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Optimal Stability Results for Interpolation by Kernel Functions

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Abstract

This paper proves lower bounds for the eigenvalues of positive definite matrices arising from interpolation of scattered data by positive definite kernels. By comparison with upper bounds for the interpolation error, it turns out that both bounds are asymptotically optimal for sufficiently dense data sets. Applications include interpolation on the sphere, the torus, and general Riemannian manifolds.

Keywords: Sobolev spaces, kernel expansions, n-sphere, n-torus, radial basis functions.

1 Introduction

`Simplified preliminary copy of the introduction of our RiP paper`

Let $\{\varphi_j(x)\}_{j\in\mathcal{J}}$ be a complex–valued orthonormal basis of $L_2(\Omega)$, where $\mathcal{J}$ is a countable index set, Ω is a bounded domain in $\mathbb{R}^{dim}$, or a compact n–dimensional Riemannian manifold [7]; the n-sphere $\mathbb{S}^{dim}$ and the n-torus $\mathbb{T}^{dim}$ are manifolds of special interest.

Expansions of functions $f \in L_2(\Omega)$ with respect to $\{\varphi_j(x)\}_{j\in\mathcal{J}}$ will be written as

`function`

$$f = \sum_{j\in\mathcal{J}} \hat{f}(j)\varphi_j, \quad \hat{f}(j) := (f, \varphi_j)_2, \tag{1}$$

The symbols c and C will stand for generic constants.

We shall study interpolation of functions $f \in L_2(\Omega)$ by linear combinations of functions $\Phi(\cdot, y)$, where $y \in \Omega$ and $\Phi : L_2(\Omega) \times L_2(\Omega) \to \mathbb{R}$ is a *symmetric positive definite kernel* (see e.g. [7, 12, 13]) having an expansion

`kernelphi`

$$\Phi(x, y) := \sum_{j\in\mathcal{J}} \hat{\Phi}(j)\varphi_j(x)\overline{\varphi_j(y)} \tag{2}$$

with the coefficients $\hat{\Phi}(j)$ being strictly positive. Such a framework may be viewed as the natural analogue in Ω of RBF approximation on all of $\mathbb{R}^{dim}$. The smoothness of the kernel and the summability of the above series is usually controlled by conditions on the decay of $\hat{\Phi}(j)$ of the form

`phidecay`

$$c\|j\|^{-\tau} \le \hat{\Phi}(j) \le C\|j\|^{-\tau} \tag{3}$$

for $\|j\| \to \infty$, where $\|j\|$ will be a norm on the index set. The precise inequalities in (3) will be provided later in specific cases.

We call a kernel of the form (2,`kernelphi`) *admissible*, if the sequence $\{\hat{\Phi}(j)\}_{j\in\mathcal{J}}$ satisfies

$$\sum_j \hat{\Phi}(j)|\varphi_j(x)|^2 \le C < \infty$$

for all $x \in \Omega$. According to the overview given in [13], there are many admissible kernels arising from positive integral operators

`posintop`

$$v \mapsto \int_\Omega v(x)\Phi(\cdot, x)dx \tag{4}$$

having $\{\phi_j\}_{j\in\mathcal{J}}$ as a complete orthonormal set of eigenfunctions with eigenvalues $\hat{\Phi}(j)$.

Any admissible kernel generates a Hilbert subspace sobolev

$$\mathcal{S}_\Phi := \left\{ f = \sum_{j\in\mathcal{J}} \hat{f}(j)\varphi_j,\ \|f\|_\Phi^2 := \sum_{j\in\mathcal{J}} \frac{|\hat{f}(j)|^2}{\hat{\Phi}(j)} < \infty \right\}, \tag{5}$$

called the *native space* for Φ. There is a well-developed theory for interpolation of functions f in the native space (see [7, 3, 4] for the torus and the sphere).

`New text from here on...`

Given a set $X := \{x_1, \ldots, x_N\}$ of N distinct points of Ω and real–valued data $y_1, \ldots, y_N$ one can use functions of the form

eqapp

$$s(x) := \sum_{j=1}^{N} \alpha_j \Phi(x, x_j) \tag{6}$$

to solve the interpolation problem

$$s(x_j) = y_j,\ 1 \leq j \leq N.$$

This way of interpolation has various optimality properties among all other linear recovery processes that reconstruct functions of the native space S_Φ from these data. In practice, it requires solving a linear system with the symmetric matrix

$$A_{\Phi,X} := (\Phi(x_j, x_k))_{1\leq j,k\leq N}$$

whose condition necessarily must be bad when data points come close. However, the propagation of absolute errors from the data vector $y \in \mathbb{R}^N$ into the coefficient vector $\alpha \in \mathbb{R}^N$ of (6,`eqapp`) is not influenced by condition, but rather by the smallest eigenvalue of $A_{\Phi,X}$ via

$$\|\alpha\|_2 \leq \lambda_{min}^{-1}(A_{\Phi,X})\|y\|_2.$$

This follows from

$$\begin{array}{rcl} A_{\Phi,X}\alpha & = & y \\ \alpha^T A_{\Phi,X}\alpha & = & \alpha^T A y \\ \lambda_{min}(A_{\Phi,X})\|\alpha\|_2^2 & \leq & \alpha^T A y \leq \|\alpha\|_2\|y\|_2, \end{array}$$

and implies that upper bounds for the stability of the interpolation process with respect to absolute errors are provided via lower bounds for $\lambda_{min}(A_{\Phi,X})$. The standard theory for such bounds in the case $\Omega = \mathbb{R}^{dim}$ started with papers by Ball, Narcowich, and Ward [1, 2, 8, 9, 10] with a generalization by Schaback in [11]. The resulting bounds are of the form

Gbound

$$\lambda_{min}(A_{\Phi,X}) \geq G_\Phi(q_X) \tag{7}$$

with the separation index

$$q_X := \min_{1\leq j<k\leq N} \|x_j - x_k\|_2$$

depending on the data locations only, while the function G_Φ depends on Φ and is independent of the data.

More precisely, the function G_Φ is determined by smoothness properties of Φ. In the translation–invariant case $\Phi(x,y) = \phi(x-y)$ on $\mathbb{R}^{dim}$, this is quantified by the decay of the Fourier transform of ϕ in the form

$$\hat{\phi}(\omega) = \mathcal{O}(\|\omega\|_2^{-dim-\beta}) \text{ for } \|\omega\|_2 \to \infty.$$

Then one can prove

$$G_\Phi(q_X) \geq c_\Phi q_X^\beta$$

(see Table 2 in [11]), and the order β in this bound cannot be improved. Moreover, the function G_Φ decays exponentially to zero whenever the Fourier transform of ϕ decays exponentially at infinity. This is the theoretical background for the bad numerical behavior of multiquadrics and Gaussians on dense data.

In this paper, we want to carry these results over to cases where the kernel functions come from series expansions. The smoothness of the kernel will be measured by a decay condition like (3,`phidecay`) with an exponent τ measuring the smoothness, and the result will then be of the form

`Grate`

$$\lambda_{min}(A_{\Phi,X}) \geq G_\Phi(q_X) \geq c_\Phi q_X^{\tau-dim}, \tag{8}$$

where d is the dimension of Ω, and where the order $\tau - d$ cannot be improved. The proof technique will be different from the previous literature, and it will use a scale of kernels with small support, which are of interest themselves.

2 Optimality of Stability Orders

To assess the optimality of the exponent in (8,`Grate`), we need upper bounds for the smallest eigenvalue of $A_{\Phi,X}$. Such bounds are provided by the uncertainty relation of [11] together with upper bounds for the power function, which are byproducts when proving error bounds for interpolation. We can omit most of the background theory, if we look at standard error bounds for interpolants s of the form (6,`eqapp`) to functions f from the native space (5,`sobolev`) on data sets X using a kernel Φ. These bounds have the form

`intbound`

$$\|f - s\|_{\infty,\Omega}^2 \leq F_\Phi(h_X)\|f\|_\Phi^2, \tag{9}$$

where h_X stands for the fill distance

$$h_X := \sup_{x\in\Omega} \min_{x_j\in X} \|x - x_j\|_2$$

and where F_Φ is a function that depends on the smoothness of Φ in the sense of (3,`phidecay`). In particular,

`betasimple`

$$F_\Phi(h) = \mathcal{O}(h^{\tau - dim}) \text{ for } h \downarrow 0 \tag{10}$$

in case of a kernel Φ satisfying (3,`phidecay`), and where dim is the dimension of Ω. Special instances of such results are in [5].

Now the uncertainty relation of [11] relates F_Φ and G_Φ by

$$G_\Phi(h) \leq c_1 F_\Phi(c_2 h)$$

for $h \downarrow 0$, and by comparison of (10,`betasimple`) and (8,`Grate`) we see that both the error orders and the stability orders are optimal.

3 Lower Bounds for Eigenvalues

We introduce a new technique for proving bounds of the form (7,`Gbound`) via a perturbation of Φ that does not spoil the positive definiteness of Φ while modifying the matrix $A_{\Phi,X}$ on the diagonal only.

Theorem 3.1 *Assume that there is a not necessarily positive definite symmetric and admissible kernel g such that*

`gcond`

$$\begin{array}{rcl} \hat{\Phi}(j) - \hat{g}(j) & \geq & 0 \text{ for all } j \in \mathcal{J} \\ g(x,y) & = & 0 \text{ for all } \|x-y\|_2 \geq 2q_X. \end{array} \tag{11}$$

Then we have

$$\lambda_{min}(A_{\Phi,X}) \geq g(0).$$

Proof: The kernel $\Phi - g$ is positive semidefinite due to the first condition of (11,`gcond`), and we get

$$0 \leq \lambda_{min}(A_{\Phi-g,X}) = \lambda_{min}(A_{\Phi,X} - g(0)I) = \lambda_{min}(A_{\Phi,X}) - g(0)$$

because the second condition of (11,`gcond`) makes the matrices differ only on the diagonal. □

It is an interesting problem to ask for a positive definite kernel g satisfying (11,`gcond`) and maximizing $g(0)$. We address this question in [6].

4 Lower Bounds for Convolutions

The following assumes that everything takes place in R^{dim}, but it generalizes easily to sphere caps and the torus.

Theorem 4.1 *Let two nonnegative compactly supported functions g and h satisfy*

$$g(x) \geq g_0 \chi_{B_r(0)}(x),\ h(x) \geq h_0 \chi_{B_s(0)}(x)$$

for positive values g_0, h_0 and radii r, s of balls $B_r(0), B_s(0)$ around zero. Then the convolution is nonnegative and satisfies

$$(g * h)(x) \geq g_0 h_0 \ vol\ (B_t) \chi_{B_t(0)}(x)$$

for $t = \min(r, s)/2$.

Proof: In fact, if the shift distance $\|x\|_2$ is at most t, then the ball $B_t(x/2)$ is contained in both $B_r(0)$ and the shifted ball $B_s(x)$. This follows from

$$\begin{array}{rcccl} \|y\|_2 & \leq & \|y - x/2\|_2 + \|x/2\|_2 & \leq & t + \|x\|_2/2 < 2t \leq r \\ \|z - x\|_2 & \leq & \|z - x/2\|_2 + \|x/2\|_2 & \leq & t + \|x\|_2/2 < 2t \leq s \end{array}$$

for $y, z \in B_t(x/2)$. Then

$$(g * h)(x) = \int g(y) h(x - y) dy \geq g_0 h_0 \int_{B_r(0) \cap B_s(x)} dy \geq g_0 h_0 \text{ vol } (B_t(x/2)).$$

□

We now use this lower bound for successive convolutions of $g = \chi_{B_\epsilon(0)}$. We get

$$\begin{array}{rcl} (g * g)(x) & \geq & \text{vol } (B_{\epsilon/2}) \chi_{B_{\epsilon/2}(0)}(x) \\ (g * g * g)(x) & \geq & \text{vol } (B_{\epsilon/2}) \text{ vol } (B_{\epsilon/4}) \chi_{B_{\epsilon/4}(0)}(x) \\ (\underbrace{g * \ldots * g}_{m-1 \text{ times}})(x) & \geq & \chi_{B_{\epsilon/2^{m-1}}(0)}(x) \prod_{j=1}^{m-1} \text{ vol } (B_{\epsilon/2^j}) \\ (\underbrace{g * \ldots * g}_{m-1 \text{ times}})(0) & \geq & c(m, dim) \epsilon^{dim(m-1)}, \end{array}$$

which is what we need later. This argument works likewise for the circle, the torus, and caps of spheres.

5 The Circle Case

Our comparison function g for the circle will be constructed by convolution of the 2π–periodic continuation B_ϵ^1 of the characteristic function $\chi_{[-\epsilon,\epsilon]}$ for $0 < \epsilon < \pi$. We get the L_2–convergent representation

$$B_\epsilon^1(x) = \frac{\epsilon}{\pi} + \frac{2\epsilon}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\epsilon}{n\epsilon} \cos(nx)$$

and convolve $B^1_{\epsilon/m}$ with itself $m-1$ times. This yields

$$\begin{aligned}
B^m_\epsilon(x) &:= (B^1_{\epsilon/m} * \ldots * B^1_{\epsilon/m})(x)\\
&= \left(\frac{\epsilon}{\pi}\right)^m (2\pi)^{m-1} + \left(\frac{2\epsilon}{\pi}\right)^m \pi^{m-1} \sum_{n=1}^\infty \left(\frac{\sin n\epsilon}{n\epsilon}\right)^m \cos nx\\
&= \frac{1}{2\pi}(2\epsilon)^m + \frac{1}{\pi}(2\epsilon)^m \sum_{n=1}^\infty \left(\frac{\sin n\epsilon}{n\epsilon}\right)^m \cos nx,
\end{aligned}$$

where we remark that each convolution introduces a factor 2π and π in the first and the remaining terms, respectively. If the given kernel has the form

`cirker`

$$\Phi(x,y) := \frac{\rho_0}{2} + \sum_{n=1}^\infty \rho_n \cos n(x-y) \tag{12}$$

with the property

`rhobnd`

$$\rho_n \geq \rho n^{-m} \text{ for all } n \geq 0 \tag{13}$$

with a fixed positive ρ, then $\Phi - c_\Phi B^m_\epsilon$ is positive semidefinite for

$$c_\Phi = \min\left(\pi\rho 2^{-m}, \pi\rho_0 2^{-m}\pi^{-m}\right).$$

In fact,

$$c_\Phi \frac{1}{2\pi}(2\epsilon)^m \leq \pi\rho_0 2^{-m}\pi^{-m}\frac{1}{2\pi}(2\epsilon)^m \leq \pi\rho_0 2^{-m}\pi^{-m}\frac{1}{2\pi}(2\pi)^m \leq \rho_0/2,$$

$$c_\Phi \frac{1}{\pi}(2\epsilon)^m \left(\frac{\sin n\epsilon}{n\epsilon}\right)^m \leq \pi\rho 2^{-m}\frac{1}{\pi}(2\epsilon)^m \left(\frac{1}{n\epsilon}\right)^m \leq \rho n^{-m} \leq \rho_n.$$

Thus we apply Theorem 3.1 for $g := c_\Phi B^m_{2q_X}$ and we have to evaluate $g(0) = c_\Phi B^m_{2q_X}(0)$ or find a positive lower bound. Theorem 4.1 provides a lower bound of order ϵ^{m-1}. We summarize:

Theorem 5.1 *If a symmetric positive definite kernel Φ on the circle has the smoothness order m defined in* (13,`rhobnd`)*, then there is a positive constant γ depending on Φ but not on the data, such that*

$$\lambda_{min}(A_{\Phi,X}) \geq \gamma q_X^{m-1}$$

holds for sufficiently dense data sets X, and this order is best possible.

6 The Sphere Case

The kernels considered here have the form

`pdkernelsph`

$$\Phi(p,q) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)} \hat{\Phi}(\ell,k)Y_{\ell,k}(p)\overline{Y_{\ell,k}(q)}, \qquad p,q\in\mathbb{S}^{dim}, \quad \hat{\Phi}(\ell,k)>0, \quad (14)$$

where the $Y_{\ell,k}$'s are spherical harmonics of order ℓ, and

$$N(dim,\ell) = \frac{2\ell+dim-1}{\ell}\binom{\ell+dim-2}{\ell-1} = \mathcal{O}(\ell^{dim-1}) \quad \text{for} \quad \ell\geq 1.$$

The spherical harmonic $Y_{\ell,k}$ is an eigenfunction of the the Laplace-Beltrami operator on $\mathbb{S}^{dim}$ corresponding to the eigenvalue $\lambda_\ell = \ell(\ell+dim-1)$, $\ell\geq 0$. The set $\{Y_{\ell,k}\}_{k=1}^{N(\ell,dim)}$ is chosen to be an an orthonormal basis for $\mathcal{E}_\ell$, the eigenspace of the Laplace-Beltrami operator on $\mathbb{S}^{dim}$ corresponding to the eigenvalue λ_ℓ. Collectively, the $Y_{\ell,k}$'s form an orthonormal basis for $L_2(\mathbb{S}^{dim})$. We describe the smoothness of Φ via

$$\hat{\Phi}(\ell) \geq c_\Phi \ell^{-\tau}$$

for some $\tau > dim$ and $\ell\to\infty$, using

$$\hat{\Phi}(\ell) := \max_{1\leq k\leq N(dim,\ell)} \hat{\Phi}(\ell,k).$$

and following (3,`phidecay`). For such kernels, we can cite (10,`betasimple`) from [5] with slightly different notation.

We now want to come up with a scale of zonal kernels with small support. For simplicity, let us consider the 2–sphere only. Zonality then means $\hat{\Phi}(\ell) = \hat{\Phi}(\ell,k)$ for all k, and $N(2,\ell) = 2\ell+1$. Due to

`YYp`

$$\sum_{k=-\ell}^{\ell} Y_{\ell,k}(p)\overline{Y_{\ell,k}(q)} = \frac{2\ell+1}{4\pi}P_\ell(\cos\varphi) \qquad (15)$$

where φ is the angle between points p and q on the sphere, i.e. $\cos\varphi = p^Tq$, the kernel can be rewritten as

$$\begin{aligned}
\Phi(p,q) &= \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\hat{\Phi}(\ell)P_\ell(\cos\varphi)\\
&= \sum_{\ell=0}^{\infty}\hat{\phi}(\ell)P_\ell(\cos\varphi)\\
\hat{\phi}(\ell) &= \frac{2\ell+1}{4\pi}\hat{\Phi}(\ell).
\end{aligned}$$

Note that then (3,`phidecay`) turns into phidecnew

$$c\|j\|^{-\tau+1} \le \hat{\phi}(j) \le C\|j\|^{-\tau+1}. \tag{16}$$

We shall distinguish between the notations $\hat{\Phi}$ and $\hat{\phi}$ in the sequel. The latter sense will be assumed for the function g to be constructed.

Our zonal kernel should have the form

gkernel

$$g(p,q) = g(\cos\varphi) = \sum_{\ell=0}^{\infty} \hat{g}(\ell) P_\ell(\cos\varphi), \qquad \varphi \in [0,\pi] \tag{17}$$

where φ is the angle between points p and q on the sphere, i.e. $\cos\varphi = p^T q$. The transform can be recovered via classical Legendre polynomial theory as

recov

$$\begin{aligned} \hat{g}(\ell) &= \frac{2\ell+1}{2} \int_0^\pi g(\cos\varphi) P_\ell(\cos\varphi) \sin\varphi d\varphi \\ &= \frac{2\ell+1}{2} \int_{-1}^1 g(t) P_\ell(t) dt. \end{aligned} \tag{18}$$

If we want g to be linear for angles smaller than $\epsilon > 0$, then we get a function g_ϵ^1 with

$$\begin{aligned} \hat{g}_\epsilon^1(\ell) &= \frac{2\ell+1}{2} \int_0^\epsilon P_\ell(\cos\varphi) \left(1 - \frac{1-\cos\varphi}{1-\cos\epsilon}\right) \sin\varphi d\varphi \\ &= \frac{2\ell+1}{2} a_\ell(\cos\epsilon), \\ a_\ell(x) &:= \int_x^1 \frac{t-x}{1-x} P_\ell(t) dt. \end{aligned}$$

Before we proceed further to evaluate the transform, let us look at convolution $G * H$ of zonal functions

$$\begin{aligned} G(p^T q) &= \sum_{\ell=0}^\infty \hat{G}(\ell) P_\ell(p^T q) \\ H(p^T q) &= \sum_{\ell=0}^\infty \hat{H}(\ell) P_\ell(p^T q) \\ (G * H)(p^T r) &:= \int_S G(p^T q) H(q^T r) d\mu(r) \\ &= \sum_{\ell=0}^\infty \frac{4\pi}{2\ell+1} \hat{G}(\ell) \hat{H}(\ell) P_\ell(p^T r) \end{aligned}$$

which follows from (15,`YYp`) and the orthonormality of the spherical harmonics. We convolve g_ϵ^1 with itself $m-1$ times to get a new function g_ϵ^m with transform

$$\begin{aligned} \hat{g}_\epsilon^m(\ell) &= \left(\frac{2\ell+1}{2}\right)^m a_\ell^m(\cos\epsilon) \left(\frac{4\pi}{2\ell+1}\right)^{m-1} \\ &= (2\pi)^{m-1} \frac{2\ell+1}{2} a_\ell^m(\cos\epsilon). \end{aligned}$$

We have to pick m in such a way that if Φ satisfies (16,`phidecnew`), then first `gl`

$$\hat{g}_\epsilon^m(\ell) \le \mathcal{O}(\ell^{-\tau+1}) \tag{19}$$

for $\ell \to \infty$ uniformly for small ϵ, and second

$$g_\epsilon^m(0) = (2\pi)^{m-1} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} a_\ell^m(\cos\epsilon) \ge \mathcal{O}(\epsilon^{\tau-2})$$

to reach full optimality. To this end, we need more information on $a_\ell(\cos\epsilon)$. We shall apply

$$P_\ell = \frac{1}{2\ell+1}(P'_{\ell+1} - P'_{\ell-1})$$

and

$$P_\ell(\cos\epsilon) = \frac{2}{\pi}\int_0^\epsilon \frac{\cos(\ell+1/2)\varphi}{(2(\cos\varphi - \cos\epsilon))^{1/2}} d\varphi.$$

We first do integration by parts on

$$\begin{aligned}
a_\ell(x) &:= \int_x^1 \frac{t-x}{1-x} P_\ell(t)dt. \\
&= \frac{1}{2\ell+1}\int_x^1 \frac{t-x}{1-x}(P'_{\ell+1}(t) - P'_{\ell-1}(t))dt \\
&= \frac{1}{2\ell+1}\left(\frac{t-x}{1-x}(P_{\ell+1}(t) - P_{\ell-1}(t))\right)\Big|_x^1 \\
&\quad -\frac{1}{(2\ell+1)(1-x)}\int_x^1 (P_{\ell+1}(t) - P_{\ell-1}(t))dt \\
&= -\frac{1}{(2\ell+1)(1-x)}\left(\frac{1}{2\ell+3}\,(P_{\ell+2} - P_\ell)|_x^1 - \frac{1}{2\ell-1}\,(P_\ell - P_{\ell-2})|_x^1\right) \\
(1-x)a_\ell(x) &= -P_{\ell+2}(x)\frac{1}{(2\ell+1)(2\ell+3)} \\
&\quad +2P_\ell(x)\frac{1}{(2\ell-1)(2\ell+3)} \\
&\quad -P_{\ell-2}(x)\frac{1}{(2\ell-1)(2\ell+1)}
\end{aligned}$$

for $\ell \ge 3$, caring about small ℓ later. We apply the Christoffel–Darboux formula

$$\frac{1}{n+1}\sum_{\nu=0}^{n}(2\nu+1)P_\nu(x) = \frac{P_n(x) - P_{n+1}(x)}{1-x}$$

twice to get

$$\frac{2n+1}{n+1}P_n(x) + \frac{2n+1}{n(n+1)}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x) = \frac{P_{n-1}(x) - P_{n+1}(x)}{1-x}.$$

Inserting this int the formula for $a_\ell(x)$, we get

$$\begin{aligned}
(2\ell+1)a_\ell(x) &= \frac{1}{2\ell+3}\left(\frac{2\ell+3}{\ell+2}P_{\ell+1}(x)+\frac{2\ell+3}{(\ell+1)(\ell+2)}\sum_{\nu=0}^{\ell}(2\nu+1)P_\nu(x)\right)\\
&\quad -\frac{1}{2\ell-1}\left(\frac{2\ell-1}{\ell}P_{\ell-1}(x)+\frac{2\ell-1}{(\ell-1)\ell}\sum_{\nu=0}^{\ell-2}(2\nu+1)P_\nu(x)\right)\\
&= \frac{1}{\ell+2}P_{\ell+1}(x)+\frac{1}{(\ell+1)(\ell+2)}\sum_{\nu=0}^{\ell}(2\nu+1)P_\nu(x)\\
&\quad -\frac{1}{\ell}P_{\ell-1}(x)-\frac{1}{(\ell-1)\ell}\sum_{\nu=0}^{\ell-2}(2\nu+1)P_\nu(x)\\
&= \frac{1}{\ell+2}P_{\ell+1}(x)+\frac{2\ell+1}{(\ell+1)(\ell+2)}P_\ell(x)\\
&\quad +\left(\frac{2\ell-1}{(\ell+1)(\ell+2)}-\frac{1}{\ell}\right)P_{\ell-1}(x)\\
&\quad +\left(\frac{1}{(\ell+1)(\ell+2)}-\frac{1}{(\ell-1)\ell}\right)\sum_{\nu=0}^{\ell-2}(2\nu+1)P_\nu(x)\\
&= \frac{1}{\ell+2}P_{\ell+1}(x)+\frac{2\ell+1}{(\ell+1)(\ell+2)}P_\ell(x)\\
&\quad +\frac{\ell^2-4\ell-2}{\ell(\ell+1)(\ell+2)}P_{\ell-1}(x)\\
&\quad -\frac{2\ell+1}{(\ell-1)\ell(\ell+1)(\ell+2)}\sum_{\nu=0}^{\ell-2}(2\nu+1)P_\nu(x)
\end{aligned}$$

and this yields a bound of the form

$$|a_\ell(x)| \leq C(\ell+1)^{-2}$$

for all $\ell \geq 0$, where C is independent of $x \in [-1,1]$.

These asymptotics of $a_\ell(x)$ for $\ell \to \infty$ are sufficient to handle (3,`phidecay`) with $\tau = 2m$ for general m. Theorem 4.1 provides a lower bound of order ϵ^{2m-2}. We summarize:

Theorem 6.1 *If a symmetric positive definite kernel Φ on the 2–sphere has the smoothness order $\tau = 2m$ defined in* (3,`phidecay`)*, then there is a positive constant γ depending on Φ but not on the data, such that*

$$\lambda_{min}(A_{\Phi,X}) \geq \gamma q_X^{2m-2}$$

holds for sufficiently dense data sets X, and this order is best possible.

7 The Euclidean Case

We now apply our technique to the Euclidean case. There, optimality of the orders of lower bounds for minimal eigenvalues and upper bounds for the power function are well known, see e.g. the summary in [11]. The methods of this paper, however, are different and allow a considerably shorter proof in case of algebraic decay of the Fourier transform.

We assume a conditionally positive definite translation–invariant kernel

$$\Phi(x, y) = \phi(x - y)$$

with an even function ϕ on $\mathbb{R}^{dim}$ whose generalized Fourier transform satisfies

$$\hat{\phi}(\omega) \geq c\|\omega\|^{-dim-\beta}$$

for positive c, β and for $\|\omega\|_2 \geq 1$. Then optimal error bounds have functions F and G of the form h^β in the sense of section 2.

To prove such bounds with the techniques of this paper, we consider the function

$$g^m := \underbrace{\chi_1 * \ldots * \chi_1}_{m-1 \text{ times}}$$

where χ_1 is the characteristic function of the unit ball in $\mathbb{R}^{dim}$. It has Fourier transform

$$\hat{\chi}_1(\omega) = \|\omega\|_2^{-dim/2} J_{dim/2}(\|\omega\|_2)$$

with decay order $-(dim + 1)/2$ at infinity. If we convolve the function $m - 1$ times with itself and scale it in such a way that the support is proportional to ϵ, we get a Fourier transform with behavior

$$\|\omega\|_2^{-m(dim+1)/2} \epsilon^{dim-m(dim+1)/2}$$

at infinity and ϵ^{dim} at zero. This operation can be done for noninteger values of

$$m = \frac{2(dim + \beta)}{dim + 1}.$$

to generate the same Fourier transform decay at infinity as of ϕ. Since the value of this function at zero still is one, we find a lower bound for the eigenvalues of the order

$$\epsilon^{-dim+m(dim+1)/2} = \beta.$$

`(must be polished...)`

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