On Global GC^2 Convexity Preserving Interpolation of Planar Curves by Piecewise Bezier Polynomials

by

R. Schaback

Abstract

For sequences of planar data points whose piecewise linear interpolant has the property that two successive direction changes add up to at most an acute angle, there is a unique global GC^2 interpolant consisting of convex quadratic polynomial pieces where the data allow a convex interpolant. Cubic pieces are used where the data require inflection points. Collinear data points are interpolated by straight lines embedded on both sides by cubic pieces. Not–a–knot–boundary conditions are possible.

§1. Global parametric spline interpolation

In the nonparametric case the standard construction of a polynomial spline function $s(x)$ interpolating planar data

$$b_i = (x_i, y_i),\ 1 \le i \le n,\ x_i < x_{i+1} \text{ for } 1 \le i \le n-1$$

is carried out by taking tha data points as "knots" or "breakpoints". The degree of the polynomial pieces between the knots then is determined by continuity requirements for s and its derivatives. For C^2 continuity of s this leads to cubic polynomials, but two additional boundary conditions must be specified to yield a fully determined solution.

In the parametric case an anologous "standard" interpolant to a given sequence of data points

$$b_i \in \mathbb{R}^2,\ 1 \le i \le n,\ b_i \ne b_{i+1} \text{ for } 1 \le i \le n-1 \tag{1}$$

should also have "breakpoints" at the data points b_i, and the degree of the parametric polynomial curve between b_i and b_{i+1} should also be determined by the continuity requirements. For "visual" or "geometric" C^2 continuity two scalar conditions at each inner breakpoint have to be satisfied (continuity of tangent direction and curvature) and therefore the parametric interpolating piece should have only two additional degrees of freedom between the specified endpoints b_i and b_{i+1}. This implies that the interpolant should be a piecewise *quadratic* Bezier polynomial defined by triplets of control points $b_i, \widetilde{b_i}, b_{i+1}$ for $1 \leq i \leq n-1$. As in the nonparametric case, two remaining degrees of freedom have to be fixed by certain boundary conditions. For example, tangent directions at b_1 and b_n can be prescribed by two additional points $b_0 \neq b_1$ and $b_{n+1} \neq b_n$ with the requirement that the lines $L(b_0, b_1)$ and $L(b_n, b_{n+1})$ through b_0, b_1 and b_n, b_{n+1}, respectively, are tangents to the solution at b_1 and b_n.

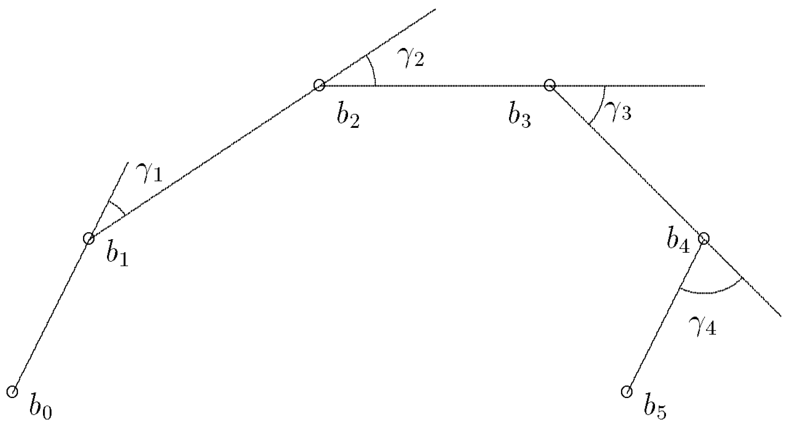

Fig. 1. Data set ($n = 4$) with prescribed tangent directions.

The following facts about the interpolant were proven in [3]:

1. A solution of the interpolation problem can not have an inflection point, and therefore the *chord angles*

$$\gamma_i = \angle(b_{i+1} - b_i, b_i - b_{i-1}),\ 1 \leq i \leq n,$$

   measured by arc length values in $(-\pi, \pi]$, must necessarily be all in $(0, \pi)$ or all in $(-\pi, 0)$. That is, the piecewise linear interpolant of the data always "turns left" or always "turns right" in the sense of an observator moving along the curve. This is a necessary condition depending on the data only. For the rest of this section, we restrict ourselves to the case $\gamma_i \in (0, \pi)$.
2. The interpolation problem is *solvable*, if

$$\gamma_i + \gamma_{i+1} < \pi \text{ for } 1 \leq i \leq n-1. \quad (2)$$

   There are unsolvable cases where (2) is not satisfied.
3. The interpolation problem is *uniquely* solvable, if

$$\gamma_i + \gamma_{i+1} < \pi/2 \text{ for } 1 \leq i \leq n-1. \quad (3)$$

   There are cases with multiple solutions where (3) is not satisfied.

This paper extends [3] by dealing with

1. cubic pieces to be inserted where the data require inflection points;
2. straight lines interpolating sections of collinear data, with cubic pieces at ends;
3. not–a–knot boundary conditions.

The interpolation problem can be written as a system of equations involving the *tangent angles*

$$\alpha_i = \text{ angle between chord } \ b_{i+1} - b_i \ \text{ and tangent at } \ b_i$$

and *chord lengths* $h_i = \|b_{i+1} - b_i\|_2$. Equating curvature at both sides of b_i (see [3]) yields the tridiagonal system

$$\frac{\sin(\gamma_i - \alpha_i)\sin^2(\gamma_i - \alpha_i + \alpha_{i-1})}{2h_{i-1}\sin^2\alpha_{i-1}} = \frac{\sin\alpha_i \sin^2(\gamma_{i+1} - \alpha_{i+1} + \alpha_i)}{2h_i \sin^2(\gamma_{i+1} - \alpha_{i+1})} \qquad (4)$$

for $2 \le i \le n-1$. The tangent angles α_i are variables for $2 \le i \le n-1$ satisfying $0 < |\alpha_i| < |\gamma_i|$, while the boundary conditions fix $\alpha_1 = \gamma_1, \ \alpha_n = 0$. An equivalent formulation of the interpolation problem can be given by fixing $b_1, \ldots, b_n$ as interpolation points and using two angles $\gamma_1 = \alpha_1$ and γ_n to describe directions of boundary tangents, leaving $\alpha_2, \ldots, \alpha_{n-1}$ as variables in the system (4) with $\alpha_n = 0$.

If (4) is solved, the interior control point $\widetilde{b_i}$ of a quadratic piece defined by control points $b_i, \widetilde{b_i}, b_{i+1}$ simply is constructed as the intersection of the tangents at b_i and b_{i+1} defined by the corresponding tangent angles. However, (4) was not used in [3] to prove existence and uniqueness of a solution; certain geometrical arguments ("shooting strategy") proved to be more powerful.

§2. Cubic pieces at inflection points

Definition 2.1. *A section $b_{i-1}b_i$, $3 \le i \le n-1$, of a data set (1) is called an inflection piece, if the chord angles satisfy $\gamma_{i-1} \cdot \gamma_i < 0$.*

We intend to use a cubic Bezier polynomial to interpolate in an inflection piece, but we want to retain the standard construction for the rest of the interpolation. This will automatically yield a convexity preserving interpolant. In view of (2) and (3) we tacitly assume all chord angles γ_i at inflection pieces to be acute.

Algorithm 2.2.

Step 1. *Let $b_{i-1}b_i$ be an inflection piece for the data set (1). Fix angles*

$$\alpha_j^* = \gamma_j h_j/(h_j + h_{j-1}), \ j = i-1, \ i \qquad (5)$$

for boundary tangents at b_{i-1} and b_i. This defines two subproblems by

a) interpolation points $b_1, \ldots, b_{i-1}$ *and tangent directions* $\alpha_1^*, \alpha_{i-1}^*$,
b) interpolation points $b_i, \ldots, b_n$ *and tangent directions* α_i^*, γ_n.

Step 2. *Assume* $|\alpha_i^*| \le |\alpha_{i-1}^*|$. *If the condition*

$$\|b_i - b_{i+1}\| \sin|\alpha_i^*| \sin^2(|\alpha_{i-1}^*| - |\alpha_i^*|) < \frac{3}{4}\|b_i - b_{i-1}\| \sin^2|\alpha_i^*| \sin(|\gamma_i| - |\alpha_i^*|) \quad (6)$$

is satisfied, solve both subproblems and proceed to Step 3. Otherwise replace α_{i-1}^* *by a value that satisfies both restrictions, e.g.* $\alpha_{i-1}^* := -\alpha_i^*$, *and proceed as before.*

Step 3. *Find a cubic Bezier polynomial that interpolates data points, tangent directions and curvature values of the two partial solutions of Step 2 at* b_{i-1} *and* b_i. *This can be done by solving a system of two quadratic equations, as will be shown later.*

Remark : Asymptotically,

$$\alpha_j = \gamma_j h_j/(h_j + h_{j-1}) + \mathcal{O}(h^2)$$

$$\tan\alpha_j = 1/2 \cdot h_j \cdot \kappa(b_j) + \mathcal{O}(h^2)$$

hold for sufficiently dense data samples from smooth curves, where $h := \max_j h_j$ tends to zero and where $\kappa(b_j)$ is the curvature of the curve at the data point b_j. This makes the choice (5) of tangent angles in Step 1 reasonable and implies that (6) is automatically satisfied for large data sets sampled with bounded mesh ratio h_j/h_{j+1} from a smooth curve.

Step 3 requires the solution of a Hermite–type interpolation problem by a cubic Bezier polynomial, where function values, tangent directions, and curvature values at two points A and D have to be reproduced. As was already pointed out by deBoor, Höllig, and Sabin in [1], a solution must not necessarily exist. Unfortunately, the approach of [1] does not carry over directly to this situation (in their terminology we would need $\rho_0 \cdot \rho_1 < 0$), but we use a similar method to prove

Theorem 2.3. *Under the conditions of Step 2, a unique solution of Step 3 exists.*

Proof: Let A, B, C, D be the control points of the required cubic Bezier polynomial that interpolates at A and D and has tangent angles α_A and α_D defined as in Fig. 2.

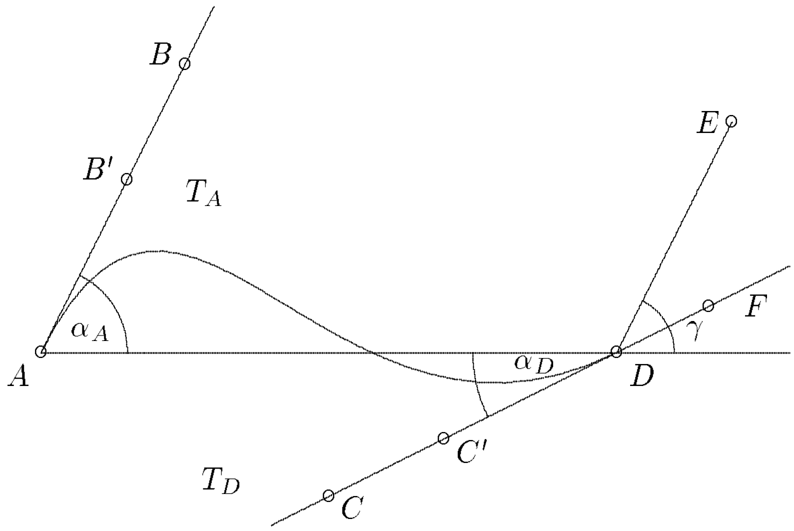

Fig. 2. Local insertion of a cubic piece with an inflection point.

Curvature values κ_A and κ_D at A and D are expressed by two points B' and C' on the tangents T_A and T_D at A and D such that the quadratic Bezier polynomials with control points A, B', D and A, C', D attain κ_A and κ_D at A and D, respectively. That is,

$$|\kappa_A| = \frac{\|A-D\| \sin\alpha_A}{2\|A-B'\|^2}, \ |\kappa_D| = \frac{\|A-D\| \sin\alpha_D}{2\|D-C'\|^2}.$$

Now the curvature of the cubic piece must be

$$|\kappa_A| = \frac{2 \ \mathrm{dist}(C, T_A)}{3\|A-B\|^2}, \ |\kappa_D| = \frac{2 \ \mathrm{dist}(B, T_D)}{3\|D-C\|^2}.$$

Introducing the variables

$$x := \|A-B\|/\|A-B'\|, \ y := \|D-C\|/\|D-C'\|$$

and the constants

$$u := \frac{\|D-C'\| \sin(\alpha_A - \alpha_D)}{\|A-D\| \sin\alpha_A}, \ v := \frac{\|A-B'\| \sin(\alpha_A - \alpha_D)}{\|A-D\| \sin\alpha_D}$$

a little calculation produces the system

$$\frac{3}{4}x^2 - 1 = -uy, \ \frac{3}{4}y^2 - 1 = vx. \tag{7}$$

In case of $\alpha_A = \alpha_D$ we have $u = v = 0$ and find $x = y = 2/\sqrt{3}$ as unique positive solutions.

If we assume $\alpha_A > \alpha_D$, a straightforward discussion of the parabolae (7) yields existence of a unique positive solution whenever $u < \sqrt{3}/2$. Since

$$u^2 = \frac{\sin\alpha_D \sin^2(\alpha_A - \alpha_D)}{\|A - D\| \sin^2\alpha_A \cdot 2|\kappa_D|}$$

depends on κ_D, we try to express κ_D by angles. If D, F, E are the control points of the next quadratic piece (see Fig. 2), we have

$$|\kappa_D| = \frac{\|D - E\| \sin(\gamma - \alpha_D)}{2\|D - F\|^2} \geq \frac{\sin(\gamma - \alpha_D)}{2\|D - E\|},$$

because (3) makes the angle at the control point F obtuse. Then

$$u^2 \leq \frac{\|D - E\| \sin\alpha_D \sin^2(\alpha_A - \alpha_D)}{\|A - D\| \sin^2\alpha_A \sin(\gamma - \alpha_D)},$$

and $u < \sqrt{3}/2$ is satisfied if (6) holds in the form

$$\|D - E\| \sin\alpha_D \sin^2(\alpha_A - \alpha_D) < \frac{3}{4}\|A - D\| \sin^2\alpha_A \sin(\gamma - \alpha_D).$$

■

When an inflection point is enforced by a boundary condition (i.e. $i = 2$ or $i = n$ in Definition 2.1) a similar strategy is possible.

§3. Straight sections

Definition 3.1. *If a data set (1) contains collinear points*

$$b_i, b_{i+1}, \ldots, b_{i+k},\ k \geq 2,\ 1 \leq i \leq n - k, \tag{8}$$

we call $b_i, \ldots, b_{i+k}$ a straight section of the data set.

Straight sections should be interpolated by straight lines. Each straight section splits the interpolation problem and requires "patching" to a neighboring standard solution.

Algorithm 3.2. *Assume $i > 1$ for a straight section (8) and the solvability of the piecewise quadratic interpolation problem in $b_1, \ldots, b_i$ with prescribed tangent direction $b_{i+1} - b_i$ at b_i.*

First, solve this problem. The last piece of the solution has control points $b_{i-1}, \widetilde{b_{i-1}}, b_i$. Then, with

$$\widehat{b_{i-1}} := \frac{1}{4}\widetilde{b_{i-1}} + \frac{3}{4}b_i,$$

replace the last section of the solution by the cubic Bezier polynomial defined by control points $b_{i-1}, \widetilde{b_{i-1}}, \widehat{b_{i-1}}, b_i$.

Theorem 3.3. *The algorithm produces a geometrically C^2 patch between the first $i-2$ pieces of the interpolant of $b_1, \ldots, b_i$ and the linear interpolant of $b_i, \ldots, b_{i+k}$.*

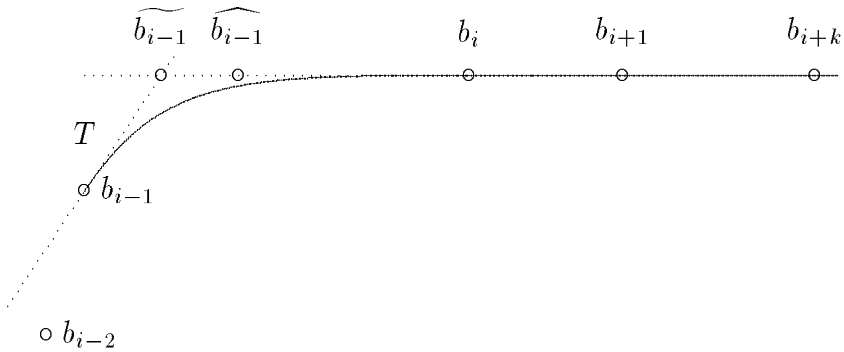

Fig. 3. Local insertion of a cubic piece near a straight section.

Proof: If T is the tangent at b_{i-1}, i.e. the line through b_{i-1} and $\widetilde{b_{i-1}}$, the absolute value κ of the curvature at b_{i-1} can be expressed as

$$\kappa = \frac{1}{2} \frac{\operatorname{dist}(b_i, T)}{\|b_{i-1} - \widetilde{b_{i-1}}\|^2} = \frac{2}{3} \frac{\operatorname{dist}(\widehat{b_{i-1}}, T)}{\|b_{i-1} - \widetilde{b_{i-1}}\|^2},$$

using a quadratic and a cubic piece. The algorithm's choice of the additional control point $\widehat{b_{i-1}}$ for the cubic piece on the line through $b_i, \ldots, b_{i+k}$ guarantees

$$\operatorname{dist}(\widehat{b_{i-1}}, T) = \frac{3}{4} \operatorname{dist}(b_i, T),$$

as required for reproduction of κ at b_{i-1}. ■

§4. Not–a–knot boundary conditions

If no tangent direction in the first point of a data set (1) is available, one can try to interpolate b_1, b_2, b_3 by a single quadratic Bezier polynomial, placing the first breakpoint at b_3. We do not prescribe the parameter t at which b_2 is to be interpolated.

Theorem 4.1. *Let b_1, b_2, b_3 be three different and non–collinear points, and let b_1, B, b_3 be the control points of a quadratic Bezier polynomial Q that interpolates b_1, b_2, b_3 successively. Then B lies on the hyperbola*

$$b_2 + u(b_1 - b_2) + v(b_3 - b_2), \; u, v \in \mathbb{R}, \; u \cdot v = 1/4, \; u < 0. \tag{9}$$

Conversely, $Q(t) = b_2$ *holds for* $t \in (0,1)$*, if* B *is taken as*

$$B(t) = b_2 - \frac{1-t}{2t}(b_1 - b_2) - \frac{t}{2(1-t)}(b_3 - b_2). \qquad (10)$$

Proof: Write the condition $Q(t) = b_2$ in the barycentric coordinates used in (9). ■

Theorem 4.2. *For a data set satisfying (3) the usual boundary conditions may be replaced by not–a–knot–conditions, and there will still be a unique solution.*

Proof: We use the "shooting" technique of [3] and consider the image $F(B(t))$ of the hyperbola (10) under the mapping F defined by the property that the quadratic Bezier polynomials with control points b_1, B, b_3 and $b_3, F(B), b_4$ are geometrically C^2 continuous at b_3. Some simple monotonicity arguments imply that $F(B(t))$ is a (radially) monotonic transversal curve in the sense of [3] contained in the cone C_3^+ (see Fig. 4). This proves feasibility of the not–a–knot–condition at one end of the data set.

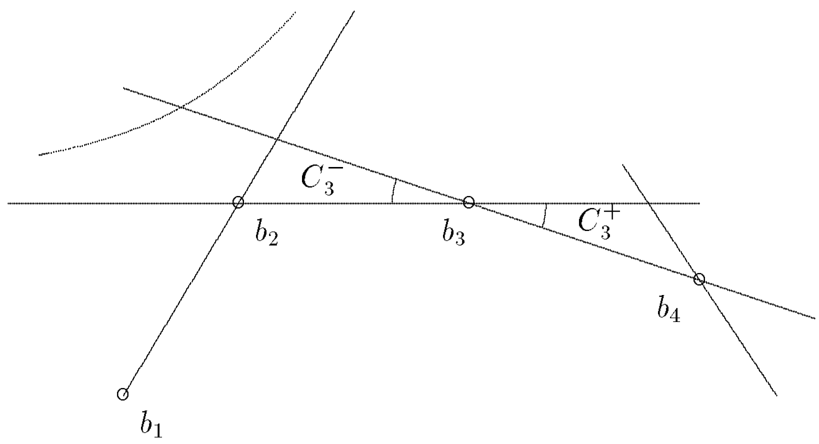

Fig. 4. Local behavior of the shooting method.

If a shooting strategy is carried out from the other end, a curve starting from b_3 results, extending monotonically (in the sense of [3]) into the cone C_3^-, and reaching the line through b_3, b_4 asymptotically. Such a curve uniquely intersects $B(t)$, proving feasibility of the not–a–knot–condition at both ends. ■

Since the "shooting" strategy is numerically unstable, we have to reformulate the not–a–knot boundary condition in terms of the system (4). For $i = 3$ the left–hand–side of (4) has to be replaced by the curvature κ_3 at b_3 of the quadratic piece interpolating b_1, b_2, and b_3. Then t has to be expressed by the variable α_3. Introducing the angle $\delta = \gamma_3 - \alpha_3$ in Fig. 5, we want to write $B(t)$ as a function $B(t(\delta))$ of δ. Then the left–hand side of (4) for $i = 3$ becomes

$$\kappa_3 = \frac{\|b_3 - b_1\| \sin(\beta + \gamma_3 - \alpha_3)}{2\|B(t(\gamma_3 - \alpha_3)) - b_3\|^2},$$

where we used the notations defined in Fig. 5.

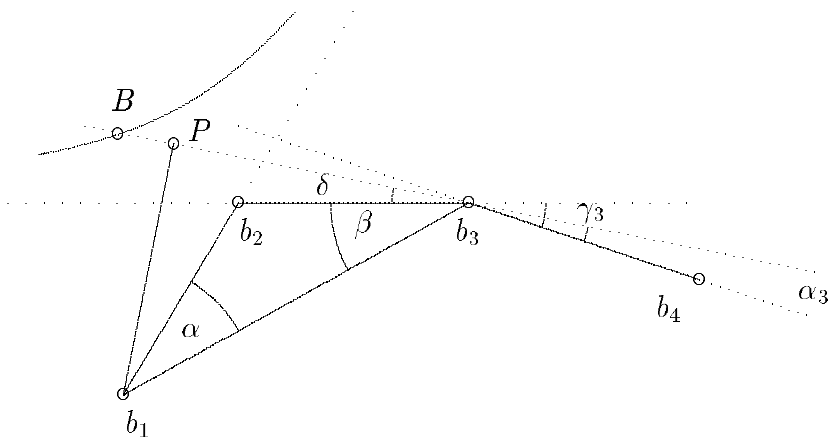

Fig. 5.

We drop the arguments δ and t for simplicity and first use the fact that B, b_3, and the projection P of b_1 to the tangent are collinear:

$$B - b_3 = \lambda(P - b_3). \tag{11}$$

The we express B and P in barycentric coordinates

$$B = b_2 + u_B(b_1 - b_2) + v_B(b_3 - b_2),\ P = b_2 + u_P(b_1 - b_2) + v_P(b_3 - b_2),$$

eliminate λ from (11) as

$$\lambda = \frac{u_B}{u_P} = \frac{1 - v_B}{1 - v_P}$$

and use $u_B \cdot v_B = 1/4$ to express u_B and v_B as functions of u_P and v_P. With $w_P := u_P/(1 - v_P) = u_B/(1 - v_B) < 0$ we get

$$u_B = \frac{1}{2} w_P (1 + \sqrt{1 - w_P^{-1}}),\ 1 - v_B = \frac{1}{2}(1 + \sqrt{1 - w_P^{-1}}).$$

Having eliminated B we are left with P, and some trigonometric reasoning gives the result

$$w_P = \frac{-\sin\delta}{\cos(\pi/2 - \alpha - \beta - \delta)} \frac{\|b_3 - b_2\|}{\|b_1 - b_2\|}.$$

This can be used to express $B - b_3$ via u_B and v_B as functions of δ. ∎

References

1. deBoor, C., K. Höllig, and M. Sabin, High Accuracy Geometric Hermite Interpolation, University of Wisconsin, Madison, Dept. of Computer Science, Report **692** (1987)
2. Schaback, R., Adaptive Rational Splines, NAM–Bericht **60** (1988), to appear in Constr. Approx.
3. Schaback, R., Interpolation in $\mathbb{R}^2$ by piecewise quadratic visually C^2 Bezier polynomials, NAM–Bericht **61** (1988)

Robert Schaback
Institut für Numerische und Angewandte Mathematik
Universität Göttingen, D–3400 Göttingen, FRG