Native Hilbert Spaces for Radial Basis Functions II

Robert Schaback, Göttingen

Abstract. This contribution continues an earlier survey [20] over the native spaces associated to (not necessarily radial) basis functions. After recalling the basics, the relation to L_2 spaces is studied. This leads to a new formulation of the theory of radial basis functions in the context of integral operators. Instead of Fourier transforms, the most important tools now are expansions into eigenfunctions. This unifies the theory of radial basis functions in $\mathbb{R}^d$ with the theory of zonal functions on the sphere S^{d-1} and the theory of kernel functions on Riemannian manifolds. New characterizations of native spaces and positive definite functions are provided within this context.

§0. Introduction

Since this paper is an extension of [20], we first recall the introduction to [20] with slight modifications. The final three paragraphs will go over to the current paper.

For the numerical treatment of functions of many variables, *radial basis functions* are useful tools. They have the form $\phi(\|x-y\|_2)$ for vectors $x, y \in \mathbb{R}^d$ with a *univariate* function ϕ defined on $[0, \infty)$ and the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^d$. This allows to work efficiently for large dimensions d, because the function boils the multivariate setting down to a univariate setting. Usually, the multivariate context comes back into play by picking a large number M of points $x_1, \ldots, x_M$ in $\mathbb{R}^d$ and working with linear combinations

$$s(x) := \sum_{j=1}^{M} \lambda_j \phi(\|x_j - x\|_2).$$

In certain cases, low–degree polynomials have to be added, and these complications are dealt with in section 5 of [20], while section 6 shows how to get rid of these. However, in the definitions and in the notation we still keep a space $\mathcal{P}$ which plays the role of polynomials.

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Besides the classical radial basis functions on the whole space $\mathbb{R}^d$, the survey [20] also covers *zonal* functions on the $(d-1)$–dimensional sphere $S^{d-1} \subset \mathbb{R}^d$. These have the form $\phi(x^T y) = \phi(\cos(\alpha(x,y)))$ for points x, y on the sphere spanning an angle of $\alpha(x,y) \in [0,\pi]$ at the origin. Here, the symbol ^{T} denotes vector transposition, and the function ϕ should be defined on $[-1,1]$. Periodic multivariate functions can also be treated, e.g. by reducing them to products of univariate periodic functions. Another very important case are basis functions on Riemannian manifolds, as introduced by Narcowich [13] and investigated by Dyn, Narcowich, and Ward [2]. Here, we consider symmetric functions $\Phi : \Omega \times \Omega \to \mathbb{R}$ on some domain $\Omega \subseteq \mathbb{R}^d$, covering the above situations.

All of these cases of *basis functions* share a common theoretical foundation. The functions all have a unique associated "native" Hilbert space $\mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$ of functions in which they act as a generalized reproducing kernel. The different special cases (radiality, zonality) are naturally related to geometric invariants of the native spaces. The first part of the survey [20] thus starts in section 2 with reproducing kernel Hilbert spaces and looks at geometric invariants later in section 3.

But most basis functions are constructed directly and do not easily provide information on their underlying native space. Their main properties are symmetry and (strict) positive definiteness (SPD) or conditionally positive definiteness (CPD). These notions are defined without any relation to a Hilbert space, and one then has to construct the native space, prove its uniqueness, and find its basic features. The survey [20] does this for SPD functions in section 4 and for CPD functions in section 5. The results mostly date back to classical work on reproducing kernel Hilbert spaces and positive definite functions (see e.g. [12,17]). We finished [20] with a short account of optimal recovery of functions in native spaces from given data, and provided the corresponding error bounds based on power functions.

There are different ways to define native spaces (see [10] for comparisons), but in the first part [20] of the survey we wanted to provide just one technique that is general enough to unify different constructions (e.g. on the sphere [3] or on Riemannian manifolds [2,13]). But we avoided advanced tools like Fourier transforms or expansions into series of spherical harmonics or eigenfunctions of the Laplace–Beltrami operator.

In this continuation of [20], we start with embedding of native spaces into $L_2(\Omega)$. This provides a very useful link to the theory of integral operators and their eigenfunction expansions. We use these expansions as a replacement for transforms in the classical variations of the theory. Consequently, we get new characterizations of native spaces via such expansions, and new construction techniques for positive definite functions.

The notation and numbering will simply extend from [20] in a straightforward way. We strongly advise the reader to have a copy of [20] available, since we cannot recall all definitions and results here. The references, however, are repeated for convenience of the reader.

§12. Connection to L_2 spaces: Overview

This section starts an analysis of native spaces directed towards the well–known representation of the "energy inner product" of classical splines in the form

$$(f,g)_\Phi := (Lf, Lg)_{L_2(\Omega)} =: (Lf, Lg) \tag{12.1}$$

with some linear differential operator L. Natural univariate splines of odd degree $2n-1$ are related to $L = d^m/dx^m$ on $\Omega = [a,b] \subset \mathbb{R}$. Furthermore, the fundamental work of Duchon [1] on thin–plate and polyharmonic splines is based strongly on the use of $L = \Delta^m$. For general (not necessarily radial) basis functions Φ, there is no obvious analogue of such an operator. However, we want to take advantage of (12.1) and thus proceed to work our way towards a proper definition of L, starting from the bilinear form $(.,.)_\Phi$ that we defined in [20], and which led us to the notion of the native space $\mathcal{N}_{\Phi,\mathcal{P}}(\Omega) = \mathcal{F}_{\Phi,\mathcal{P}}(\Omega) + \mathcal{P}$ of a conditionally positive definite function Φ on a domain $\Omega \subseteq \mathbb{R}^d$ with respect to a finite–dimensional space $\mathcal{P}$.

Since the procedure is somewhat complicated, we give an overview here, and point out the reasons for certain arguments that may look like unnecessary detours. We first have to relate the native space somehow to $L_2(\Omega)$. To achieve this, we simply imbed the major part $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$ of the native space $\mathcal{N}_{\Phi,\mathcal{P}}(\Omega) = \mathcal{F}_{\Phi,\mathcal{P}}(\Omega) + \mathcal{P}$ into $L_2(\Omega)$. Then we study the adjoint C of the embedding, which turns out to be a convolution–type integral operator with kernel Φ that finally will be equal to $(L^*L)^{-1}$. We thus have to form the "square root" of the operator C and invert it to get L. Taking the square root requires nonnegativity of C in the sense of integral operators. This is a property that is intimately related to (strict) positive definiteness of the kernel Φ, and thus in section 16 we take a closer look at the relation of these two notions. In between, section 15 will provide a first application of the technique we develop here: we can generalize a proof of an increased convergence order, replacing Fourier transforms by eigenfunction expansions. Finally, we give a characterization of the native space and of positive definite functions. In the notation we shall always use $(\cdot,\cdot)$ to denote the inner product in $L_2(\Omega)$.

§13. Embedding into L_2

There is an easy way to imbed a native space into an L_2 space.

Lemma 13.1. *Let Φ be symmetric and conditionally positive definite (CPD) with respect to $\mathcal{P}$ on Ω, and let Ψ be the normalized kernel with respect to Φ as defined in section 6. Assume*

$$C_2^2 := \int_\Omega \Psi(x,x)dx < \infty. \tag{13.1}$$

Then the Hilbert space $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega) \subseteq \mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$ for Φ has a continuous linear embedding into $L_2(\Omega)$ with norm at most C_2.

Proof: Conditional positive definiteness clearly implies that the integrand

$$\Psi(x,x) = (\delta_{(x)}, \delta_{(x)})_\Phi = \|\delta_{(x)}\|_\Phi^2$$

is positive when forming (13.1). □

Now for all $f \in \mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$ and all $x \in \Omega$ we can use the reproduction property (5.11) to get

$$\begin{aligned} f(x)^2 &= (f, \Psi(x,\cdot))_\Phi^2 \\ &\leq \|f\|_\Phi^2 \|\Psi(x,\cdot)\|_\Phi^2 \\ &= \|f\|_\Phi^2 \Psi(x,x), \end{aligned}$$

where we used $\Pi_{\mathcal{P}} f = 0$ for the functions $f \in \mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$. Then the assertion follows by integration over Ω. □

By the way, the above inequality shows in general how upper bounds for functions in the native space can be derived from the behaviour of Ψ on the diagonal of $\Omega \times \Omega$. And, sometimes, the related geometric mean inequality

$$\Psi(x,y)^2 \leq \Psi(x,x)\Psi(y,y)$$

is useful, following directly from (6.1) or via $f(x) := \Psi(x,y)$ from the above argument.

§14. The convolution mapping from L_2 into $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$

We now go the other way round and map $L_2(\Omega)$ into the native space.

Theorem 14.1. *Assume (13.1) to hold for a CPD function Φ on Ω. Then the integral operator*

$$C(v)(x) := \int_\Omega v(t)\Psi(x,t)dt \tag{14.1}$$

of generalized convolution type maps $L_2(\Omega)$ continuously into the Hilbert space $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega) \subseteq \mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$. It has norm at most C_2 and satisfies

$$(f,v) = (f, C(v))_\Phi \text{ for all } f \in \mathcal{F}_{\Phi,\mathcal{P}}(\Omega),\ v \in L_2(\Omega), \tag{14.2}$$

i.e. it is the adjoint of the embedding of the Hilbert subspace $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$ of the native space $\mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$ into $L_2(\Omega)$.

Proof: We use the definition of $\mathcal{M}_{\Phi,\mathcal{P}}(\Omega)$ in Theorem 8.1 and pick some finitely supported functional $\lambda \in L_{\mathcal{P}}(\Omega)$ to get

$$\begin{aligned} \lambda(C(v)) &= \int_\Omega v(t)\lambda^x\Psi(x,t)dt \\ &\leq \|v\|\|\lambda^x\Psi(x,\cdot)\| \\ &\leq C_2\|v\|\|\lambda\|_\Phi \end{aligned}$$

for all $v \in L_2(\Omega)$. In case of $f(t) := \Psi(x,t)$ with arbitrary $x \in \Omega$, equation (14.2) follows from the definition of the operator C and from the reproduction property. The general case is obtained by continuous extension. □

Of course, equation (14.2) generalizes to

$$(f - \Pi_{\mathcal{P}} f, v) = (f - \Pi_{\mathcal{P}} f, C(v))_\Phi = (f, C(v))_\Psi \text{ for all } f \in \mathcal{N}_{\Phi,\mathcal{P}}(\Omega),\ v \in L_2(\Omega)$$

on the whole native space $\mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$. We add two observations following from general properties of adjoint mappings:

Corollary 14.2. *The range of the convolution map C is dense in the Hilbert space $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$. The latter is dense in $L_2(\Omega)$ iff C is injective.* $\square$

To prove criteria for injectivity of C or, equivalently, for density of the Hilbert space $\mathcal{F}_{\Phi,\mathcal{P}}(\Omega)$ in $L_2(\Omega)$, is an open problem, at least in the general situation. For SPD functions $\Phi(x,y) = \phi(x-y)$ on $\Omega = \mathbb{R}^d$ with a strictly positive d–variate Fourier transform $\hat{\phi}$ there is a neat argument due to A.L. Brown that does the job. In fact, if there is some $v \in L_2(\Omega)$ such that $(v, \Phi(x,\cdot))_{L_2(\Omega)} = 0$ for all $x \in \Omega$, then $\hat{v} \cdot \hat{\phi} = 0$ must hold on $\mathbb{R}^d$, and then $v = 0$ in $L_2(\Omega)$.

We finally remark that the above problem is related to the specific way of defining an SPD or CPD function via finitely supported functionals. Section 16 will shed some light on another feasible definition, and we can revisit the problem in section 20 after we have replaced Fourier transforms by eigenfunction expansions.

§15. Improved convergence results

The space $C(L_2(\Omega))$ allows an improvement of the standard error estimates for reconstruction processes of functions from native spaces. Roughly speaking, the error bound can be "squared".

Theorem 15.1. *If an interpolatory recovery process in the sense of Theorem 11.1 is given, then there is a bound*

$$|f(x) - s_f^*(x)| \leq P^*(x)\|P^*\|\|v\|$$

for all $f - \Pi_{\mathcal{P}} f = C(v) \in \mathcal{N}_{\Phi,\mathcal{P}}(\Omega)$, $x \in \Omega$, $v \in L_2(\Omega)$. Here, we denote the optimized power function for the special situation in Theorem 11.1 by P^*.

Proof: Taking the L_2 norm of the standard error bound in Theorem 10.3, we get

$$\|f - s_f^*\| \leq \|P^*\|\|f - s_f^*\|_\Psi.$$

Now we use (14.2) and the orthogonality relation from Theorem 11.3:

$$\begin{aligned}
\|f - s_f^*\|_\Psi^2 &= (f - s_f^*, f - s_f^*)_\Psi \\
&= (f - s_f^*, f)_\Psi \\
&= (f - s_f^*, C(v))_\Psi \\
&= (f - s_f^*, v) \\
&\leq \|f - s_f^*\|\|v\| \\
&\leq \|P^*\|\|f - s_f^*\|_\Phi\|v\|.
\end{aligned}$$

Cancelling $\|f - s_f^*\|_\Phi$ and inserting the result into the error bound of Theorem 10.3 proves the assertion. $\square$

An earlier version of this result, based on Fourier transforms and restricted to functions on $\Omega = \mathbb{R}^d$ was given in [21]. Note that Theorem 15.1

holds only for functions in the range of the convolution map C, i.e. in a subspace of the native space. The study of the range of C is a challenging task, because there are numerical reasons to suggest that certain boundary effects are involved. We shall come back to this issue in section 19.

§16. Positive integral operators

We now look at the operator C from the point of view of integral equations. The compactness of C as an operator on $L_2(\Omega)$ will be delayed somewhat, because we first want to relate our definition of a positive definite function to that of a positive integral operator. The latter property will be crucial in later sections.

Definition 16.1. *An operator C of the form (14.1) is* `positive` *(*`nonnegative`*), if the bilinear form*

$$(w, C(v)),\ v, w \in L_2(\Omega)$$

is symmetric and positive (nonnegative) definite on $L_2(\Omega)$.

In our special situation we can write

$$(w, C(v)) = (C(w), C(v))_\Phi,\ v, w \in L_2(\Omega)$$

and get

Theorem 16.2. *If a symmetric and positive semidefinite function Φ on Ω satisfies (13.1), then the associated integral operator C is nonnegative. If this holds, positivity is equivalent to injectivity.* □

Theorem 16.3. *Conversely, if C is a nonnegative integral operator of the form (13.1) with a symmetric and continuous function $\Phi : \Omega \times \Omega \to \mathbb{R}$, then Φ is positive semidefinite on Ω.*

Proof: We simply approximate point evaluation functionals δ_x by functionals on $L_2(\Omega)$ that take a local mean. Similarly, we approximate finitely supported functionals by linear combinations of the above form. The rest is standard, but requires continuity of Φ. □

Unfortunately, the above observations do not allow to conclude positive definiteness of Ψ from positivity of the integral operator C. It seems to be an open problem to bridge this gap. However, due to the symmetry of Ψ, the integral operator C is always self–adjoint.

¿From here on, we will restrict ourselves to the strictly positive definite (SPD) case. The main reason is to keep the presentation technically simple. The general case can be treated either by going over to the regularized kernel given by (6.6) or by carefully rewriting the material of the following sections.

§17. Compact nonnegative self–adjoint integral operators

To apply strong results from the theory of integral equations, we still need that C is compact on $L_2(\Omega)$. This is implied by the additional condition

$$\int_\Omega \int_\Omega \Phi(x,y)^2 dxdy < \infty \tag{17.1}$$

which is automatically satisfied if our SPD function Φ is continuous and Ω is compact. Note the difference to (13.1), which is just enough to ensure embedding of the native space into $L_2(\Omega)$. Note further that (17.1) rules out certain familiar cases like the Gaussian on $\mathbb{R}^d$. It is an open problem to handle this situation, and here may be a subtle difference between working on bounded or unbounded domains.

¿From now on, we assume Φ to be an SPD kernel satisfying (13.1) and (17.1). Then C is a compact self–adjoint nonnegative integral operator. Now spectral theory and the theorem of Mercer [18] imply the following facts:

1. There is a finite or countable set of positive real eigenvalues $\rho_1 \geq \rho_2 \geq \ldots > 0$ and eigenfunctions $\varphi_1, \varphi_2, \ldots \in L_2(\Omega)$ such that
$$C(\varphi_n) = \rho_n \varphi_n,\ n = 1, 2, \ldots.$$
2. The eigenvalues ρ_n converge to zero for $n \to \infty$, if there are infinitely many.
3. There is an absolutely and uniformly convergent representation
$$\Phi(x,y) = \sum_n \rho_n \varphi_n(x) \varphi_n(y),\ x, y \in \Omega. \tag{17.2}$$
4. The functions φ_n are orthonormal in $L_2(\Omega)$.
5. Together with an orthonormal basis of the kernel of C, the functions φ_n form a complete orthonormal system in $L_2(\Omega)$.
6. There is a nonnegative self–adjoint operator $\sqrt[*]{C}$ such that $C = \sqrt[*]{C}\sqrt[*]{C}$ and with an absolutely and uniformly convergent kernel representation
$$\sqrt[*]{\Phi}(x,y) := \sum_n \sqrt{\rho_n} \varphi_n(x) \varphi_n(y),\ x, y \in \Omega, \tag{17.3}$$
where
$$\sqrt[*]{C}(v)(x) := \int_\Omega v(t) \sqrt[*]{\Phi}(x,t) dt,\ x \in \Omega,\ v \in L_2(\Omega).$$

We use the symbol $\sqrt[*]{\Phi}$ to denote the "convolution square–root", because

$$\Phi(x,y) = \int_\Omega \sqrt[*]{\Phi}(x,t) \sqrt[*]{\Phi}(t,y) dt \tag{17.4}$$

is a generalized convolution. We remark that this equation can be used for construction of new positive definite functions by convolution, and we provide details in section 20.

The situation of finitely many eigenvalues cannot occur for the standard case of continuous SPD kernels on bounded domains with infinitely many points and linearly independent point evaluations. Otherwise, the rank of matrices of the form $(\Phi(x_j, x_k))_{1 \leq j,k \leq N}$ would have a global upper bound.

§18. The native space revisited

The action of C on a general function $v \in L_2(\Omega)$ can now be rephrased as

$$C(v) = \sum_n \rho_n (v, \varphi_n) \varphi_n,$$

and it is reasonable to define an operator L such that $(L^*L)^{-1} = C$ formally by

$$L(v) = \sum_n (\rho_n)^{-1/2} (v, \varphi_n) \varphi_n. \tag{18.1}$$

We want to show that this operator nicely maps the native space into $L_2(\Omega)$, but for this we have to characterize functions from the native space in terms of expansions with respect to the functions φ_n.

Theorem 18.1. *The native space for an SPD function Φ which generates a nonnegative compact integral operator on $L_2(\Omega)$ can be characterized as the space of functions $f \in L_2(\Omega)$ with $L_2(\Omega)$–expansions*

$$f = \sum_n (f, \varphi_n) \varphi_n$$

such that the additional summability condition

$$\sum_n \frac{(f, \varphi_n)^2}{\rho_n} < \infty$$

holds.

Proof: We first show that on the subspace $C(L_2(\Omega))$ of the native space $\mathcal{N}_\Phi(\Omega)$ we can rewrite the inner product as

$$\begin{aligned}(C(v), C(w))_\Phi &= (v, C(w)) \\ &= \sum_n (v, \varphi_n)(C(w), \varphi_n) \\ &= \sum_n \frac{(C(v), \varphi_n)(C(w), \varphi_n)}{\rho_n}\end{aligned}$$

But this follows from $(C(v), \varphi_n) = \rho_n (v, \varphi_n)$ for all $v \in L_2(\Omega)$. Since $C(L_2(\Omega))$ is dense in $\mathcal{N}_\Phi(\Omega)$ due to Corollary 14.2, and since $\mathcal{N}_\Phi(\Omega)$ is embedded into $L_2(\Omega)$, we can rewrite the inner product on the whole native space as

$$(f, g)_\Phi = \sum_n \frac{(f, \varphi_n)(g, \varphi_n)}{\rho_n} \text{ for all } f, g \in \mathcal{N}_\Phi(\Omega). \tag{18.2}$$

The rest is standard. □

Corollary 18.2. *The functions $\sqrt{\rho_n}\varphi_n$ are a complete orthonormal system in the native space $\mathcal{N}_\Phi(\Omega)$.*

Proof: Orthonormality immediately follows from (18.2), and Theorem 18.1 allows to rewrite all functions from the native space in the form of an orthonormal expansion

$$f = \sum_n (f, \sqrt{\rho_n}\varphi_n)_\Phi \sqrt{\rho_n}\varphi_n$$

with respect to the inner product of the native space. □

Corollary 18.3. *The operator L defined in (18.1) maps the native space $\mathcal{N}_\Phi(\Omega)$ into $L_2(\Omega)$ such that (12.1) holds. It is an isometry between its domain $\mathcal{N}_\Phi(\Omega)$ and its range $L_2(\Omega)/\ker C = \text{clos}(\text{ span } \{\varphi_n\}_n)$.*

Corollary 18.4. *The operator $\sqrt[*]{C}$ defined in (17.3) maps $L_2(\Omega)$ onto the native space $\mathcal{N}_\Phi(\Omega)$. Its inverse on $\mathcal{N}_\Phi(\Omega)$ is L. Any function f in the native space has the integral representation*

$$f = \int_\Omega v(t) \sqrt[*]{C}(\cdot, t) dt \tag{18.3}$$

with a function $v \in L_2(\Omega)$.

Corollary 18.5. *The range of the mapping C consists of the functions f in $L_2(\Omega)$ such that the summability condition*

$$\sum_n \frac{(f, \varphi_n)^2}{\rho_n^2} < \infty$$

holds. It is an interesting open problem to generalize results for the radial case on $\Omega = \mathbb{R}^d$ to this setting, replacing Fourier transforms by eigenfunction expansions.

§19. Implications for numerical techniques

The reconstruction of a function f on Ω from function values $f(x_k)$ on centers $\{x_1, \ldots, x_M\}$ via a function

$$s(x) := \sum_{j=1}^M \lambda_j \Phi(x_j, x)$$

is a recovery problem in the sense of section 10, whose optimal solution in the sense of section 11 for functions $f \in \mathcal{N}_\Phi(\Omega)$ is provided by interpolation, i.e. by a solution of the system

$$f(x_k) = \sum_{j=1}^M \lambda_j \Phi(x_j, x_k) \tag{19.1}$$

for the coefficients λ_j. We now look at this numerical problem from the viewpoint of integral operators, and our goal is to show that we get some new hints for further research.

In view of Corollary 18.4 and (18.3) we can write

$$\int_\Omega v(t)\sqrt[*]{C}(x_k,t)dt = \int_\Omega \sqrt[*]{C}(x_k,t)\sum_{j=1}^M \lambda_j \sqrt[*]{C}(x_j,t)dt$$

to see that we are recovering v from the functions $\sqrt[*]{C}(x_j,t)$ via best approximation in $L_2(\Omega)$. The coefficients λ_j in the system (19.1) have a natural interpretation via the approximation

$$v(t) \approx \sum_{j=1}^M \lambda_j \sqrt[*]{C}(x_j,t).$$

The above argument is a simple implication of the fact that all functions f from the native space are solutions of the operator equation

$$f = \sqrt[*]{C}(v),\ v \in L_2(\Omega).$$

Since this is (under certain assumptions) an integral equation of the first kind, numerical problems will automatically arise whenever the function f is not in the range of the operator $\sqrt[*]{C}$, i.e. if f is not in the native space. But we see what actually happens: the numerical process is a best approximation in $L_2(\Omega)$ with respect to the functions $\sqrt[*]{C}(x_j,t)$ and thus always numerically executable. The above argument also sheds some light on why in [19] the treatment of functions f outside the native space actually worked after truncation of the Fourier transform. The applied technique suitably regularizes the ill–posed integral equation problem, and it still guarantees optimal approximation orders for given smoothness of f.

We now make things worse and turn to the operator equation

$$f = C(v),\ v \in L_2(\Omega).$$

Again, this is an integral equation of the first kind, and its solvability requires that f be in the range of C. This is precisely the situation of Theorem 15.1, and we get some explanation for the improved convergence rate. The interpretation of the coefficients λ_j in the system (19.1) now is somewhat different:

$$f(x_k) = \int_\Omega v(t)\Phi(x_k,t)dt = \sum_{j=1}^M \lambda_j \Phi(x_j,x_k)$$

makes it reasonable to compare with a quadrature formula

$$\int_\Omega g(t)dt \approx \sum_{j=1}^M \beta_j g(x_j)$$

to arrive at

$$\lambda_j \approx \beta_j v(x_j).$$

This implies that for smooth f and fairly regular configurations the coefficients at nearby points should be similar, and it provides a first technique to prolong values of coarse approximations to coefficients regarding finer center distributions. This observation (in a somewhat different form) was made by Jörg Wenz [23].

Another possible progress from here is the investigation of multilevel techniques, taking the eigensystem of C into account. Research in this direction is currently going on.

§20. Construction of positive definite functions

We now know that many strictly positive definite functions Φ on a domain Ω induce a positive integral operator in $L_2(\Omega)$ and have a representation (17.2). But we can turn things upside down and **define** Φ by (17.2), starting with a complete orthonormal system $\{\varphi_n\}_n$ in $L_2(\Omega)$ and a sequence $\{\rho_n\}_n$ of nonnegative numbers, converging to zero. In some sense, this approach is more general than the original one, because discontinuous or singular functions may result, depending on the decay of ρ_n for $n \to \infty$. Furthermore, the orthonormal systems arising from eigenfunction expansions are somewhat special, because they are smoother than general L_2 functions. We thus have to expect a wider class of functions Φ when starting from (17.2).

To actually carry out the construction, we first observe that Φ defined by (17.2) is a generalized positive semidefinite function in the sense that

$$(\lambda, \mu)_\Phi := \sum_n \rho_n \lambda(\varphi_n)\mu(\varphi_n) \tag{20.1}$$

is a continuous bilinear form on the dual of $L_2(\Omega)$. We cannot use the standard definition, because point evaluations are not continuous. Note here that for any functional λ in the dual of $L_2(\Omega)$ we have

$$\|\lambda\|^2 = \sum_n \lambda(\varphi_n)^2 < \infty$$

and thus can bound the bilinear form by

$$(\lambda, \mu)_\Phi^2 \le \left(\sum_n \sqrt{\rho_n}\lambda(\varphi_n)^2\right)\left(\sum_n \sqrt{\rho_n}\mu(\varphi_n)^2\right).$$

The bilinear form is an inner product, if all ρ_n are positive. Now we can define the future native space via Theorem 18.1 and provide it with the bilinear form (18.2). The Riesz map $R_{\Phi,\Omega}$ comes out to be

$$R_{\Phi,\Omega}(\lambda) = \lambda^x \Phi(x, \cdot) = \sum_n \rho_n \lambda(\varphi_n)\varphi_n$$

as expected, and the dual of the native space will be the closure of all functionals λ in the dual of $L_2(\Omega)$ under the inner product (20.1). Naturally, the dual of the native space will be larger than the dual of $L_2(\Omega)$, i.e. $L_2(\Omega)$ itself.

If some of the ρ_n are zero, we see that we get something like a generalized conditionally positive definite case, and regularization of the kernel along the lines of section 6 just does the right thing. Finally, it now is somewhat more clear why conditions for injectivity of C are nontrivial: one may be in a situation where some of the ρ_n are zero, and then everything has to be done modulo the kernel of C or, equivalently, the span of the φ_n with $\rho_n = 0$.

A look at (17.4) reveals another technique to construct positive semidefinite functions. In fact, if some function $P : \Omega \times \Omega \to \mathbb{R}$ has the property $P(x, \cdot) \in L_2(\Omega)$ for all $x \in \Omega$, we can form the generalized convolution

$$\Phi(x,y) := \int_\Omega P(x,t)P(y,t)dt.$$

The two construction techniques of this section have not yet been exploited to generate new and interesting basis functions. For the radial case, a toolbox was provided by [22], but there is no generalization so far.

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Robert Schaback
Universität Göttingen, Lotzestraße 16-18
D-37083 Göttingen, Germany
`schaback@math.uni-goettingen.de`