Operators On Radial Functions

Robert Schaback and Z. Wu

Abstract. A general theory is provided that allows to write multivariate Fourier transforms or convolutions of radial functions as very simple univariate operations. As a byproduct, an interesting group of operators $\{I_\alpha\}_{\alpha\in\mathbb{R}}$ with $I_{\alpha+\beta} = I_\alpha \circ I_\beta = I_\beta \circ I_\alpha$ is defined. It contains the classical derivatives as $I_{-1} = \frac{d}{dr}$ and is intimately connected to the Fourier transform. Applications to the construction of new positive definite radial functions and to new identities for special functions are included.

§1. Radial functions

Among all functions $g : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ we consider those which can be written as a univariate function of the Euclidean norm $\|x\|_2$ on $\mathbb{R}^d$.

Definition 1.1. A function $g : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is *radial* if there is a function $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$g(x) = f(\|x\|_2^2/2), \qquad x \in \mathbb{R}^d \setminus \{0\}. \tag{1.1}$$

Remark 1.2. Functions defined on all of $\mathbb{R}^d$ are treated similarly. Note that we do not use $g(x) = f(\|x\|_2)$ for reasons that will soon be apparent. Any univariate function $f : \mathbb{R}_{>0} \to \mathbb{R}$ will induce a radial function on each $\mathbb{R}^d \setminus \{0\}$ via (1.1), and we shall always assume (1.1) when going over from f to a radial function g on $\mathbb{R}^d \setminus \{0\}$.

Our major goal is to provide a toolbox of univariate operators that allows to calculate multivariate transforms of radial functions effectively, including multivariate Fourier transforms and convolutions. The results are somewhat related to the work of S.E. Trione [6] based on Laplace transforms, and of M.A. Pinsky et al. [3] on Fourier series of radial functions. They were applied in several papers that constructed compactly supported positive definite functions [7], [8], [9].

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§2. Fourier transforms

Here we define the multivariate Fourier transform symmetrically as

$$\hat{g}(\omega) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(x) e^{-i\omega^T x} dx$$

and

$$\check{g}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} g(\omega) e^{+i\omega^T x} d\omega$$

for $g \in L_1(\mathbb{R}^d)$, where ^{T} stands for vector transposition. Now if $g(x) = f(\|x\|^2/2)$ is a radial function, then the d–variate Fourier transform is (see Stein and Weiss [5], p. 155 with a slightly different normalization of the Fourier transform)

$$\begin{aligned}
\hat{g}(\omega) &= \|\omega\|_2^{-\frac{d-2}{2}} \int_0^\infty f(s^2/2) s^{d/2} J_{\frac{d-2}{2}}(s \cdot \|\omega\|_2) ds \\
&= \int_0^\infty f\left(\frac{s^2}{2}\right) \left(\frac{s^2}{2}\right)^{\frac{d-2}{2}} \left(\frac{s \cdot \|\omega\|_2}{2}\right)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(s \cdot \|\omega\|_2) s \, ds \\
&= \int_0^\infty f\left(\frac{s^2}{2}\right) \left(\frac{s^2}{2}\right)^{\frac{d-2}{2}} H_{\frac{d-2}{2}}\left(\frac{s^2}{2} \cdot \frac{\|\omega\|_2^2}{2}\right) s \, ds
\end{aligned}$$

with the functions J_ν and H_ν defined by

$$\left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = H_\nu(z^2/4) = \sum_{k=0}^\infty \frac{(-z^2/4)^k}{k!\Gamma(k+\nu+1)} = \frac{F_1(\nu+1; -z^2/4)}{\Gamma(\nu+1)}$$

for $\nu > -1$. If we substitute $t = s^2/2$, we find

$$\begin{aligned}
\hat{g}(\omega) &= \int_0^\infty f(t) t^{\frac{d-2}{2}} H_{\frac{d-2}{2}}\left(t \cdot \frac{\|\omega\|^2}{2}\right) dt \\
&=: \left(F_{\frac{d-2}{2}} f\right)(\|\omega\|^2/2)
\end{aligned}$$

with the general operator

$$(F_\nu f)(r) := \int_0^\infty f(t) t^\nu H_\nu(tr) dt$$

for $\nu > -1$ defined on all $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$f(t) \cdot t^{\nu+\frac{1}{2}} \in L_1(\mathbb{R}_{>0}). \tag{2.1}$$

Note that both F_ν and H_ν generalize to arbitrary $\nu \in I\!R$, provided that restrictions like (2.1) hold. Furthermore, by symmetry of radial functions and our definition of Fourier transforms we have

$$F^{-1}_{\frac{d-2}{2}} = F_{\frac{d-2}{2}} \qquad \text{for } d \in I\!N.$$

We shall see later that this generalizes to $F_\nu^{-1} = F_\nu$ for all $\nu \in I\!R$, wherever both operators are defined.

In analogy to the classical space of tempered test functions we shall often use the space $\mathcal{S}$ of all functions on $I\!R_{\geq 0}$ that are infinitely differentiable such that any derivative vanishes faster than polynomially at infinity. We call such functions *tempered radial test functions* and remark that $f(r) = e^{-r}$ is in $\mathcal{S}$ but not in Schwartz' space. We can easily consider F_ν as an operator on $\mathcal{S}$ for all $\nu \in I\!R$, and we shall prove $F_\nu^2 = Id$ there. However, we postpone the extension of F_ν for $\nu \leq -1$ somewhat, because there will be a more handy definition. For convenience, we shall mainly work on the function space $\mathcal{S}$, but particular results will be extendable by continuity to much more general radial functions, including some with a singularity at zero.

Our basic tool will be the following lemma which provides a formula for the action of F_μ on the special function H_ν:

Lemma 2.1. *For $\nu > \mu > -1$ and all $r, s > 0$ we have*

$$(F_\mu H_\nu(s\cdot))(r) = \frac{s^{-\nu}(s-r)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}$$

with the usual definition of the truncated power function.

Proof: The assertion is a consequence of the Weber–Schafheitlin integral (see Abramowitz and Stegun [1], p. 487, 11.4.41) after substitutions of the type $t = s^2/2$. In detail, we have

$$\begin{aligned}
&\left(F_\mu H_\nu\left(\frac{u^2}{2}\cdot\right)\right)\left(\frac{r^2}{2}\right)\\
=&\int_0^\infty t^\mu H_\mu\left(\frac{r^2}{2}t\right)H_\nu\left(\frac{u^2}{2}t\right)dt\\
=&\int_0^\infty \left(\frac{s^2}{2}\right)^\mu\cdot s\cdot H_\mu\left(\frac{r^2}{2}\cdot\frac{s^2}{2}\right)H_\nu\left(\frac{u^2}{2}\cdot\frac{s^2}{2}\right)ds\\
=&\int_0^\infty 2^{-\mu}s^{2\mu+1}\left(\frac{rs}{2}\right)^{-\mu}\left(\frac{us}{2}\right)^{-\nu}J_\nu(us)ds\\
=&2^\nu r^{-\mu}r^{-\nu}\int_0^\infty s^{\mu-\nu+1}J_\mu(rs)J_\nu(us)ds\\
=&\frac{2^\nu r^{-\mu}u^{-\nu}2^{\mu-\nu+1}r^\mu(u^2-r^2)_+^{\nu-\mu-1}}{u^\nu\Gamma(\nu-\mu)}\\
=&\frac{1}{\Gamma(\nu-\mu)}\left(\frac{u^2}{2}\right)^{-\nu}\left(\frac{u^2}{2}-\frac{r^2}{2}\right)_+^{\nu-\mu-1}.
\end{aligned}$$

■

Theorem 2.2. *Let $\nu > \mu > -1$. Then for all functions $f : \mathbb{R}_{>0} \to \mathbb{R}$ with*

$$f(t)\cdot t^{\nu-\mu-1/2} \in L_1(\mathbb{R}_{>0}) \tag{2.2}$$

it follows that

$$F_\mu \circ F_\nu = I_{\nu-\mu} \tag{2.3}$$

where the integral operator I_α is given by

$$(I_\alpha f)(r) = \int_0^\infty f(s)\,\frac{(s-r)_+^{\alpha-1}}{\Gamma(\alpha)}\,ds, \qquad r>0,\ \alpha>0.$$

Proof of Theorem 2.2: For any tempered radial test function $f \in \mathcal{S}$ we evaluate $(F_\mu \cdot F_\nu)f(r)$ by means of Lemma 2.1 to obtain

$$\begin{aligned}
&\int_0^\infty H_\mu(tr)t^\mu\int_0^\infty H_\nu(st)s^\nu f(s)ds\ dt\\
=&\int_0^\infty s^\nu f(s)\int_0^\infty t^\mu H_\mu(tr)H_\nu(ts)dt\ ds\\
=&\int_0^\infty s^\nu f(s)\cdot F_\mu(H_\nu(s\cdot))(r)ds\\
=&\int_0^\infty f(s)\,\frac{(s-r)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\,ds = (I_{\nu-\mu}f)(r).
\end{aligned}$$

By standard continuity arguments this proof extends to any f with (2.2). ■

As a step towards proving $F_\mu^{-1} = F_\mu$ in general, we need

Lemma 2.3. *For $\nu > \mu > -1$ and all $s, r > 0$ we have*

$$F_\mu\left(\frac{s^{-\nu}(s-\cdot)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\right)(r) = F_\mu^2(H_\nu(s\cdot))(r) = H_\nu(rs).$$

Proof: We directly calculate the assertion and use Lemma 4.13 of Stein and Weiss (p. 170). In detail,

$$\begin{aligned}
&F_\mu\left(\frac{s^{-\nu}(s-\cdot)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\right)(r)\\
&=\int_0^\infty t^\mu\,\frac{s^{-\nu}(s-t)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\,H_\mu(tr)dt\\
&=\frac{s^{-\nu}}{\Gamma(\nu-\mu)}\int_0^s t^\mu(s-t)^{\nu-\mu-1}H_\mu(tr)dt\\
&=\frac{s^{-\nu}}{\Gamma(\nu-\mu)}\int_0^s t^\mu(s-t)^{\nu-\mu-1}J_\mu(2\sqrt{rt})(rt)^{-\mu/2}dt,
\end{aligned}$$

and by substitution $t = su^2$, we get

$$\begin{aligned}
&=\frac{s^{-\nu}}{\Gamma(\nu-\mu)}\int_0^1 s^\mu u^{2\mu}s^{\nu-\mu-1}(1-u^2)^{\nu-\mu-1}J_\mu(2\sqrt{rs}u)(rsu^2)^{-\mu/2}2su\;du\\
&=\frac{2(rs)^{-\mu/2}}{\Gamma(\nu-\mu)}\int_0^1 u^{\mu+1}(1-u^2)^{\nu-\mu-1}J_\mu(2\sqrt{rs}u)du\\
&=\frac{2(rs)^{-\mu/2}}{\Gamma(\nu-\mu)}\,\frac{2^{\nu-\mu-1}\Gamma(\nu-\mu)}{(2\sqrt{rs})^{-\nu-\mu}}\,J_\nu(2\sqrt{rs})\\
&=(\sqrt{rs})^{-\nu}J_\nu(2\sqrt{rs})\\
&=H_\nu(rs).
\end{aligned}$$

■

Corollary 2.4. *For all $\nu > \mu > -1$ and all $s, r > 0$*

$$\begin{aligned}
&F_\mu^2\left(\frac{s^{-\nu}(s-\cdot)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\right)(r)\\
&=F_\mu(H_\nu(s\cdot))(r)\\
&=\frac{s^{-\nu}(s-r)_+^{\nu-\mu-1}}{\Gamma(\nu-\mu)}\,.
\end{aligned}$$

Theorem 2.5. *$F_\mu^2 f = f$ holds for all $\mu > -1$ and on all tempered or compactly supported C^∞ test functions, and on all continuous and locally L_1 functions f for which $F_\mu^2 f$ is well–defined.*

Proof: The proof follows by density of truncated power functions and by continuity arguments. ∎

§3. Properties of integral operators

We now generalize (2.2) for all $\nu, \mu \in I\!R_{>-1}$. To do this, we define

$$\begin{aligned} (I_0 f)(r) &:= f(r), & f \in C(I\!R_{>0}) \\ (I_{-1} f)(r) &:= -f'(r), & f \in C^1(I\!R_{>0}) \\ I_{-n} &:= I_{-1}^n, & n > 0 \\ I_{-\alpha} &:= I_{n-\alpha} \circ I_{-n}, 0 < \alpha \le n = \lceil \alpha \rceil \end{aligned}$$

and get

Theorem 3.1. *The operators I_α for $\alpha \in I\!R$ form an Abelian group isomorphic to $(I\!R, +)$ via $\alpha \mapsto I_\alpha$, if the operators are restricted to functions in $\mathcal{S}$. Furthermore, they preserve compact supports for all α, and they are monotonic in the sense*

$$\begin{aligned} & (I_\alpha f)(r) \ge 0 \text{ for all } r > 0 \\ \text{if} \quad & f(r) \ge 0 \text{ for all } r > 0, \end{aligned}$$

provided that $\alpha \ge 0$.

Proof: For $\alpha, \beta, \gamma \ge 0$ we have

$$\begin{aligned} I_\alpha \circ I_\beta &= F_\gamma F_{\gamma+\alpha} F_{\gamma+\alpha} F_{\gamma+\alpha+\beta} \\ &= F_\gamma F_{\gamma+\alpha+\beta} = I_{\alpha+\beta} \\ &= I_\beta \circ I_\alpha \end{aligned}$$

because of $F_\gamma^{-1} = F_\gamma$ for $\gamma > -1$. For $n \in I\!N$ and $0 \le n \le \alpha$ it is easy to see by explicit differentiation that

$$I_\alpha \circ I_{-n} = I_{\alpha-n} = I_{-n} \circ I_\alpha$$

holds. The definition $I_{-\alpha} := I_{n-\alpha} \circ I_{-n}$ for $0 < \alpha \le n \in I\!N$ is independent of n, since for any $k \in I\!N_{\ge 0}$ we have

$$\begin{aligned} (I_{n-\alpha} \circ I_{-n})(f) &= (-1)^n I_{n-\alpha} f^{(n)} \\ &= (-1)^{n+k} I_{n-\alpha} I_k f^{(n+k)} \\ &= I_{n+k-\alpha} I_{-n-k} f. \end{aligned}$$

Thus the identity

$$I_{-\alpha} = I_{n-\alpha} \circ I_{-n} = I_{-n} \circ I_{n-\alpha}$$

is valid for all $0 < \alpha < n$, and

$$I_\alpha \circ I_{-\alpha} = I_\alpha \circ I_{n-\alpha} \circ I_{-n} = I_n \circ I_{-n} = I_0.$$

Likewise,

$$I_{-\alpha} \circ I_\alpha = I_{n-\alpha} \circ I_{-n} \circ I_\alpha = I_{n-\alpha} \circ I_\alpha \circ I_{-n} = I_0$$

for all $0 < \alpha < n$. If $\alpha, \beta \in \mathbb{R}$ are general, take $k \geq -\alpha,\ n \geq -\beta$ to find

$$\begin{aligned} I_{\alpha+\beta} &= I_{\alpha+\beta+n+k} \circ I_{-n-k} = I_{\alpha+k} \circ I_{\beta+n} \circ I_{-n} \circ I_{-k} \\ &= I_{\alpha+k} \circ I_{-k} \circ I_{\beta+n} \circ I_{-n} = I_\alpha \circ I_\beta \end{aligned}$$

and similarly $I_{\alpha+\beta} = I_\beta \circ I_\alpha$. The remaining properties of I_α follow from the definition. ■

Note that the operator group $\{I_\alpha\}_{\alpha \in \mathbb{R}}$ nicely interpolates between all classical derivatives and integrals, which occur as special cases I_n for $n \in \mathbb{Z}$. Writing a radial function in the form (1.1) instead of $g(x) = f(\|x\|_2)$ pays off by getting nice forms of I_1 and I_0. The identity $f = (I_n \circ I_{-n})f$ is

$$f(r) = (-1)^n \int_r^\infty f^{(n)}(s)\, \frac{(s-r)^{n-1}}{(n-1)!}\, ds, \quad r > 0$$

for functions $f \in C^n(\mathbb{R}_{>0})$ with $f^{(n)}(t) \cdot t^{n+\frac{1}{2}} \in L_1(\mathbb{R}_{>0})$. This is Taylor's formula at infinity, and for this reason the group $\{I_\alpha\}_{\alpha \in \mathbb{R}}$ should be called Taylor's group.

Theorem 3.2. *On test functions in $\mathcal{S}$ the identity*

$$I_{\nu-\mu} = F_\mu \cdot F_\nu \tag{3.1}$$

holds for all $\nu, \mu > -1$.

Proof: We only have to treat the case $\mu > \nu > -1$, where

$$\begin{aligned} I_{\nu-\mu} &= (I_{\mu-\nu})^{-1} = (F_\nu \circ F_\mu)^{-1} \\ &= F_\mu^{-1} \circ F_\nu^{-1} = F_\mu \circ F_\nu. \end{aligned}$$

■

Definition 3.3. The operators F_μ may be defined for all $\mu \in I\!R$ and on $\mathcal{S}$ by

$$F_\mu := I_{\nu-\mu} \circ F_\nu \tag{3.2}$$

where some $\nu > -1$ is taken.

Theorem 3.4. *With the above definition,*

$$I_{\nu-\mu} = F_\mu \circ F_\nu \tag{3.3}$$

holds for all $\mu, \nu \in I\!R$. Another useful identity is

$$F_\nu \circ I_\alpha = I_{-\alpha} \circ F_\nu \tag{3.4}$$

for all $\alpha, \nu \in I\!R$.

Proof: We first assert that (3.3) holds for all $\nu > -1$, $\alpha \in I\!R$. It follows from (3.2) for $\nu + \alpha > -1$ or $\alpha > 0$. The case $\alpha < 0$ is then implied by

$$I_{-\alpha} \circ F_\nu \circ I_{-\alpha} = I_{-\alpha} \circ I_\alpha \circ F_\nu = F_\nu.$$

The independence of (3.4) from the choice of $\nu > -1$ is easily implied by (3.2). Thus F_μ for $\mu \leq -1$ is well–defined by (3.4) and (3.1) now holds for $\mu \in I\!R$, $\nu > -1$. For $\mu, \nu \in I\!R$ we define

$$F_\mu := I_{\rho-\mu} \circ F_\rho, \quad F_\nu := I_{\omega-\nu} \circ F_\omega$$

for $\rho, \omega > -1$ and get

$$\begin{aligned} F_\mu \circ F_\nu &= I_{\rho-\mu} \circ F_\rho \circ I_{\omega-\nu} \circ F_\omega \\ &= I_{\rho-\mu} \circ F_\rho \circ F_\omega \circ I_{\nu-\omega} \\ &= I_{\nu-\mu}. \end{aligned}$$

Finally, for $\mu \in I\!R$ and $F_\mu = I_{\nu-\mu} \circ F_\nu$ with $\nu > -1$ we find

$$\begin{aligned} F_\mu \circ I_\alpha &= I_{\nu-\mu} \circ F_\nu \circ I_\alpha \\ &= I_{\nu-\mu} \circ I_{-\alpha} \circ F_\nu \\ &= I_{-\alpha} \circ I_{\nu-\mu} \circ F_\nu \\ &= I_{-\alpha} \circ F_\mu. \end{aligned}$$

■

For $\alpha > 0$ we can use the definition of $\Gamma(\alpha)$ to see that

$$I_\alpha(e^{-r}) = e^{-r}$$

holds, and this easily generalizes to all $\alpha \in I\!R$, since also $I_{-1}e^{-r} = -(e^{-r})' = e^{-r}$. By direct calculation we get

$$F_\nu(e^{-r}) = e^{-r}$$

for all $\nu > -1$ and this also generalizes to all $\nu \in I\!R$. Thus the exponential function e^{-r} acts as a universal nontrivial fixed point for the two operator families $\{I_\alpha\}_{\alpha\in I\!R}$ and $\{F_\nu\}_{\nu\in I\!R}$ which deserve further study as actions on $\mathcal{S}$. For each function $f \in \mathcal{S}$ the orbits

$$\begin{array}{ll} \alpha \mapsto f_\alpha := I_\alpha f & (\text{“Taylor orbit”}) \\ \nu \mapsto g_\nu := F_\nu f & (\text{“Fourier orbit”}) \end{array}$$

define curves on $\mathcal{S}$. They degenerate into single points for $f(r) = 0$ and $f(r) = e^{-r}$, and these two points are the only cases where the curves $\{f_\alpha\}_{\alpha\in I\!R}$ and $\{g_\nu\}_{\nu\in I\!R}$ can intersect at all:

Theorem 3.5. *If for some tempered test function $f : I\!R_{\geq 0} \to I\!R$ and some $\alpha, \nu \in I\!R$ the identity*

$$I_\alpha f = F_\nu f$$

is valid, then it holds for all α and ν, while f necessarily coincides with e^{-r} or the zero function.

Proof: We consider

$$\begin{aligned} F_\mu I_\beta f &= F_\mu I_{\beta-\alpha} I_\alpha f \\ &= F_\mu I_{\beta-\alpha} F_\nu f \\ &= F_\mu F_\nu I_{\alpha-\beta} f \\ &= I_{\nu-\mu+\alpha-\beta} f \end{aligned}$$

and get $F_\mu I_\beta f = I_\beta f$ whenever $\nu-\mu+\alpha = 2\beta$. Thus for $\beta = \frac{1}{2}\left(\nu+\alpha+\frac{1}{2}\right)$ and $\mu = -1/2$ we have that $I_\beta f$ is a fixed point of the univariate symmetric “radial” Fourier transform in the sense of Section 2. Since $e^{-r^2/2}$ is the only nontrivial fixed point of the univariate classical Fourier transform, the assertion follows. ■

§4. Recursion of Fourier transforms

We can use (3.3) in the form

$$F_\mu = I_{\nu-\mu} F_\nu = F_\nu I_{\mu-\nu}$$

to express Fourier transforms via other Fourier transforms. With $\mathcal{F}_d := F_{\frac{d-2}{2}}$, the d–variate Fourier transform written as a univariate operator on radial functions, we find

$$\mathcal{F}_n = I_{(m-n)/2}\mathcal{F}_m = \mathcal{F}_m I_{(n-m)/2}$$

for all space dimensions $m, n \geq 1$. Recursion through dimensions can be done in steps of two via

$$\mathcal{F}_{m+2} = I_{-1}\mathcal{F}_m = \mathcal{F}_m I_1$$

and in steps of one by

$$\mathcal{F}_{m+1} = I_{-1/2}\mathcal{F}_m = \mathcal{F}_m I_{1/2}.$$

Note that the operators I_1, I_{-1}, and $I_{1/2}$ are much easier to handle than the Hankel transforms F_μ and $\mathcal{F}_m$. This allows simplified computations of Fourier transforms of multivariate radial functions, if the univariate Fourier transforms are known. Furthermore, these operators map compactly supported functions to compactly supported functions, and the I_α operators for $\alpha \geq 0$ are monotone, i.e. they map nonnegative functions to nonnegative functions. This was successfully used by Wu [8] to construct piecewise polynomial compactly supported radial functions with nonnegative Fourier transforms and to characterize all radial functions with nonnegative multivariate Fourier transforms [9].

To be somewhat more explicit, consider odd-dimensional Fourier transforms

$$\mathcal{F}_{2m+1} = I_{-1}\mathcal{F}_{2m-1} = I_{-1}^m\mathcal{F}_1.$$

Thus, if g is a univariate function that describes the univariate Fourier transform of a radial function f after substitution $t \mapsto t^2/2$, then

$$I_{-1}^m g = (-1)^m g^{(m)}$$

is the radial Fourier transform of f in $2m+1$ dimensions. This very simple fact should be mentioned in all standard tables of Fourier transforms. Note that Fourier transforms in even dimensions also nicely boil down to bivariate transforms, but the transition between bivariate and univariate transforms does not use a plain derivative, but instead the semi–derivative

$$(I_{-1/2}f)(r) = (I_{1/2}I_{-1}f)(r) = -\int_r^\infty f'(s)\,\frac{(s-r)^{-1/2}}{\Gamma(1/2)}\,ds \tag{4.1}$$

§5. Convolutions

Two functions $g, h \in L_1(\mathbb{R}^d)$ can be convolved in $\mathbb{R}^d$ via

$$(g *_d h)(x) = \int_{y \in \mathbb{R}^d} g(y)h(x-y)dy.$$

If g and h are radial, the result is again a radial function, whose d–variate Fourier transform is the product of the d–variate Fourier transforms of g and h.

Definition 5.1. The operator

$$C_\nu : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$$

defined by

$$C_\nu(f, g) = F_\nu((F_\nu f) \cdot (F_\nu g))$$

is a generalized convolution operator on radial functions. For $\nu = \dfrac{d-2}{2}$ it coincides with the operator that takes d–variate convolutions of radial functions and rewrites the result in radial form.

We can now use Theorem 3.2 to describe transforms between two convolutions:

Theorem 5.2. *For* $\nu, \mu \in \mathbb{R}$ *we have*

$$C_\nu(f, g) = I_{\mu-\nu} C_\mu(I_{\nu-\mu} f, I_{\nu-\mu} g) \tag{5.1}$$

for all $f, g \in \mathcal{S}$.

Proof: By definition and (3.1) we find

$$\begin{aligned} F_\nu C_\nu(f, g) &= F_\nu f \cdot F_\nu g \\ = F_\mu I_{\nu-\mu} C_\nu(f, g) &= (F_\mu I_{\nu-\mu} f) \cdot (F_\mu I_{\nu-\mu} g) \\ &= F_\mu C_\mu(I_{\nu-\mu} f, I_{\nu-\mu} g) \end{aligned}$$

and get

$$I_{\nu-\mu} C_\nu(f, g) = C_\mu(I_{\nu-\mu} f, I_{\nu-\mu} g)$$

and

$$C_\nu(f, g) = I_{\mu-\nu} C_\mu(I_{\nu-\mu} f, I_{\nu-\mu} g).$$

■

The main application of Theorem 5.2 consists in the reduction of multivariate convolutions to univariate convolutions. In fact, for dimensions $d \geq 1$ we have

$$C_{\frac{d-2}{2}}(f,g) = I_{\frac{1-d}{2}} C_{-\frac{1}{2}}\left(I_{\frac{d-1}{2}} f, I_{\frac{d-1}{2}} g \right).$$

If d is odd, the d–variate convolution of radial functions finally just boils down to a derivative of a univariate convolution of integrals of f and g. For instance,

$$\begin{aligned} f *_3 g &= I_{-1}((I_1 f) *_1 (I_1 g)) \\ &= -\frac{d}{dr}\left(\left(\int_r^\infty f \right) *_1 \left(\int_r^\infty g \right) \right). \end{aligned}$$

Since those transformations preserve compactly supported piecewise polynomials, one can very easily calculate multivariate convolutions of compactly supported piecewise polynomial radial functions on odd–dimensional spaces. The same properties hold for transitions between even–dimensional spaces.

To reduce a bivariate convolution to a univariate convolution, one needs the operations

$$(I_{1/2} f)(r) = \int_r^\infty f(s) \frac{(s-r)^{-1/2}}{\Gamma(1/2)} \, ds$$

and (4.1) corresponding to integration and differentiation of order 1/2. If applied to compactly supported piecewise polynomials, these operations will generate unpleasant inverse trigonometric functions.

§6. Special functions

We now study the action of the operators I_α and F_ν on certain fixed functions f. Starting with some test function f_0, we can define

$$f_\alpha := I_\alpha f_0 \qquad (\alpha \in \mathbb{R})$$

and get a variety of integral or differential equations from application of the I_α operators via the identities

$$f_{\alpha+\beta} = I_\beta f_\alpha = I_\alpha f_\beta.$$

Furthermore, we can set $g_\nu := F_\nu f_0$ and get another series of equations

$$\begin{aligned} I_\alpha g_\nu &= I_\alpha F_\nu f_0 = F_{\nu-\alpha} f_0 = g_{\nu-\alpha} \\ F_\mu g_\nu &= F_\mu F_\nu f_0 = I_{\nu-\mu} f_0 = f_{\nu-\mu} \\ F_\mu f_\alpha &= F_\mu I_\alpha f_0 = F_{\mu+\alpha} f_0 = g_{\mu+\alpha} \end{aligned} \tag{6.1}$$

that describe the action of F_μ or I_α operators on the f_α or g_ν functions.

For $m \in \mathbb{N}_{\geq 0}$ fixed we can start with $f_0(r) = r^m e^{-r}$ and get for $\alpha > 0$

$$f_\alpha(r) = e^{-r} \cdot \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} r^{m-j} =: e^{-r} P_\alpha(r)$$

and for $\nu > -1$ by 11.4.28 of [1], p. 486

$$g_\nu(r) := (F_\nu f_0)(r) = \frac{\Gamma(m+\nu+1)}{\Gamma(\nu+1)} M(m+\nu+1, \nu+1, -r)$$

with the confluent hypergeometric function. The other cases can be constructed by taking derivatives or integrals, e.g.

$$f_\alpha(r) = (-1)^n \frac{d^n}{dr^n} \left(e^{-r} P_{\alpha+n}(r)\right) = (I_{-n} f_{\alpha+n})(r)$$

for $n > 0$. Or, for $\mu \in \mathbb{R}$ and $n > 0$ with $n + \mu > -1$, we have

$$g_\mu(r) = F_\mu f_0(r) = (I_n \circ F_{n+\mu}) f_0(r) = (I_n g_{n+\mu})(r)$$

taking integrals of the M function. For $n \in \mathbb{Z}_{<0}$ this describes the differentiation rules for M.

Now the equations (6.1) yield other relations between the f_α and g_ν, for instance

$$\begin{aligned}(F_\mu g_\nu)(r) &= \frac{\Gamma(m+\nu+1)}{\Gamma(\nu+1)} \int_0^\infty M(m+\nu+1, \nu+1, -s) s^\mu J_\mu(rs) ds \\ &= f_{\nu-\mu}(r) = e^{-r} P_{\nu-\mu}(r),\end{aligned}$$

which is not in [1]. Since the $P_{\nu-\mu}$ polynomials are positive for $r \geq 0$, this application establishes the confluent hypergeometric functions $g_\nu(r)$ as new examples of positive definite functions, i.e. as functions with a positive Fourier transform when radialized in each $\mathbb{R}^d$.

Another case that is relevant for applications stems from inverse multiquadrics or Bessel potentials

$$(c^2 + \|x\|_2^2)^{\beta/2}, \quad c > 0, \quad x \in \mathbb{R}^d$$

for $\beta < 0$. Here, we leave the restricted domain of test functions, but we shall only treat cases that can be handled via approximation by test functions. The function f_0 now will be

$$f_0(r) = (\gamma + r)^{\beta/2}$$

with $\gamma = c^2/2$, and we can apply I_α for $0 < \alpha < -\beta/2$ to get another multiquadric

$$
\begin{aligned}
f_\alpha(r) &= \int_r^\infty (\gamma+s)^{\beta/2}\,\frac{(s-r)^{d-1}}{\Gamma(\alpha)}\,ds \\
&= \frac{1}{\gamma(\alpha)}\int_0^\infty (\gamma+r+t)^{\beta/2+\alpha-1}\left(\frac{t}{t+r+\gamma}\right)^{\alpha-1} dt \\
&= \frac{(\gamma+r)^{\beta/2+\alpha}}{\Gamma(\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{-\alpha-\beta/2-1}du \\
&= (\gamma+r)^{\beta/2+\alpha}\,\frac{\Gamma(-\alpha-\beta/2)}{\Gamma(-\beta/2)}\,.
\end{aligned}
$$

Taking derivatives, this extends to α in the range $-\infty < \alpha < -\beta/2$, but I_α is not applicable to f_0 for $\alpha \geq \beta/2$.

Fourier transforms will now need Bessel or MacDonald functions K_μ via

$$
\begin{aligned}
g_\nu(r) := (F_\nu f_0)(r) &= \int_0^\infty (\gamma+t)^{\beta/2} t^\nu H_\nu(tr)dt \\
&= \frac{2^{1+\beta/2}}{\Gamma(-\beta/2)}\left(\frac{u}{\sqrt{\gamma}}\right)^{-\nu-1-\beta/2} K_{\nu+1+\beta/2}(u\sqrt{\gamma})
\end{aligned}
$$

for $r = u^2/2$ and $-1 < \nu < -\frac{1}{2} - \beta$ (see Abramowitz and Stegun [1],p. 488, 11.4.44). Other values can be obtained by

$$(F_\mu f_0)(x) = (I_{\nu-\mu}(F_\nu f_0))(r)$$

where $\mu \leq -\beta - 1 - \nu$ is possible. Further cases can be handled by (6.1) for various ranges of α, μ and ν. The identities $I_\alpha g_\nu = g_{\nu-\alpha}$ generalize the differentiation rules for functions of the form $z^{-\nu}K_\nu(z) =: L_\nu(z^2/2)$, namely $L_\nu' = L_{\nu+1}$ (see Abramowitz and Stegun [1] p. 376, 9.6.28).

§7. Compactly supported functions

We now take the characteristic function $f_0(r) = \chi_{[0,1]}(r)$ and get the truncated power function

$$(I_\alpha f_0)(r) = \int_0^1 \frac{(s-r)_+^{\alpha+1}}{\Gamma(\alpha)}\,ds = \frac{(1-r)_+^\alpha}{\Gamma(\alpha+1)} = f_\alpha(r), \qquad \alpha > 0.$$

Now Lemma 2.1 yields

$$f_\alpha = F_\mu H_\nu$$

for $\nu - \mu = \alpha + 1,\ \nu > \mu > -1$ and

$$F_\mu f_\alpha = H_{\mu+\alpha+1}$$

for $\alpha > 0, \mu > -1$. If $h_\alpha = f_\alpha * f_\alpha$ is the univariate radial convolution of two instances of f_α in the sense

$$h_\alpha(s^2/2) = \int_{-\infty}^{+\infty} f_\alpha(r^2/s) \cdot f_\alpha((s-r)^2/2)dr, \tag{6.2}$$

then

$$\mathcal{F}_1(h_\alpha) = (\mathcal{F}_1(f_\alpha))^2 = (F_{-1/2}f_\alpha)^2 = H^2_{\alpha+1/2}.$$

We now fix $0 \leq k \leq m \leq \alpha$ and get

$$\begin{aligned} \mathcal{F}_{2k+1} I_{-m} h_\alpha &= I_m F_{k-1/2} h_\alpha \\ &= I_{m-k} I_k F_{k-1/2} h_\alpha = I_{m-k} F_{-1/2} h_\alpha \\ &= I_{m-k} H^2_{\alpha+1/2} \geq 0, \end{aligned}$$

proving that the function $I_{-m}h_\alpha$ is positive definite on all $\mathbb{R}^d$ with $d \leq 2m+1$, and the corresponding radial Fourier transform is explicitly known. This generalizes the construction in [8] and provides Fourier transforms of Wu's functions in the form $\phi_{\alpha,m} = I_{-m}h_\alpha$.

Now we apply the above technique to calculate inner products

$$\begin{aligned} &(I_{-m}h_\alpha(\|\cdot - x\|_2^2/2), I_{-m}h_\alpha(\|\cdot - y\|_2^2/2))_{L_2(\mathbb{R}^{2k+1})} \\ =&C_{k-1/2}(I_{-m}h_\alpha, I_{-m}h_\alpha)(\|x-y\|_2^2/2) \end{aligned}$$

that arise when the functions $I_{-m}h_\alpha$ are used for L_2 approximation on $\mathbb{R}^{2k+1}$, $0 \leq k \leq m \leq \alpha$. Theorem 5.2 turns the above quantities into

$$I_{-k}C_{-1/2}(I_{k-m}h_\alpha, I_{k-m}h_\alpha) \tag{6.3}$$

such that one only has to

- calculate the univariate convolution of $I_{k-m}h_\alpha$ with itself in the sense of (6.2)
- take the k–th derivative.

For moderate integer values of α the explicit determination of h_α, $I_{k-m}(h_\alpha)$, and (6.3) can be easily done by any program for symbolic calculation. This has been done to provide the examples in [4].

If we use the convolution recursion again, we find the representation

$$\begin{aligned} I_{-m}h_\alpha &= I_{-m}C_{-1/2}(f_\alpha, f_\alpha) \\ &= I_{-m}C_{-1/2}(I_\alpha f_0, I_\alpha f_0) \\ &= I_{\alpha-m}C_{\alpha-1/2}(f_0, f_0) \end{aligned}$$

of Wu's functions as integrals of $(2\alpha+1)$–variate convolutions $C_{\alpha-1/2}(f_0, f_0)$ of the characteristic function f_0 of the unit ball in $\mathbb{R}^{2\alpha+1}$. These positive definite compactly supported functions can be called "Euclids Hats". An explicit construction based on results of H. Wendland is in [4], while the above argument is from Wendland [7].

Another nice construction due to H. Wendland starts with Askey's functions

$$A_\nu(x) := (1 - \|x\|_2)_+^\nu, \quad x \in \mathbb{R}^d$$

which are compactly supported and positive definite on $\mathbb{R}^d$ for $\nu \geq \lfloor d/2 \rfloor + 1$ (see Askey [2]). If they are rewritten as radial functions

$$a_\nu(r) = (1 - \sqrt{2r})_+^\nu$$

in our sense, then the functions $I_k a_\nu$ for $k \in \mathbb{N}$ are positive definite and compactly supported in $\mathbb{R}^d$ for $\nu \geq \lfloor d/2 \rfloor + k + 1$ because of

$$F_{\frac{d-2}{2}} I_k a_\nu = F_{\frac{d+2k-2}{2}} a_\nu.$$

When written as functions $\psi_{\nu,k}$ of $\|x\|_2$ again, the functions $I_k a_\nu$ are in C^{2k} and consist of a single polynomial piece of degree $\nu + 2k$ on their support.

Fixing d and k to ensure C^{2k} and positive definiteness on $\mathbb{R}^d$, the lowest possible value of ν is $\lfloor d/2 \rfloor + k + 1$, yielding a degree of $\lfloor d/2 \rfloor + 3k + 1$. Under the above restrictions on d and k, H. Wendland shows [7] that this degree is minimal.

Further examples along the lines of (6.1) and (4.1) are left to the reader. It should be clear by now that the operator families $\{F_\nu\}$ and $\{I_\alpha\}$ together with their intrinsic relations form a useful toolbox for handling radial functions.

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Robert Schaback
Institut für Numerische und Angewandte Mathematik, Universität Göttingen, Lotzestraße 16-18, D-37083 Göttingen, Germany
schaback@namu01.gwdg.de

Zong–min Wu
Department of Mathematics, Fudan University, 200433 Shanghai, China