Optimal Geometric Hermite Interpolation of Curves

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Abstract. Bernstein–Bézier two–point Hermite G^2 interpolants to plane and space curves can be of degree up to 5, depending on the situation. We give a complete characterization for the cases of degree 3 to 5 and prove that rational representations are only required for degree 3.

§1. Introduction and Overview

We consider recovery of curves from irregularly sampled data. If the curves are to be represented by NURBS, we want to generate representations which are minimal in the following sense:

1. they should have a minimal number of knots,
2. their degree should be as small as possible,
3. polynomial pieces are preferred over rational ones, and
4. the sampling and reconstruction process should be independent of the parametrization.

Then evaluation algorithms are fast, and additional knot elimination will not be necessary. Furthermore, using minimal degrees usually helps to preserve shape properties. Within the above setting, this paper continues the presentation [14] given at the Biri conference, and we solve some problems posed there. In particular, the examples of [14] showed that the degree of G^2 piecewise polynomial or rational curve interpolants must necessarily be at least five in general, while there are cases that work with degree three (de Boor, Höllig, and Sabin [1], Höllig [9]) and degree four (Peters [11]). Here, we focus on the problem of determining the minimal degree that works in each specific situation, and we give a complete classification. However, we omit Lagrange interpolation and confine ourselves to two–point Hermite interpolation in $\mathbb{R}^2$ or $\mathbb{R}^3$. The presentation will mainly be in geometric terms; it started from a complicated algebraic analysis with 19 different cases as provided by C. Schütt [16].

Mathematical Methods for Curves and Surfaces II
Morten Dæhlen, Tom Lyche, Larry L. Schumaker (eds.), pp. 1–12.

ISBN 1-xxxxx-xxx-x.

Our results will show that all degrees up to five actually occur in specific cases. If degrees four or five are necessary, one can get away with polynomial pieces. Rational pieces are sometimes required in cases that work with degree three (e.g. Höllig [9]). We provide a geometric approach to the full classification and illustrate it by numerous examples. However, the different cases depend crucially (and nonlinearly) on the sampled data, and the transition from one case to another may be discontinuous or may involve singularities. We exhibit some examples for this behavior and provide a general technique for regularization of singular situations. However, shape–preserving properties and approximation orders still are open research problems.

§2. Hermite Data for Two–point G^2 Interpolation

We want to recover a regular and smooth curve f from data at two different positions $y_0,\ y_1 \in \mathbb{R}^3$. If the curve f is locally reparametrized over $[0,1]$, we require positional interpolation in the form

$$f(i) = y_i,\ i = 0,1. \tag{1}$$

To make interpolated pieces G^1 continuous, we assume normalized tangent directions $r_0,\ r_1 \in S^2 := \{r \in \mathbb{R}^3\ :\ \|r\|_2 = 1\}$ to be given such that

$$\frac{f'(i)}{\|f'(i)\|_2} = r_i,\ i = 0,1. \tag{2}$$

Note that this is intended to generate interpolants that are regular at the data positions. Furthermore, the fixed orientation of the tangents helps to avoid cusps. The tangent directions r_i generate tangent rays R_i and tangent lines L_i defined as

$$\begin{aligned} R_i &:= \{y_i + (-1)^i \alpha_i r_i\ :\ \alpha_i \in \mathbb{R}_{>0}\} \\ L_i &:= \{y_i + \beta_i r_i\ :\ \beta_i \in \mathbb{R}\} \end{aligned}$$

at y_i for $i = 0,1$. Note that we let R_1 point “backwards” to simplify subsequent arguments.

There are several possibilities for prescribing second order data for G^2 continuity. To be compatible with the purely planar situation, we require normalized binormal vectors $\eta_0,\ \eta_1 \in S^2$ to be given together with nonnegative real curvature values κ_0, κ_1. The given vectors η_i should make sure that the orthogonality conditions

$$\eta_i \perp f'(i),\ \eta_i \perp f''(i),\ i = 0,1 \tag{3}$$

hold. Thus η_i is orthogonal to the two vectors which span the linear part of the osculating hyperplane P_i at $y_i = f(i)$, if they are linearly independent. Note, however, that we do not assume this extra condition to be satisfied. The additional scalar G^2 interpolation conditions then are

$$\kappa_f(i) = \frac{\|f'(i) \times f''(i)\|_2}{\|f'(i)\|_2^3} = \kappa_i,\ i = 0,1,$$

but it is more convenient to use the vectorial conditions

$$\frac{f'(i) \times f''(i)}{\|f'(i)\|_2^3} = \kappa_i \eta_i, \ i = 0, 1 \tag{4}$$

which also account for a proper orientation of the osculating hyperplanes with respect to the curve data. Altogether, we thus consider $\kappa_i \eta_i$ for $i = 0, 1$ to be the composite second–order data for G^2 continuity in the sense of (4). The vectors η_i should always be normalized to length one, and $\kappa_i = 0$ must hold in the degenerated case where $f'(i)$ and $f''(i)$ are linearly dependent. This may look complicated at first sight, but it has the advantage that there is always a (generalized) osculating hyperplane P_i at $y_i = f(i)$ well–defined by its normal η_i. We shall use the term osculating hyperplane in the above sense. Note that this setting also works for purely planar curves, and it models G^2 continuity of adjacent interpolating curve pieces by ensuring continuity of osculating hyperplanes.

The linear part of the osculating hyperplane P_i at y_i is spanned by tangent directions r_i and normal vectors $n_i := -r_i \times \eta_i \in S^2$ such that r_i, n_i, and $\eta_i = r_i \times n_i$ form a Frénet frame at y_i. Note that the n_i are defined indirectly here via the vectors r_i and η_i. Figure 1 shows the arrangement of full G^2 data with the osculating hyperplanes, but without the tangent rays. The prescribed curvature values are indicated by small circles in the osculating hyperplanes. These circles lie in the open halfspaces

$$H_i := \{y_i + \beta_i r_i + \gamma_i n_i \ : \ \beta_i \in \mathbb{R}, \ \gamma_i \in \mathbb{R}_{>0}\}$$

within the osculating hyperplanes P_i, containing the normal vectors n_i, and we call these halfspaces admissible for reasons that will soon be apparent. Note that the boundary of the halfspace H_i is the tangent line L_i. The admissible halfspaces in Figure 1 are indicated by the normals n_0 and n_1 pointing up and down in the osculationg planes at y_0 and y_1, respectively.

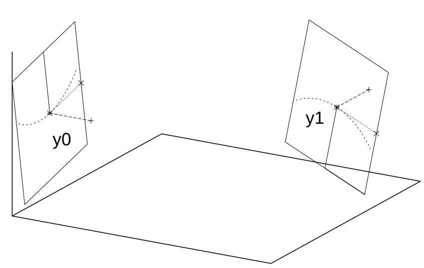

Fig. 1. Data for G^2 interpolation.

§3. Necessary Conditions

Let us first look at conditions under which there is a polynomial or rational solution of degree $n \geq 2$ in Bernstein–Bézier form. It must have control points $b_0, \ldots, b_n \in \mathbb{R}^3$ with the interpolation conditions (1) for positions satisfied by

$$b_0 = y_0, \ b_n = y_1. \tag{5}$$

For G^1 data, the conditions (2) imply that b_1 and b_{n-1} must lie on the tangent rays R_0 and R_1 at $y_0 = b_0$ and $y_1 = b_n$ in the directions r_0 and $-r_1$, respectively:

$$b_1 = y_0 + \alpha_0 r_0, \ b_{n-1} = y_1 - \alpha_1 r_1, \tag{6}$$

where the real numbers $\alpha_0, \ \alpha_1$ must be positive to ensure regularity of the Bernstein–Bézier interpolant at the data positions.

Now let us look at second–order conditions. The control points b_0, b_1, b_2 must lie in the osculating hyperplane P_0 at y_0, while b_{n-2}, b_{n-1}, b_n lie in the osculating hyperplane P_1 at y_1, respectively. This implies

$$\begin{aligned} b_2 &= y_0 + \beta_0 r_0 + \gamma_0 n_0, \\ b_{n-2} &= y_1 - \beta_1 r_1 + \gamma_1 n_1, \end{aligned} \tag{7}$$

with real numbers $\beta_i, \gamma_i, \ i = 0, 1$, and we shall see next that the γ_i must be nonnegative in order to account for (4). Using standard rational Bernstein–Bézier notation with positive weights $w_0, \ldots w_n$ from *e.g.*[2,3,10], conditions (4) are equivalent to

$$\begin{aligned} \kappa_0 &= \frac{n-1}{n} \frac{w_0 w_2}{w_1^2} \frac{\gamma_0}{\alpha_0^2}, \\ \kappa_1 &= \frac{n-1}{n} \frac{w_n w_{n-2}}{w_{n-1}^2} \frac{\gamma_1}{\alpha_1^2}, \end{aligned} \tag{8}$$

since the vectorial part is already accounted for by (7). This implies nonnegativity of the γ_i. But we must be careful in case of $\kappa_i = 0$. This implies $\gamma_i = 0$ and forces $b_{2+i(n-4)}$ to lie on the tangent line L_i at y_i. Since we want to make sure that the control points b_2 and b_{n-2} always lie in certain sets H_0 and H_1 at y_0 and y_1, respectively, we have to redefine H_i by

$$H_i := \{y_i + \beta_i r_i + \operatorname{sgn}(\kappa_i)\gamma_i n_i \ : \ \beta_i \in \mathbb{R}, \ \gamma_i \in \mathbb{R}_{>0}\}.$$

Thus H_i degenerates into the tangent line L_i at y_i in case $\kappa_i = 0$, but is an open halfspace if $\kappa_i > 0$. As a warm–up exercise let us first look at the case $n = 5$.

Theorem 1. *There always is a polynomial solution of degree 5.*

Proof: For $n = 5$, the placement of the points b_0, b_1, b_2 to satisfy the interpolation conditions at y_0 is independent of the data at y_1, since these only affect

b_3, b_4, b_5. One can pick any real value of β_0 and any two values of $\alpha_0 > 0$ and $\gamma_0 \geq 0$ that are related by (8) in the special form

$$5\kappa_0\alpha_0^2 = 4\gamma_0 \tag{9}$$

for a degree 5 polynomial. The same thing is done on the other side. Thus there always is a solution, and the degrees of freedom can be visualized by simply picking any two points b_2 and $b_3 = b_{n-2}$ in the admissible sets H_0 and H_1 at y_0 and y_1, respectively. Equations (9) will then place the points b_1 and $b_4 = b_{n-1}$ on the tangent lines to satisfy the curvature requirements. However, they allow additional degrees of freedom for these control points if zero curvature data are prescribed. □

Note that a practical solution to this interpolation problem will require some strategy to handle the excessive degrees of freedom. We shall comment on this in section 8. Furthermore, the above discussion shows that there always is a variety of polynomial solutions of degree 5 that can be used instead of solutions with smaller degrees that we construct later. Note finally that the degree 5 case can even occur in purely planar situations, namely if tangents are parallel and inflection points are required (see Figure 2). Nonplanar cases with minimal degree 5 will follow in the next section.

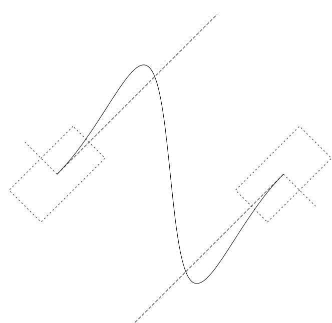

Fig. 2. Planar data for degree 5.

§4. Cases of Degree 4 or Less

Let us now derive necessary conditions for cases that can possibly be handled by polynomial or rational pieces of degree $n \leq 4$. By degree elevation, it suffices to look at $n = 4$. Then we see from the preceding discussion that the control point $b_2 = b_{n-2}$ must lie in the intersection of the admissible sets H_0 and H_1 at y_0 and y_1. This yields

Theorem 2. *The minimal degree solution is of degree 5 (and a polynomial) iff the admissible sets H_0 and H_1 do not intersect.*

Proof: It remains to show that we can construct an interpolant of degree at most 4 if there is a nonempty intersection. But this is clear from the degree

5 technique and our definition of the sets H_i. Instead of (9) we now use

$$4\kappa_0\alpha_0^2 = 3\gamma_0. \tag{10}$$

Any choice of $b_2 = b_{n-2}$ from the intersection will work. This technique is partially covered by the work [11] of Peters. □

§5. Cases of Degree 3 or Less

If there is a solution of degree 3, there is a solution of degree 4 by degree elevation. Therefore the intersection $S = H_0 \cap H_1$ is not empty. But for degree $n = 3$ we have

$$b_1 \in R_0 \cap \overline{H_1},\ b_2 \in R_1 \cap \overline{H_0}$$

and this implies

$$b_1 \in R_0 \cap \overline{S},\ b_2 \in R_1 \cap \overline{S}. \tag{11}$$

Theorem 3. *There is a solution of degree 3 or less, iff the sets $R_0 \cap \overline{S}$ and $R_1 \cap \overline{S}$ are not empty. In general, this solution is rational and uniquely dependent on the choice of points from (11).*

Proof: We have to carry out the construction of the interpolant if both sets in (11) are not empty. All control points $b_0, \ldots, b_3$ are now assumed to satisfy either (1) or (11). This defines positive values of the α_i and nonnegative values of the γ_i with $\operatorname{sgn}(\gamma_i) = \operatorname{sgn}(\kappa_i)$ for $i = 0, 1$ via (6) and (7). Thus one cannot assume (8) to be satisfied with equal weights. But when going over to a rational solution, one can fix $w_1 = w_2 = 1$ and pick positive values of w_0 and w_3 to satisfy (8). This coincides with the technique introduced by Höllig [9]. □

§6. Geometric Characterizations

To arrive at a more geometric description of the above cases, we have to look at the possible forms of nonempty intersections S of the two admissible sets H_0 and H_1. To this end, we first check the nonempty intersection of the two osculating hyperplanes P_0 and P_1. It can be a single line or a full hyperplane. The latter case implies full planarity of the problem and will be discussed later in section 7. Let the intersection of the osculating hyperplanes be a line that we call the pivot line from now on. On the pivot line, the nonempty intersection S of the two admissible sets H_0 and H_1 must be a simply connected set S. Either $\overline{S}$ consists of a single point or $\overline{S}$ contains an open subset of the pivot line. The latter case cannot occur if both curvature values are prescribed to be zero. It provides multiple solutions of degree 4.

Solvability with degree at most 3 will require both tangent rays R_0 and R_1 to hit the pivot line within the set $\overline{S}$. This will usually fix the two control points b_1 and b_2 uniquely, and the central piece of the control net will lie

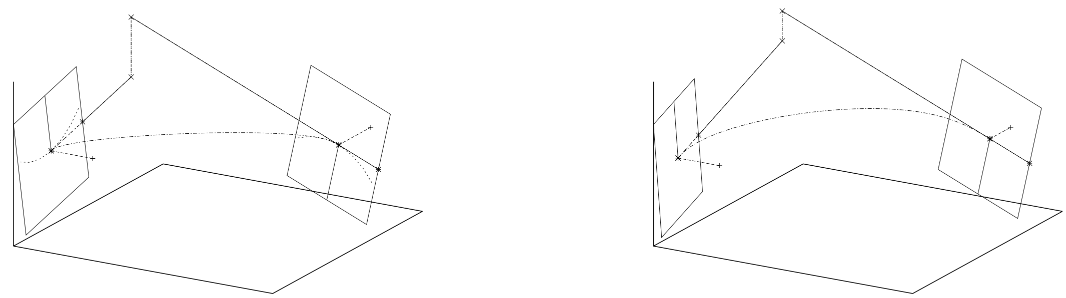

Fig. 3. Rational Solutions of Degree 3.

completely in $\overline{S}$ on the pivot line, since S is simply connected. This is why the degree 3 case usually has a unique rational solution in the nonplanar case.

For illustration, look at the left case of Figure 3 which has the same data as Figure 1. The tangent rays R_0 and R_1 are now emanating out of the osculating hyperplanes, and they hit the vertical pivot line in the background. The normals at the data locations indicate that the admissible halfspace on the left is pointing upward, while the one on the right is pointing downward. Thus the intersection S of the two halfspaces is an open bounded interval on the pivot line. The two boundary points of S lie on the tangent rays, and they precisely form the sets of (11). Consequently, there is a unique rational solution of degree 3, as drawn into the figure. The two tangent rays and the set S on the pivot line form the control polygon. This situation is typical for cases with degree 3, and note that the solution looks unexpectedly "straight". This is due to the fact that it cares for the boundary curvature by weight adjustment, introducing a tension–like effect. The picture on the right simply has smaller curvature requirements, and the tension effect disppears.

§7. Planar Data

For purely planar cases, the above theorems apply without modification. But the preceding paragraph needs another argument, since now the osculating hyperplanes always coincide and there is no pivot line. For nonzero curvature data, the set S is an intersection of two halfspaces in the plane, and this intersection will in general be a cone. Since the tangents are the boundary lines of the halfspaces, the vertex of the cone will in general be at the intersection of the tangents. The two tangents split the plane into four cones, and one of those is S, depending on the direction of the normals at the data positions. Each point of S, when used as b_2, will each lead to a degree 4 solution.

Reduction to degree 3 is based on the sets in (11), and these are parts of the bounding rays of the cone, because the tangent rays are parts of the boundary lines of the admissible halfspaces. The admissible sections of the boundary of the cone are precisely the intersections of the tangent rays with the cone's boundary.

In contrast to the space curve case, the degree 3 solution in the planar case will thus be nonunique. One can try to use the freedom to get away with a polynomial solution. But this is a hazardous task, since it leads to two quadratic equations with two unknowns that have to be solved on a restricted domain. The paper [1] of de Boor, Höllig, and Sabin treats the special case of fully convex data, while the short unpublished note [13] works for a simple inflection point. The approach of this paper can be easily applied to generate various other situations in which a degree 3 polynomial solution may exist.

Fig. 4. Planar polynomial solutions of degree 3.

Figure 4 shows two planar cases where the set S forms a nondegenerate two–dimensional cone. On the left we have the situation treated by de Boor, Höllig, and Sabin in [1]. The cone S is pointing downwards, and the control points b_1, b_2 can vary along the tangent rays up to their intersection. The picture on the right has the cone directed upwards because both normals are pointing upwards, and the control points can vary on the tangent rays from the intersection up to infinity, leading to a cusp if they are sufficiently far up. This actually happens, if they are moved to generate the shown polynomial solution.

However, since the sets of (11) are nonempty in general, one can get away with a variety of degree 3 rational solutions, and good ones will keep the control points at reasonable distances from the data positions. The papers [4,5,6,7,8] by T. N. T. Goodman and his collaborators deal with methods to pick a suitable solution with shape preserving properties. For instance, the right–hand case of Figure 4 could better be handled by a rational solution with control points near the vertex of the cone S. The cusp would be avoided, but bringing the control points too near to the vertex will result in a solution looking like a straight line, because the weights at the positions will become large in order to cope with curvature data prescribed there. See Figure 6 for a similar situation in $\mathbb{R}^3$.

§8. Regularization of Degree 5 Solutions

For picking a useful solution in case of ambiguities we adopt a very simple scheme that first distinguishes between "good" and "bad" data. Imagine that at the position y_0 a large curvature $\kappa_0 > 0$ is required. The curvature radius $R_0 = \kappa_0^{-1}$ then is very small, and the curve will have second order contact to the circle of radius R_0 with center $c_0 := y_0 + R_0 n_0$ in the osculating hyperplane. To allow some leeway, one could say that it should locally stay within a "trust region" of, say, radius $4R_0$ around the position y_0. But if the second position y_1 is further away, we should decide that this is a "bad" case. Look at Figure 5 for illustration.

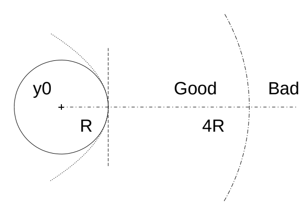

Fig. 5. Regions for good and bad y_1.

More precisely, we fix a constant $K > 1$ and say that the data are bad at y_0 if $\kappa_0 \|y_1 - y_0\|_2 > K$. The same can be done at y_1. For regularizing cases with degree 4 or 3, we cannot split the problem into two separate parts, and then we call a data set bad if

$$\sqrt{\kappa_0 \kappa_1} \|y_1 - y_0\|_2 > K.$$

The main implication of good and bad situations is that we use different strategies to handle them. The good cases are regularized towards good approximation properties for dense samples. Note that for a dense sample of data from a smooth curve we have only good pieces, since the distances get small while the curvature values are bounded.

The bad pieces are regularized for good asymptotic behavior when curvature tends to infinity. This sounds strange, but it can be done by moving control points near to the data locations, and the resulting curves simply interpolate bad data pieces by nearly straight curves with "high tension". This avoids unreasonable large loops, cusps, and wiggles, keeping the control points near the data locations.

Let's study the details for degree 5, and we can do this around y_0 locally, fixing the control points b_0, b_1, b_2 properly. For a dense sample from a smooth curve, the distances $\|b_{i+1} - b_i\|_2$ are each about $A/5$, where A is the arclength of the curve between y_0 and y_1. This arclength can be estimated by

$$a = \delta \left(1 + \frac{\kappa_0^2 \delta^2}{24}\right)$$

for small values of $\delta = \|y_0 - y_1\|_2$ up to terms of order five in δ (see [12] for details). Using (6) and (8), we take $\alpha_0 = a/5, \beta_0 = 2a/5$, and pick $\gamma_0 = a^2\kappa_0/20$ from (9). Since $a\kappa_0$ stays bounded by our criterion for "good" pieces, this works nicely for dense data samples. A proper convergence analysis would be an interesting project.

For "bad" cases we want to make α_0 proportional to κ_0^{-1} in order to let the control point b_1 move towards b_0 for extremely large curvature values. Using this proportionality and making the transition to the "good" case continuous, a short calculation yields the recipe

$$c = \frac{K}{4}\left(1 + \frac{K^2}{24}\right)$$
$$\alpha_0 = \frac{4c}{5\kappa_0}$$
$$\beta_0 = 2\alpha_0$$
$$\gamma_0 = \frac{4c^2}{5\kappa_0}$$

satisfying (9). Note that α_0, β_0, and γ_0 are proportional to κ_0^{-1}.

For true space curves, the cases of degree 3 usually are unique. For degree 4, the regularization technique needs modification, because the control point b_2 is restricted to S on the pivot line. After α_0 and γ_0 are estimated as above, one can in general find a point b_2^0 on the pivot line satisfying (8). The same can be done from the other side, resulting in some point b_2^1. But these points need not lie in S, though they often are good guesses for the actual point b_2. If just one is feasible, we take it, and if both are feasible, we take the one that is nearer to its corresponding tangent ray. If both are unfeasible (this is a rare exception), then $\overline{S}$ must be a bounded interval on the pivot line, and we take the mean of its bounds, ignoring our previous estimates. This complicated strategy works well in most cases, but improvements seem to be possible. A comparison with the strategy of Peters [11] is still missing. Details of the regularization technique will be found in [15].

§9. Computational Remarks

If data from a smooth space curve are sampled, one can observe that most pieces can be handled by rational interpolation of degree 3, *i.e.*,with Höllig's technique frm [9]. The exceptions usually are solvable by polynomials of degree 4, and it needs special tricks to produce cases where degree 5 actually is needed. The interpolation is stable, especially if the above regularization technique is used. The whole process seems to be largely shape–preserving due to the technique of picking the smallest possible degree.

§10. Transitions between Cases

It would be desirable to have an interpolation process that depends smoothly on the data. But this is not possible if we insist on minimal degrees and if we always prefer polynomial pieces over rational ones. The main reason for this is that the rational solution of degree 3 often is unique, but cannot in general go continuously over into a polynomial of degree 4 or 5. The left picture of Figure 6 shows a rational degree 3 case with an extremely small interval S on the pivot line. A small rotation of one of the osculating planes lets the interval disappear, and then one gets the degree 5 solution on the right–hand side. But this solution cannot in general be near to a degree three truly rational curve with nearly coalescing inner control points. The latter gets so much tension that it finally looks very much like a straight line. This special transition situation forces the control points b_1 and b_2 to lie nearly on the tangent rays from the opposite position. This in turn yields very small values of the α_i and γ_i, and the curvature requirements are satisfied by extremely large weights at $y_0 = b_0$ and $y_1 = b_3$, making the solution look like a straight line.

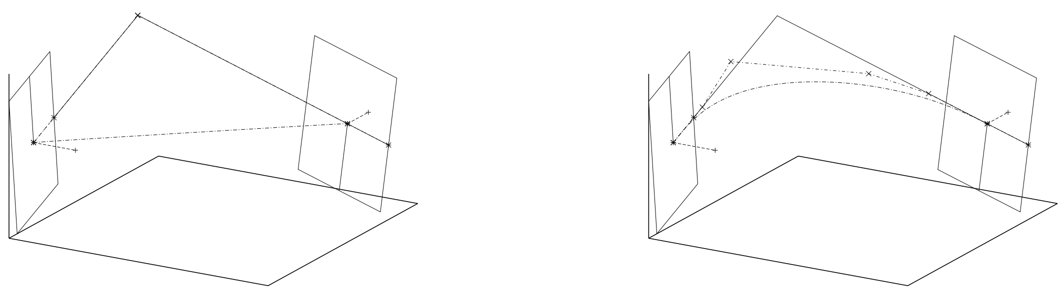

Fig. 6. Neighboring solutions of degree 3 and 5.

Note how the degree 5 solution is smoothed out by our regularization. The tangent rays make up most of the control polygon for degree 3 due to the strong tension effect. However, for degree 5 the control polygon nicely uses the leeway in the admissible halfspaces.

Acknowledgments. Help in proofreading was provided by Christoph Schröder.

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