\Phi(r)$ in the usual way, we have a better condition for $B^T A_{X,X} B$ than for $A_{X,X}$. About the dependence on k, we found that as k increases, the condition $\mathcal{K}_2(B^T A_{X,X} B)$ decreases. Here are some examples in $2D$, for $M = 100$ scattered data points.

- For thin–plate splines, we found $\mathcal{K}_2(A_{X,X}) = 8.29 \times 10^6$, while for $k = 2$ we have $\mathcal{K}_2(B^T A_{X,X} B) = 2.42 \times 10^6$ and for $k = 6$ we get $\mathcal{K}_2(B^T A_{X,X} B) = 8.84 \times 10^4$; for $k = 13$ the largest value such that $q(k) \leq M$, we have $\mathcal{K}_2(B^T A_{X,X} B) = 8.16$.

- On the same X and using the scaled multiquadric $-\sqrt{1 + (x^2 + y^2)/\delta^2}$ with $\delta = 0.01$, we get $\mathcal{K}_2(A_{X,X}) = 4.22 \times 10^6$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.20 \times 10^6$ for $k = 1$; we get $\mathcal{K}_2(B^T A_{X,X} B) = 3.52 \times 10^4$ for $k = 6$, and for $k = 13$ we get $\mathcal{K}_2(B^T A_{X,X} B) = 7.72$. For $\delta = 1$: we get $\mathcal{K}_2(A_{X,X}) = 3.83 \times 10^{18}$, $\mathcal{K}_2(B^T A_{X,X} B) = 8.23 \times 10^{16}$ with $k = 1$; we get $\mathcal{K}_2(B^T A_{X,X} B) = 3.27 \times 10^{12}$ with $k = 6$, and we get $\mathcal{K}_2(B^T A_{X,X} B) = 2.62 \times 10^4$ with $k = 13$.

The construction along (5) fits the data (X, f_X) by a combination of radial basis functions constrained to have a decay that is function of k, let us say $F_\phi(k)$, and then it calculates the polynomial of order k that fits the residual in the least squares sense. In general, as k increases, features of f are shifted from the combination of the radial basis function part constrained to decay as $F_\phi(k)$, and they are captured by the polynomial instead. It is known that the combination of the radial part, plus the polynomial of minimal order that guarantees strict positivity, can fit f with full accuracy, but the polynomial of large order might present undue oscillations in regions with scarce data and in particular at the boundary, so in general it is not recommended to take the largest k such that $q(k) \leq M$.

Our experience shows that k should not be larger than six for smooth bivariate functions, when M is of the order of one hundred. However, values of k larger than six can provide accurate results when f is well fitted by a polynomial (with the multiquadric and with δ large enough, we usually have an improvement of many orders of $\mathcal{K}_2(B^T A_{X,X} B)$ with respect to $\mathcal{K}_2(A_{X,X})$ in this case) or, when using uniformely scattered data, we have in addition a good information at the border. Here we provide examples. We use the multiquadric with suitable choices of δ and k such that the results are accurate both in terms of error and of graphical appearance. The discrete root mean squared error e_2 and the discrete maximum error e_∞, both computed at the points of a uniform grid 61×61, are provided.

Example 1: $M = 121$ mildly scattered data from $f(x, y) = (\sqrt{x^2 + y^2} - 0.6)_+^4$ within $[0, 1] \times [0, 1]$. The results for $k = 10$ and $\delta = 0.35$ are $e_2 = 3.1 \times 10^{-5}$, $e_\infty = 1 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.4 \times 10^5$ while $\mathcal{K}_2(A_{X,X}) = 1.7 \times 10^{10}$. The graphical output is shown in Fig. 1.

Example 2: $M = 161$ data of which 121 are mildly scattered within $[0, 1] \times [0, 1]$ and 40 on the boundary from Franke's "humps and dips" function. The results for $k = 10$ and $\delta = 0.35$ are $e_2 = 8.6 \times 10^{-4}$, $e_\infty = 5.6 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 10^6$. The graphical output is shown in Fig. 2. We found $\mathcal{K}_2(A_{X,X}) = 6.3 \times 10^{10}$.

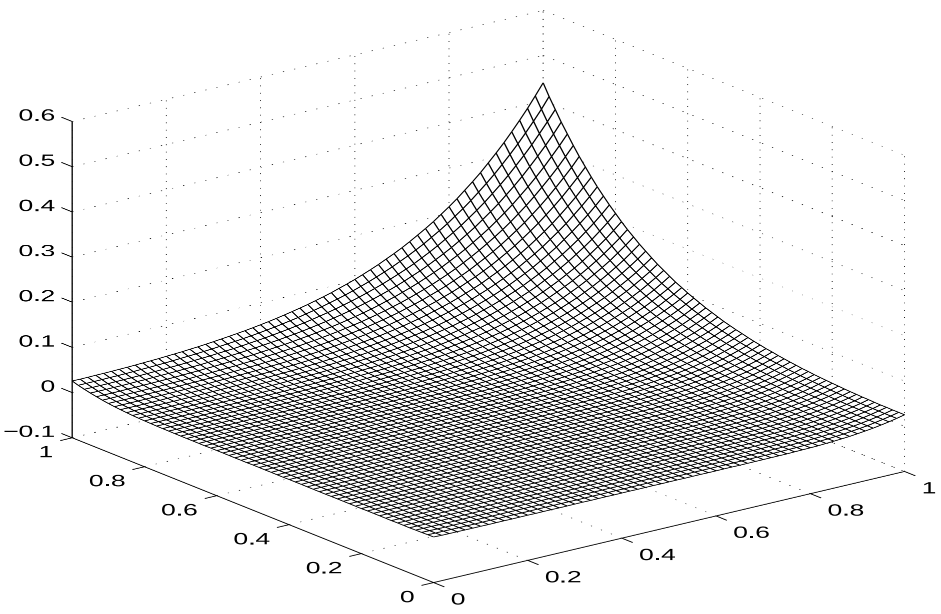

Figure 1: Example 1: reconstruction.

5 Shifted B–Spline Kernels

From here on, we use a fixed generalized finite difference scheme and shift it around.

Theorem 5.1 *Let Φ be translation–invariant and conditionally positive definite of order m on $I\!R^d$ with m minimal. Furthermore, Φ should have a generalized Fourier transform which is positive almost everywhere on $I\!R^d$. Let Y be a discrete set of points around the origin, and let $\alpha \in I\!R^{|Y|}$ be a nonzero vector with annihilation order $k \geq m$ on Y. Then the* shifted B–spline kernel

$$\Psi(x) := \sum_{y_j \in Y, y_l \in Y} \alpha_j \alpha_\ell \Phi(x - (y_j - y_\ell))$$

is positive definite on $I\!R^d$.

Proof: The Fourier transform of Ψ is

$$\begin{aligned} \hat{\Psi}(\omega) &= \hat{\Phi}(\omega) \sum_{y_j \in Y, y_l \in Y} \alpha_j \alpha_\ell e^{-i\omega^T (y_j - y_\ell)} \\ &= \hat{\Phi}(\omega) \left| \sum_{y_j \in Y} \alpha_j e^{-i\omega^T y_j} \right|^2 \end{aligned}$$

and since the singularity of the transform at zero is cancelled, the assertion follows. Here, we made use of the fact [10] that m is the smallest nonnegative integer such that $\hat{\Phi}(\omega)\|\omega\|_2^{2m}$ is integrable around the origin. $\square$.

Remark. If $k \geq m$ is not satisfied, one gets a kernel that still is conditionally positive definite of order $m - k$.

The above result allows to use shifted kernel B–splines within the standard setting of interpolation by kernel functions. The decay of the shifted kernel B–splines is useful for system solving, but it may be necessary to add polynomials to cope with global trends.

Since the error analysis and stability properties of kernels are dominated by smoothness properties, as far as orders are concerned, the shifted B–spline kernel Ψ inherits the properties of Φ. Improvements in error and stability behavior can therefore only be effected via multiplicative constants that differ from those obtained for the pure kernels. We provide a specific example later.

The condition of the matrix A of an interpolation system mainly consists of the part $\|A^{-1}\|$ contributed by the smallest eigenvalue of A. This part is a function of the separation distance of points, and its order is not affected by taking linear combinations. However, for increasing kernels the $\|A\|$ part is also relevant, and here the application of divided differences can make use of the available decay of the constructed kernel B–splines.

6 Special B–Spline Kernels

To get away with a minimal number of points for a specified polynomial order k, one should take a set Y with $|Y| = q(k) + 1$ that is in general position, i.e. the rank condition (2) holds. It appears that the decay of the resulting B–spline is closely related to the order of polynomials annihilated by the coefficient vector, as theoretically confirmed by the results of Section 3.

We take the seven points in $Y := \{y_0, y_1, \ldots, y_6\} \subset I\!R^2$ to be the origin $y_0 = 0$ and the six roots of unity $y_1, \ldots, y_6$ scaled by the factor $\rho > 0$. The coefficient vector is $\alpha = \frac{1}{6}(6, -1, -1, -1, -1, -1, -1)$, and it is easy to see that we have annihilation order 2. In fact the Fourier transform factor in (4) is

$$\begin{aligned} \sigma(\omega) &= \sum_{j=0}^{6} \alpha_j e^{-i\omega^T y_j} \\ &= 1 - \frac{4}{6}\cos(\frac{\rho\omega_1}{2})\cos(\frac{\rho\omega_2\sqrt{3}}{2}) - \frac{2}{6}\cos(\rho\omega_1) =: T(\rho\omega) \end{aligned} \tag{6}$$

by straightforward computations. It is nonnegative and vanishes only at the isolated points

$$\left(\frac{4k\pi}{\rho}, \frac{4\ell\pi}{\sqrt{3}\rho}\right)$$

$$\left(\frac{(4k+2)\pi}{\rho}, \frac{(4\ell+2)\pi}{\sqrt{3}\rho}\right)$$

for all integers k, ℓ, forming a hexagonal grid as an overlay of two standard grids. Clearly, the expansions around all of these points, including zero, vanish to second order.

When considering the scaled multiquadric as ϕ, the resulting *hexagonal multiquadric B–spline* is not radial, but close to radial for $\rho < \delta$. Furthermore, the perturbation theory of [2] applies here, because it can be easily generalized to the translation–invariant setting.

In Fig. 3 we show the behaviour of the close-to-radial hexagonal multiquadric B–spline. It was calculated with $\rho = 0.142$ and $\delta = 1$ and normalized to have maximum equal to one.

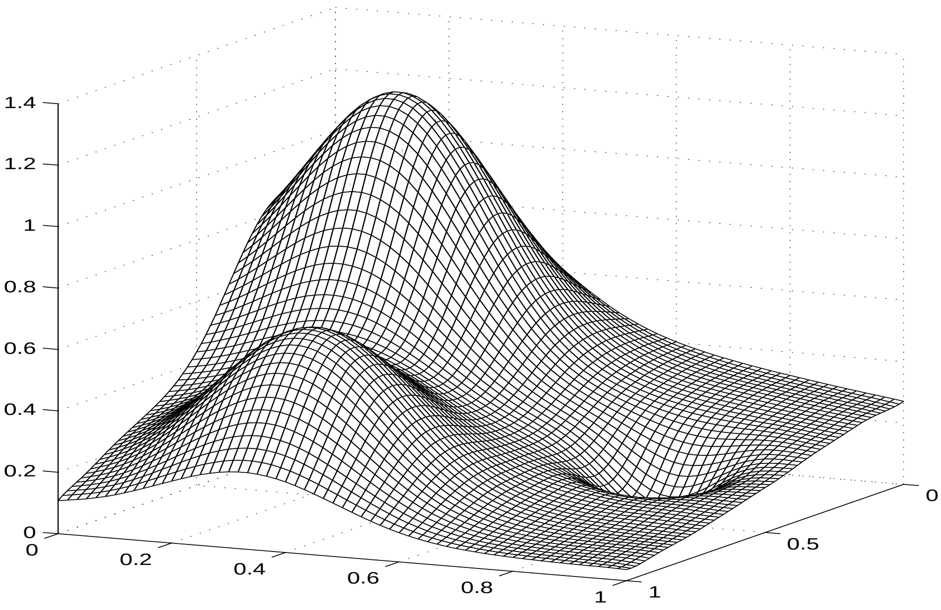

Figure 2: Example 2: reconstruction.

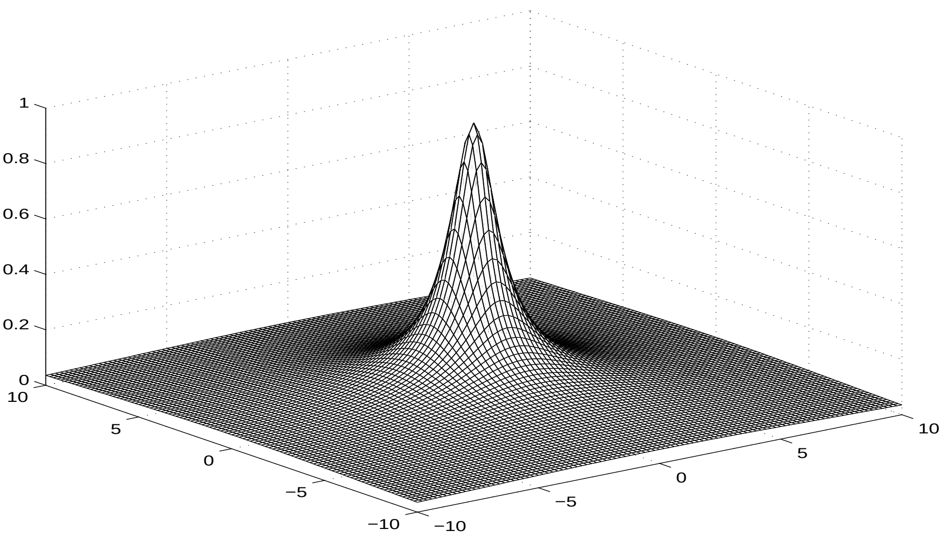

Figure 3: Hexagonal multiquadric B–spline.

Following the hexagonal multiquadric B–spline based on 7 points, we now define a *Laplacian multiquadric B–spline*. We take five points in $Y := \{y_0, y_1, \ldots, y_4\} \subset I\!R^2$ to be the origin $y_0 = 0$ and the four roots of unity $y_1, \ldots, y_4$ scaled by the factor $\rho > 0$. In this case the coefficient vector is $\alpha = \frac{1}{4}(4, -1, -1, -1, -1)$ and the Fourier transform factor in (4) is

$$\begin{aligned} \sigma(\omega) &= \sum_{j=0}^{4} \alpha_j e^{-i\omega^T y_j} \\ &= 1 - \frac{1}{2} cos(\rho\omega_1) - \frac{1}{2} cos(\rho\omega_2) := T(\rho\omega). \end{aligned} \tag{7}$$

It is nonnegative and vanishes only at the isolated points of the grid

$$(\frac{2k\pi}{\rho}, \frac{2l\pi}{\rho})$$

for all integers k and l. The expansions around all of these points, including zero, vanish to second order. The resulting B–spline is close to radial when ρ is considerably less than δ. In Fig. 4 we show the Laplacian multiquadric B–spline normalized to have maximum equal to one; here the parameters are $\rho = 0.05$ and $\delta = 1$.

7 Stability

Theorem 7.1 *If q is the minimal separation distance of the data, and if we take $\rho = 0.142q$, the smallest eigenvalue of the interpolation matrix defined via the hexagonal multiquadric B–spline is at least by a factor of $4/3$ larger than the smallest eigenvalue of the matrix for unscaled multiquadric interpolation.*

Proof. Following [10] we have the generalized Fourier transform of the 2D multiquadric as

$$\hat{\Phi}(\omega) = \frac{(1 + \|\omega\|_2) \exp(-\|\omega\|_2)}{\|\omega\|_2^3},$$

up to a constant, which is monotone decreasing and thus attains its minimum

$$\varphi_0(M) := \frac{(1 + 2M) \exp(-2M)}{8M^3}$$

on all $\|\omega\|_2 \leq 2M$. Note that this function is central in Theorem 3.1 of [7] for proving stability bounds for the multiquadric. In order to apply this theorem, we have to evaluate

$$\psi_0(M, \rho) := \inf_{\|\omega\|_2 \leq 2M} \hat{\Phi}(\omega) T(\rho\omega)$$

with T of (6). We used MAPLE to give a radial local lower bound for T by

$$T(\rho\omega) \geq 1 - \frac{1}{3}\cos(\frac{\sqrt{3}}{2}\rho\|\omega\|_2) - \frac{2}{3}\cos^2(\frac{\sqrt{3}}{4}\rho\|\omega\|_2) =: G(\rho\|\omega\|_2)$$

in the range $\rho\|\omega\|_2 \leq \frac{2\pi}{\sqrt{3}}$. MAPLE also shows that for ρ fixed, the quantity $\hat{\Phi}(\omega)G(\rho\|\omega\|_2)$ is monotone decreasing with respect to $\|\omega\|_2$. Thus we look at $\hat{\Phi}(2M)G(2M\rho)$ and get the value $\frac{4}{3}\hat{\Phi}(2M)$ for $\rho = \pi/(M\sqrt{3})$. This proves $\psi_0(M, \pi/(M\sqrt{3})) \geq \frac{4}{3}\varphi_0(M)$ for all M.

To provide a lower bound to the smallest eigenvalue λ of A, we follow the line of argument of §3 in [7] with $M := \frac{12.76}{q}$, where q is the minimal separation distance of the data locations. This value of M is the optimal one for bounding the smallest eigenvalue of the multiquadric interpolation matrix from below. Thus for $\rho = 0.142q$ we have an improvement of the lowest eigenvalue by a factor of 4/3. □

In the case of the Laplacian multiquadric B-spline, there is no improvement of the lowest eigenvalue respect to the one of the multiquadric, but because of ρ smaller than in the case of the hexagonal multiquadric B-spline, the two B-spline kernels can get equivalent stability.

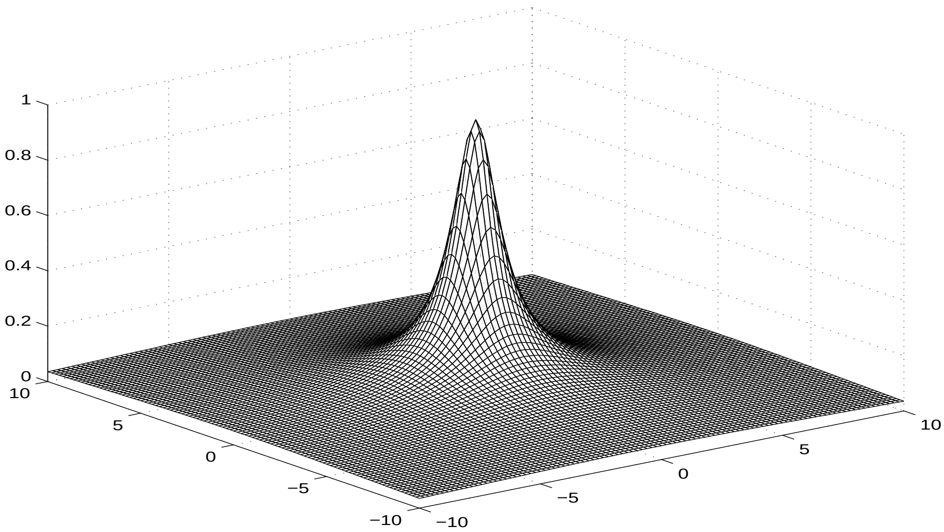

Figure 4: Laplacian multiquadric B–spline.

8 Numerical Experiments with B–Spline Kernels

In this section we show that in numerical applications actually we have a good stability together with a good recovery of the interpolated function. In all cases we have considered the hexagonal multiquadric B-spline $u = u_Y$, which gave good results when we approximate a function by few significant points obtained from a large sample [1]. First of all we consider the condition number of the interpolation matrix $A^u(\delta)$ in comparison with the condition number of the interpolation matrix $A^\Phi(\delta)$ for the classical multiquadric.

We provide the values for the case of 100 scattered data on $[0,1] \times [0,1]$.

- For $\delta = 0.01$ we have $\mathcal{K}_2(A^u(\delta)) = 840$ and $\mathcal{K}_2(A^\Phi(\delta)) = 4.2 \times 10^6$,
- For $\delta = 0.5$ we have $\mathcal{K}_2(A^u(\delta)) = 6.9 \times 10^5$ and $\mathcal{K}_2(A^\Phi(\delta)) = 1.1 \times 10^{14}$,
- For $\delta = 1$ we have $\mathcal{K}_2(A^u(\delta)) = 6.8 \times 10^{12}$ and $\mathcal{K}_2(A^\Phi(\delta)) = 3.8 \times 10^{18}$.

Now we provide two examples for the recovery of a function. As before we consider the errors e_2 and e_∞ computed on the uniform grid 61×61.

Example 1: $M = 101$ scattered data (mildly scattered data except for a cluster of two data) from the function defined as

$$\begin{cases} \Gamma := (x_\Gamma = x, y_\Gamma = 0.6 \cdot sin(\pi x/1.2)) \qquad x \in [0.3, 0.7] \\ d(x,y) := \min_\Gamma((x - x_\Gamma)^2 + (y - y_\Gamma)^2) \\ f(x,y) := 0.1\ exp(-d(x,y)) \end{cases}$$

within $[0,1] \times [0,1]$. The results for $\delta = 0.5$ are $e_2 = 6.0 \times 10^{-5}$, $e_\infty = 4.4 \times 10^{-4}$. We have $\mathcal{K}_2(A^u(\delta)) = 4.6 \times 10^{13}$ while $\mathcal{K}_2(A^\Phi(\delta)) = 5.6 \times 10^{17}$. The reconstruction is shown in the figure 5.

We get a similar value of the condition when using the kernel B-spline adapted to X and based on the multiquadric with $\delta = 0.5$, but with a slight loss of accuracy. In fact with $k = 5$ we get $e_2 = 7.4 \times 10^{-5}$, $e_\infty = 7.0 \times 10^{-4}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.3 \times 10^{14}$; with $k = 6$ we get $e_2 = 9.7 \times 10^{-5}$ and $e_\infty = 1.5 \times 10^{-3}$ and $\mathcal{K}_2(B^T A_{X,X} B) = 1.9 \times 10^{13}$.

Example 2: $M = 1024$ mildly scattered data from the "peaks" function in MATLAB. The results for $\delta = 1$ are $e_2 = 3.8 \times 10^{-5}$, $e_\infty = 1.1 \times 10^{-3}$. We have $\mathcal{K}_2(A^u(\delta)) = 4.5 \times 10^{10}$ while $\mathcal{K}_2(A^\Phi(\delta)) = 2.0 \times 10^{14}$. The reconstruction is shown in the figure 6; by the kernel B spline adapted to X and based on Φ multiquadric with $\delta = 1$ and $k = 6$, we get $e_2 = 2.3 \times 10^{-4}$ and $e_\infty = 7.1 \times 10^{-3}$, and we get $B^T A_{X,X} B = 2.3 \times 10^{11}$.

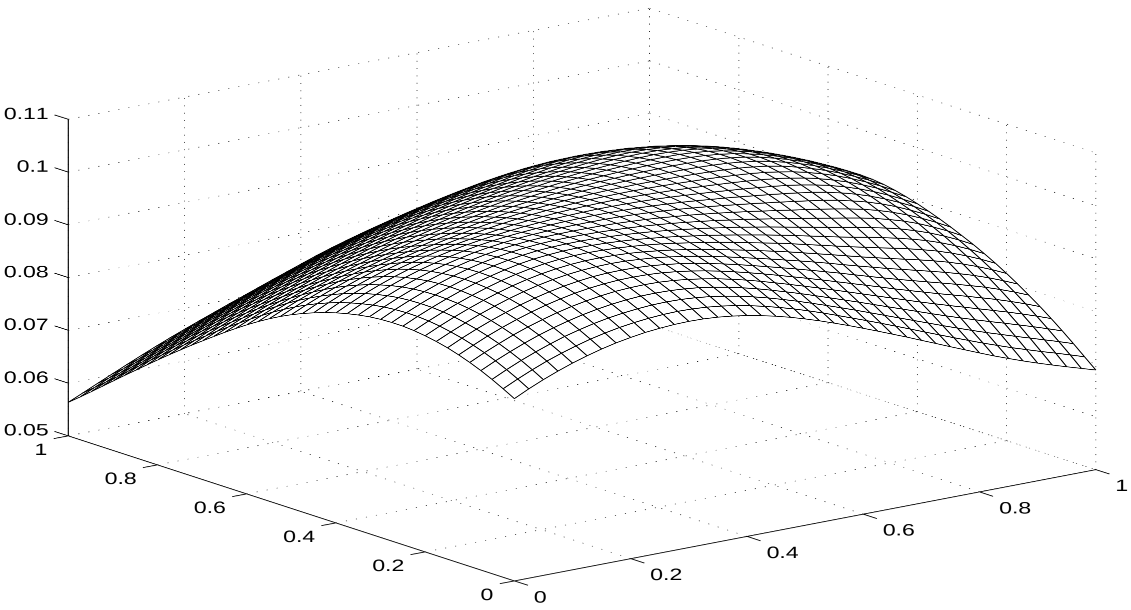

Figure 5: Example 1: reconstruction.

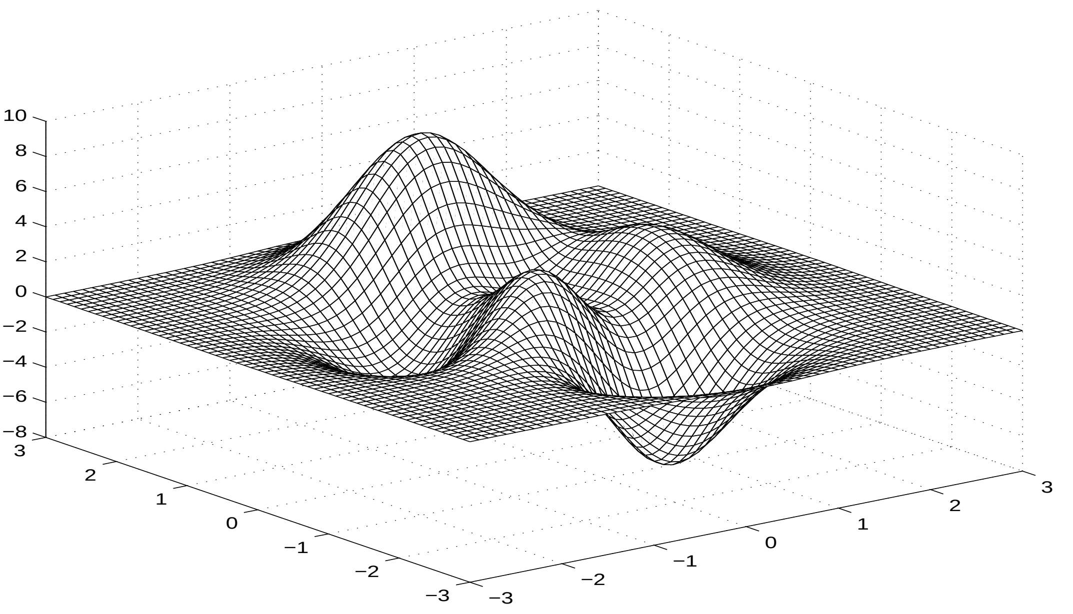

Figure 6: Example 2: reconstruction.

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