# Rational Geometric Curve Interpolation

R. Schaback

**Abstract.** This survey covers geometric (i.e. parametrization–independent) curve interpolation methods by piecewise rational functions with geometric continuity. It includes shape–preserving properties, orders of accuracy, and provides a number of examples.

## §1. Introduction

A *classical* (non–parametric) univariate interpolation problem is based on two different kinds of data: given *ordinates*

$$y_0, y_1, \dots, y_{N-1} \in \mathbb{R}^d, \quad d, N \in \mathbb{N} \tag{1}$$

are to be interpolated at given *abscissae*

$$a \le t_0 < t_1 < \dots < t_{N-1} \le b, \quad a, b \in \mathbb{R} \tag{2}$$

by a function $f_N$ from a set $F_N \subseteq C[a,b]$ of (possibly vector–valued) functions on $[a,b] \subset \mathbb{R}$ in the sense that the ordinate values $y_i$ are attained by $f_N$ on the abcissae, i.e.:

$$f_N(t_i) = y_i, \quad 0 \le i \le N-1. \tag{3}$$

In Computer–Aided Design applications, however, only the ordinates or *positions* (1) are given, and there is no information on the abscissae or *parameters* (2). The interpolation conditions still have the form (3), but the parameters $t_i$ have to be determined from the positions alone (*geometric* interpolation). If the data

$$y_i = f(s_i), \quad 0 \le i \le N-1$$

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are sampled from a curve

$$f : [a, b] \to \mathbb{R}^d$$

for unknown values

$$0 \leq s_0 < s_1 < \ldots < s_{N-1} \leq b$$

of the parameter $s$ of $f$, then the positions $y_i$ are considered as being independent of changes of the parametrization. The geometric information contained in the range $R = f([a, b])$ is the only available source for the determination of the unknown parameters $t_i$ or $s_i$.

This survey will concentrate on *geometric* curve interpolation. Recent accounts of the state–of–the–art of nonparametric univariate interpolation are by Gregory [10], Schmidt [25] and Späth [26].

In view of the well-known negative results on polynomial interpolation the sets $F_N$ of interpolating curves are usually chosen as spaces of *piecewise* polynomial or rational functions. Each piece should be defined with as little dependence on parametrization as possible, and the local connections between pieces should also not depend on parametrization. The latter is achieved by the notion of $GC^k$ (i.e. geometric $C^k$) continuity of two curves

$$f[a, b] \to \mathbb{R}^d, \quad g : [b, c] \to \mathbb{R}^d$$

at their common parameter $b$. It is required that there should exist strictly monotonic $C^k$ reparametrization functions

$$\varphi : [a, b] \to [\alpha, \beta], \quad \psi : [b, c] \to [\beta, \gamma]$$

such that the functions $f \circ \varphi^{-1}$ and $g \circ \psi^{-1}$ coincide up to their $k$–th derivative at $\beta$.

Local polynomial or rational curve pieces in *Bernstein–Bézier representation* have the advantage of being at least independent of linear or rational reparametrizations. They involve $n + 1$ vectors $b_0, \ldots, b_n$ called *control points* and represent a vector–valued polynomial as

$$BB[b_0, \ldots, b_n](t) := \sum_{j=0}^{n} \beta_{j,n}(t) b_j \tag{4}$$

of degree $\leq n$, blending the control points $b_j$ by Bernstein polynomials

$$\beta_{j,n}(t) := \binom{n}{j} \left(\frac{b - t}{b - a}\right)^{n-j} \left(\frac{t - a}{b - a}\right)^{j}, \quad 0 \leq j \leq n.$$

*Rational* pieces in Bernstein–Bézier representation additionally use $n + 1$ positive weights $w_0, w_1, \ldots, w_n$ to define

$$R[w_0, b_0, \ldots, w_n, b_n](t) := \frac{BB[w_0 b_0, \ldots, w_n b_n](t)}{BB[w_0, \ldots, w_n](t)}. \tag{5}$$

Basic facts on polynomial Bernstein–Bézier representations (4) can be found in any textbook on Computer–Aided Design, e.g. Farin [3] or Hoschek/Lasser [12], while a comprehensive account of rational curves and surfaces is given by Farin in [2].

One of the standard properties of rational Bernstein–Bézier representations (5) for $n \geq 1$ is that by a suitable reparametrization two of the weights can be chosen to be equal to one. For rational quadratics, usually $w_0 = w_2 = 1$ leaves $w_1$ to cover the whole range of conics by

$$\begin{array}{rcll} w_1 & < & 1 & : \text{ elliptic curve,} \\ w_1 & = & 1 & : \text{ parabolic curve,} \\ w_1 & > & 1 & : \text{ hyperbolic curve.} \end{array}$$

Details can be found in the survey of Lee [13] on rational Bézier representations of conics. For rational cubics it is somewhat more convenient to use $w_1 = w_2 = 1$, leaving $w_0$ and $w_3$ as free weights with the special cases

$$\begin{array}{rcccll} & & w_0 & = & w_3 & : \text{ denominator of degree } \leq 2, \\ 1 & = & w_0 & = & w_3 & : \text{ constant denominator, polynomial cubic.} \end{array}$$

Unfortunately, there is no survey of the special geometric properties of Bernstein–Bézier representations of rational cubics. From the available sources on rational curves we note without further explanation that successive subdivision of rational curves takes successive means of weights, causing all weights to tend to one, if two of the weights are normalized to be exactly one. Thus some precautions to fixed choices of weights have to be made; for sufficiently dense data the interpolation methods using rational pieces should allow for variable weights tending to one.

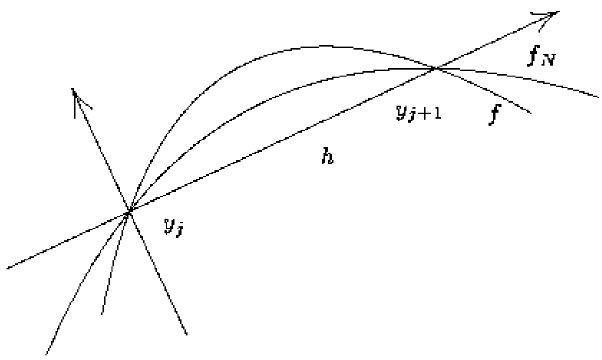


**Figure 1.** Error between curve and interpolant.

To measure the error of an interpolant $f_N$ to a curve $f$ that supplied the interpolation data, usually $f$ is assumed to be smooth and regular in the sense

$$f'(t) \neq 0 \text{ for all } t \in [a, b].$$

Then the behavior of $f$ and $f_N$ between two interpolation points $y_j \neq y_{j+1}$ is considered (see Figure 1).

If the data sample is dense enough in the sense that

$$h := \max_j \|y_{j+1} - y_j\|$$

is sufficiently small, the chord $y_j y_{j+1}$ defines a local coordinate system, along which the curve $f$ can be uniquely reparametrized by a monotonic function $\varphi_j$. The interpolant $f_N$ must be proven to be regularly reparametrizable between $y_j$ and $y_{j+1}$ by a monotonic function $\psi_j$. For instance, if $f$ and $f_N$ are regular, then on the unique hyperplane $H(t)$ perpendicular to $y_{j+1} - y_j$ and containing the point

$$y(t) = (1-t)y_j + ty_{j+1}, \; t \in [0,1]$$

there are two uniquely defined points $f(\varphi_j(t))$ and $f_N(\psi_j(t))$, and the local error between $y_j$ and $y_{j+1}$ can be measured as

$$\max_{0 \le t \le 1} \|f(\varphi_j(t)) - f_N(\psi_j(t))\|,$$

if $h$ is sufficiently small. Taking another maximum over $j$, the overall error can be evaluated. If the error has the behavior $\mathcal{O}(h^m)$ for $h \to 0$, the interpolation method is said to be of *order* $m$ or of *accuracy* $h^m$.

This survey will focus on geometric rational curve interpolation in $\mathbb{R}^d$ for dimensions $d = 2$ or $d = 3$ with geometric $GC^k$ continuity for $k = 1$ or $k = 2$, but some detours into piecewise polynomial cases will be necessary. There is a vast literature on methods with additional "tension" parameters which can be used for local shape control in design applications. This survey excludes these methods and concentrates on interpolants which are completely determined by the given data. The reader is tacitly assumed to be familiar with the basics of Bernstein–Bézier representations. Because a number of existing interpolation methods treats variations of two–point Hermite interpolation problems, these will be considered first, being followed later by Lagrange interpolation problems. Some examples will conclude the survey.

## §2. Two–point $GC^2$ Hermite Interpolation

In contrast to classical derivatives the notions of *tangent direction*

$$r(t) := f'(t)/\|f'(t)\|_2, \quad t \in [a,b] \tag{6}$$

and *curvature*

$$\kappa(t) = \|f'(t) \times f''(t)\|_2 / \|f'(t)\|_2^3, \quad t \in [a,b] \tag{7}$$

of smooth curves $f : [a,b] \to \mathbb{R}^d$ are independent of the parametrization. Thus they are ideal candidates for specifying parametrization–independent Hermite interpolation data. To make (6) and (7) well–defined, the underlying curve should be regular and the classical cross–product is employed for $d = 3$, while for $d = 2$ we use

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

to define *signed* curvature as

$$\kappa(t) = (f'(t) \times f''(t))/\|f'(t)\|_2^3, \quad t \in [a, b].$$

A two–point $GC^1$ Hermite interpolation problem will specify two positions $y_0 \neq y_1$ together with two tangent directions $r_0, r_1$ to be attained by the interpolant at $y_0, y_1$. In $\mathbb{R}^2$, two curvature values for a two–point $GC^2$ Hermite interpolation problem can be simply specified as two (signed) real numbers $\kappa_0$ and $\kappa_1$, while in $\mathbb{R}^3$ one should specify two vectors $\kappa_0\eta_0$ and $\kappa_1\eta_1$ with positive curvature values $\kappa_0$ and $\kappa_1$. Here, $\eta_0$ and $\eta_1$ are (normalized) *binormal* vectors, being orthogonal to the osculating planes and to the tangent directions $r_0$ and $r_1$ at $y_0$ and $y_1$, respectively.

If the data $y_0, y_1, r_0$ and $r_1$ lie on a line $L$ in $\mathbb{R}^2$, while the curvature data, illustrated in Figure 2 by small circular arcs at $y_0$ and $y_1$, require different signs of curvature, then any polynomial or rational interpolant in Bernstein–Bézier representation will in general need at least six control points of the form

$$\begin{array}{rclrcl} b_0 & = & y_0, & b_5 & = & y_1 \\ b_1 & = & y_0 + \alpha_0 r_0 & b_4 & = & y_1 - \alpha_1 r_1 \\ b_2 & = & y_0 + \beta_0 \eta_0 & b_3 & = & y_0 - \beta_1 \eta_1 \end{array}$$

with positive real constants $\alpha_i, \beta_i$ and vectors $\eta_0, \eta_1$ not parallel to $L$ (see Figure 2 for the $\mathbb{R}^2$ situation, but a similar construction with two disjoint osculating planes will suffice for $\mathbb{R}^3$). This proves

**Theorem 1.** *A general strategy to solve a two–point $GC^2$ Hermite interpolation problem by a rational or polynomial function in Bernstein–Bézier representation will require a degree of at least five.* ■

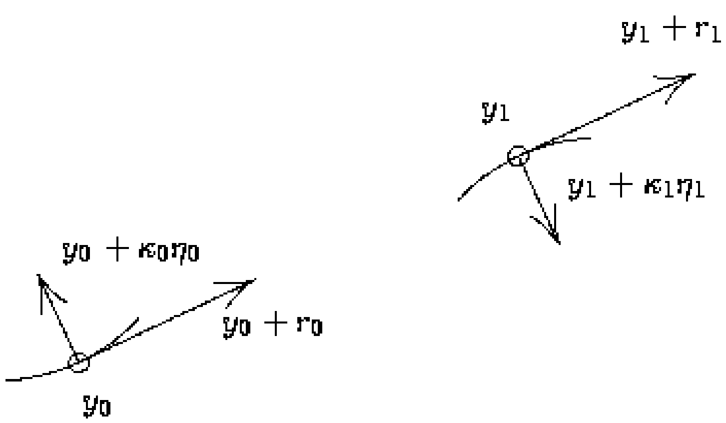


**Figure 2.** A special two–point $GC^2$ Hermite problem.

The same degree would suffice to produce a $C^2$ solution, if the parameters $t_0, t_1$ were known together with $f^{(j)}(t_i)$ for $i = 0,1$ and $j = 0,1,2$. But a simple comparison of the dimension of the problem and the degrees of freedom gives a quite different result. In $\mathbb{R}^3$, the problem dimensions are

| | | | |
|---|---|---|---|
| 3 | + | 3 | for positions $y_0, y_1$ |
| 2 | + | 2 | for tangent directions $r_0, r_1$ with $\|r_i\|_2 = 1$ |
| 2 | + | 2 | for $\kappa_i \eta_i$ values with $\eta_i \perp r_i,\ i = 0,1$ |

summing up to 14, while the degrees of freedom are

| | |
|---|---|
| 18 | for polynomial quintics |
| 15 | for polynomial quartics |
| 14 | for rational cubics |

in Bernstein–Bézier representation.

Thus the degree five interpolant, as required by Theorem 1 for the general case, has at least four superfluous degrees of freedom. However, no method is known that handles the general case by a quintic polynomial and makes a clever use of its free parameters under special circumstances. A method based on polynomial quartics, if it were available, could only handle a restricted class of subproblems and still would have one degree of freedom in a number of cases.

However, there is no additional degree of freedom if rational cubics are used, though they will not be able to handle the general problem. In [11], Höllig gives a solution and proves

**Theorem 2.** *If data are sampled from a smooth and regular curve in $\mathbb{R}^3$ with nonvanishing curvature and torsion, there is a unique solution to the $GC^2$ Hermite interpolation problem with accuracy $\mathcal{O}(h^6)$, if $h$ is sufficiently small.* ■

Höllig solves the problem via a linear system. A direct and explicit solution can be given by writing the interior control points $b_1$ and $b_2$ as

$$\begin{aligned} b_{1+2i} &= b_{3i} + (-1)^i \delta_i r_i, & i = 0,1, \\ \delta_i &= \|b_{1+2i} - b_{3i}\|_2 > 0, & i = 0,1, \end{aligned}$$

and observing that the biorthogonal vector $\eta_i$ must be perpendicular to both $b_1 - b_{3i}$ and $b_2 - b_{3i}$, yielding

$$\delta_{1-i} = \frac{\eta_i^T (y_1 - y_0)}{\eta_i^T r_{1-i}}, \quad i = 0,1 \tag{8}$$

independent of the prescribed curvature. Positivity of the solution requires $y_1 - y_0$ and $r_{1-i}$ to be in the same half–space defined by the osculating plane at $y_i$.

If the data are not coplanar and if the above condition is not satisfied, then a cubic solution cannot exist. Since we can resort to an $\mathbb{R}^2$ method for coplanar data, equations (8) characterize the solvability of the problem by cubics. Curvature continuity is achieved by choosing $w_1 = w_2 = 1$ and determining $w_0$ and $w_3$ directly from

$$\kappa_i \eta_i = \frac{2}{3}\, w_{3i} (-1)^i r_i \times (y_1 - y_0 - \delta_{1-i} r_{1-i}), \quad i = 0, 1 \tag{9}$$

(see Höllig [11]), which is possible if the data are not coplanar. Equations (8) and (9) provide an easy implementation of the method. However, there still is no satisfactory embedding of this special case into the general situation.

In $\mathbb{R}^2$, the dimensions of a two–point $GC^2$ Hermite interpolation problem are

$$\begin{array}{rcll} 2 & + & 2 & \text{for positions } y_0, y_1, \\ 1 & + & 1 & \text{for tangent directions } r_0, r_1 \text{ with } \|r_i\|_2 = 1, \\ 1 & + & 1 & \text{for (scalar) curvature values} \end{array}$$

which sum up to 8, and the degrees of freedom are

$$\begin{array}{rl} 6 & \text{for polynomial quadratics,} \\ 7 & \text{for rational quadratics,} \\ 8 & \text{for polynomial cubics,} \\ 10 & \text{for rational cubics} \\ 10 & \text{for polynomial quartics,} \\ 12 & \text{for polynomial quintics.} \end{array}$$

Again, there is an excess of 4 degrees of freedom in the polynomial quintics, which are required by Theorem 1 to handle the general case. The "natural" interpolants for a class of subproblems will consist of piecewise polynomial cubics, and this case is treated by de Boor, Höllig, and Sabin in [1]. They show

**Theorem 3.** *If the data are sampled from a smooth and regular curve in $\mathbb{R}^2$ with nonvanishing curvature, there is a unique solution to the $GC^2$ Hermite interpolation problem with accuracy $\mathcal{O}(h^6)$, if $h$ is small enough.* ■

The underlying system is a special case of (9) for planar data, i.e.:

$$\frac{3}{2}\,\delta_i^2 \kappa_i = c_i - c_0 \delta_{1-i}, \qquad i = 0, 1 \tag{10}$$

with

$$\begin{array}{rcl} \eta c_i & = & (-1)^i r_i \times (y_1 - y_0) \\ \eta c & = & r_0 \times r_1 \end{array}$$

for $\eta = \eta_0 = \eta_1 \perp r_0, r_1, y_1 - y_0$, if everything is written in $\mathbb{R}^3$ for compatibility with (9). These two quadratic equations with two unknowns may have several or no solutions, and it is quite surprising that for $h \to 0$ there are three coalescing positive solutions, if the data are sampled from a smooth and regular curve with nonvanishing curvature.

Unfortunately, the method does not work for general $h$, even if the above assumption holds. For $h$ small enough, it even works for general data sets, but it might produce up to three local solutions. Furthermore, the accuracy may go down to $\mathcal{O}(h^4)$ in cases with zeros of curvature. In [23] Theorem 3 is extended to the case of curves which may have simple zeros of curvature. As is shown in [1] by an example, the accuracy necessarily must do down to $\mathcal{O}(h^4)$ near *double* zeros of curvature. Thus for $h \to 0$ the behavior of the method is fully characterized, leaving its feasibility and numerical treatment for coarse data as an open question.

The approach of Goodman, Unsworth and others [5], [6], [9] uses rational cubics to interpolate planar two–point Hermite data with $GC^2$ continuity. Counting the degrees of freedom yields two free parameters per interval which the cited papers fix in a way that maintains convex interpolants to convex data and preserves circular arcs.

To be more specific, equations (9) and (10) in this case become

$$\frac{3}{2}\,\kappa_i\delta_i^2 = w_{3i}(c_i - c\delta_{1-i}), \quad i = 0, 1$$

and are satisfied for each choice of $\delta_0, \delta_i$ by determination of $w_{3i}$. This leaves $\delta_0$ and $\delta_1$ as free parameters, and in [5] Goodman suggests

$$\begin{aligned}
\delta_0 &= \frac{2\|y_1 - y_0\||\sin\alpha_1|}{2\lambda|\sin\alpha_1| + (1-\lambda)\|y_1 - y_0\|\cdot|\kappa_1| + 2|\sin(\alpha_0+\alpha_1)|}\\
\delta_1 &= \frac{2\|y_1 - y_0\||\sin\alpha_0|}{2\mu|\sin\alpha_0| + (1-\mu)\|y_1 - y_0\|\cdot|\kappa_0| + 2|\sin(\alpha_0+\alpha_1)|}
\end{aligned}$$

with parameters $\lambda, \mu \in (0, 1)$ and for angles

$$\begin{array}{llcll}
\alpha_0 & \text{between} & y_1 - y_0 & \text{and} & r_0,\\
\alpha_1 & \text{between} & r_1 & \text{and} & y_1 - y_0,
\end{array}$$

respectively. This choice is made for convex data and will yield a convex interpolant, while

$$\delta_0 = \gamma\|y_1 - y_0\|, \qquad \delta_1 = \delta\|y_1 - y_0\|$$

is suggested with $0 < \gamma, \delta < \gamma + \delta < 1$ for the other cases. Note that $\gamma = \delta = 1/3$ holds in the $h/3$–rule given by (10). So far there is no order of accuracy known for this method. Its advantage is its generality, providing a widely applicable $GC^2$ solution for planar data.

Goodman and Unsworth give an interactive extension in [9] which allows linear pieces to be inserted, and provide a detailed formulation of the algorithm together with a series of numerical examples. The paper [6] of Goodman, Ong, and Unsworth additionally allows to keep the interpolant away from a number of given lines which restrict the convex solution in the plane.

In [7], Goodman and Unsworth provide a $GC^n$ interpolation method for planar data using piecewise polynomials of degree $\leq n+1$. It solves the Lagrange problem via a Hermite problem with estimated tangent data and zero curvature data at the breakpoints. Though it is convexity–preserving and allows general data, its curvature properties restrict it to be of at most second order.

## §3. Two-point $GC^1$ Hermite Interpolation

Here the given data only consist of two positions $y_0, y_1$ and correspondent tangent directions $r_0, r_1$ with normalization $\|r_i\|_2 = 1$. If the rays $y_i + (-1)^i \alpha r_i$ for $\alpha \geq 0$ do not intersect, a $GC^1$ solution will necessarily need at least four control points (see Figure 3).

**Theorem 4.** *A general strategy to solve a two–point $GC^1$ Hermite interpolation problem by a rational or polynomial function in Bernstein–Bézier representation will require a degree of at least three.* ■

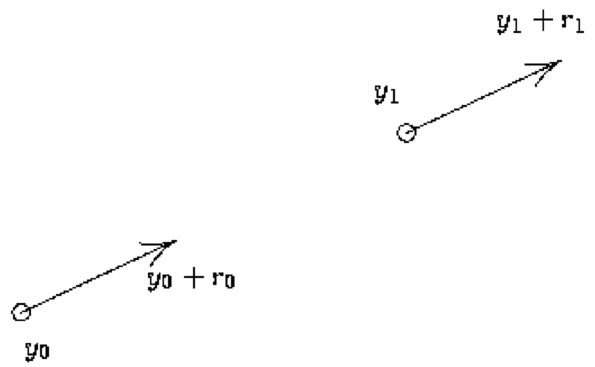


**Figure 3.** A special two–point $GC^1$ Hermite problem

All existing methods to solve this problem have additional degrees of freedom, which can be used, for instance,

1. to increase smoothness or
2. for certain reproduction properties or
3. for higher accuracy or
4. for additional interpolation conditions.

If the problem is posed in $\mathbb{R}^3$ with non-coplanar data, no partial solution by a quadratic method can exist because it would be confined to a plane defined by its three control points. Cubic polynomials will have two excess degrees of freedom, and no strategy seems to be known that makes clever use of this fact. The methods of [1] show that the simple $h/3$-rule

$$
\begin{array}{rclcrcl}
h & = & \|y_1 - y_0\| & & & & \\
b_1 & = & y_0 + \frac{h}{3}\, r_0\,, & & b_2 & = & y_1 - \frac{h}{3}\, r_1 \\
b_0 & = & y_0\,, & & b_3 & = & y_1
\end{array}
\tag{10}
$$

constructs a polynomial cubic $GC^1$ interpolant with $\mathcal{O}(h^4)$ accuracy. It works in any dimension and seems to be rather robust.

Rational cubics in $\mathbb{R}^3$ have 4 excess degrees of freedom and were considered by Piegl in [17]. When written in the form (5), Piegl's interpolant $y_P$ gets weights $(1,\ 1/3\ ,\ 1/3\ ,1)$, because it has the denominator $(1-t)^2+t^2$. If compared to a two–point cubic Hermite interpolant $y_H$, the condition $y_H(\ 1/2\ ) = y_P(\ 1/2\ )$ is imposed, leading to control points

$$
\begin{array}{rclcrcl}
b_0 & = & y_0 & & b_3 & = & y_1 \\
b_1 & = & y_0 + \frac{h}{2}\, r_0 & & b_2 & = & y_1 - \frac{h}{2}\, r_1
\end{array}
$$

for (5) with $h = \|y_1 - y_0\|$ and weights as above. Now, if the data are sampled from a smooth curve $f$ with arclength parametrization in the sense

$$
\begin{array}{rclcrcl}
y_0 & = & f(0) & & y_1 & = & f(t) \\
r_0 & = & f'(0) & & r_1 & = & f'(t),
\end{array}
$$

then curvature of Piegl's interpolant $y_P$ at zero will be

$$
\frac{8}{h^3}\left(2\cdot\frac{h}{2}\, r_0 \times \left(f(t) - \frac{h}{2}\, f'(t) - f(0)\right)\right) = \mathcal{O}(h)
$$

because of

$$
y_P'(0) = b_1 - b_0\,, \quad y_P''(0) = 2(b_2 - b_0)\,, \quad \text{and} \quad t = h + \mathcal{O}(h^3)
$$

for $h \to 0$. Thus Piegl's interpolant will have small curvature at the breakpoints for dense data. Its order of convergence cannot exceed two. Nevertheless, there may be visually pleasing results for coarse data.

In $\mathbb{R}^2$, polynomial cubics again have two additional degrees of freedom, which were used by de Boor, Höllig, and Sabin in [1] to increase smoothness to $GC^2$ and accuracy to $\mathcal{O}(h^6)$, if curvature data are known. The approach of [22] shows how to generate such auxiliary data, and then Theorem 3 can be carried over to the $GC^1$ case.

Rational or polynomial quadratics become feasible for planar data. A quadratic Bernstein-Bézier polynomial yields a $GC^1$ interpolant to Hermite data from convex curves by simply constructing the missing control point $b_1$ by intersection of tangents (see Figure 4). The methods of [1] easily supply a proof of

**Theorem 5.** *If the data are sampled from a smooth and regular curve in* $\mathbb{R}^2$ *with nonvanishing curvature, there is a unique solution to the* $GC^1$ *Hermite interpolation problem with accuracy* $\mathcal{O}(h^4)$. ■

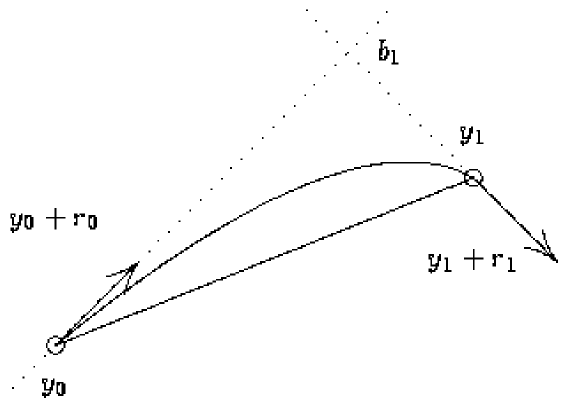


**Figure 4.** Quadratic $GC^1$ Hermite Interpolant.

Going to rational quadratics, there will be an excess degree of freedom in the choice of the weight $w_1$, which controls the type of the conic represented by the rational quadratic. Since three points and two tangents uniquely define a conic (this is a classical result of projective geometry), an additional data point $y_{1/2}$ within the triangle $y_0\ b_1\ y_1$ can be prescribed and there still will be a unique solution. For an explicit construction and further details see [24].

Both methods work wherever the data allow the formation of triangles like in Figures 4 and 5, and they give visually pleasing results. Note that the method using rational quadratics will reproduce any kind of conic.

## §4. Lagrange Interpolation via Hermite Interpolation

It is very convenient to use two-point Hermite interpolation sequentially on large data sets, because the methods are purely local, the complexity grows linearly with the sample size, and one can work on a data stream with only local storage, calculating secondary information (derivatives, bending energy etc.) "on the fly". But then one needs good estimates for derivative data. As a rule of thumb, which is made precise in [22], a Hermite interpolation method of final accuracy $h^m$ requires accuracy $h^{m-j}$ for its $j$-th derivative input. Unfortunately, for purely positional data the lack of information on the parametrization makes it hard to design methods of high order which construct data like tangent directions or values of curvature or torsion with sufficiently high accuracy.

Interpolating positional data $y_i \in \mathbb{R}^d$ at successive chordlength abscissae $t_i$ with

$$t_{i+1} - t_i = \|y_{i+1} - y_i\|_2$$

cannot yield approximations of order more than three in general, but if chordlength is taken with respect to a fixed center $y_\ell$, then

$$t_i = t_\ell + \|y_i - y_\ell\|_2 \ \mathrm{sgn}\,(i-\ell),\ i \text{ around } \ell$$

does not spoil approximation orders, but still requires elimination of the parametrization effects. A detailed analysis of this approach can be found in [22] together with methods for $GC^{k-1}$ interpolation by piecewise polynomials of degree $\leq 2k-1$ with accuracy $\mathcal{O}(h^{2k})$ for arbitrary $k$, provided that the data are sampled from a sufficiently smooth and regular curve.

## §5. Direct Lagrange Interpolation in $\mathbb{R}^3$

This is a field in which still a lot of work must be done, though it is most important for applications. To see why this case is difficult to handle, we again compare the number of parameters of the problem with the degrees of freedom of certain families of interpolating functions. The simplest conceivable and useful piecewise polynomial interpolant (apart from the trivial case of classical Lagrange interpolation at estimated parameter values) seems to be a $GC^2$ piecewise cubic. Given purely positional data $y_0, \ldots, y_{N-1} \in \mathbb{R}^3$, we are left with two free control points per section, if we place the breakpoints at the "interior" data points $y_1, \ldots, y_{N-2}$. Then $GC^1$ continuity enforces collinearity of three control points around each interior data point, yielding $2(N-2)$ scalar conditions. Furthermore, $GC^2$ continuity requires coplanarity of five points around these data points, plus a scalar curvature continuity condition on each osculating plane. Again, this amounts to $2(N-2)$ scalar conditions, which still leaves $6(N-1) - 4(N-2) = 2N+2$ unused degrees of freedom. No algorithm is known which makes a clever use of these degrees of freedom. For instance, additional data points $y_{i+1/2}$ for $i = 0, \ldots, N-2$ might be prescribed in each section joining $y_i$ and $y_{i+1}$. If the local parameter $t_{i+1/2}$ with

$$f_N(t_{i+1/2}) = y_{i+1/2}\,, \quad 0 \leq i \leq N-1$$

is not prescribed, this will fix $2(N-1)$ degrees of freedom, allowing two additional boundary tangent directions at $y_0$ and $y_{N-1}$ to be specified. No solution to this problem is known so far.

Polynomial quadratics can only form $GC^1$ interpolants, because the osculating planes will be piecewise constant. Still they will have $N+1$ degrees of freedom, and again there seems to be no method available in the literature. Going over to rational pieces does not improve the situation.

## §6. Direct Lagrange Interpolation in $\mathbb{R}^2$

In contrast to the situation in $\mathbb{R}^3$, where there is no straightforward way to formulate a well-posed problem for a suitable family of interpolating functions, there are several possibilities to handle *planar* Lagrange data. Polynomial quadratics can form a $GC^2$ interpolant to $y_0, y_1, \ldots, y_{N-1}$ in $\mathbb{R}^2$ with breakpoints at $y_1, \ldots, y_{N-2}$, because the $N-1$ additional control points $b_1, \ldots, b_{N-1}$ needed to form interpolating Bernstein-Bézier pieces $BB[y_{i-1}, b_i, y_i]$ for $i = 1, \ldots, N-1$ yield $2N-2$ degrees of freedom which are enough to account for the $2(N-2)$ conditions for $GC^2$ continuity at the

$N-2$ interior points. This leaves two degrees of freedom to be used for boundary conditions in the form of prescribed tangent directions at $y_0$ and $y_{N-1}$. However, a $GC^2$ interpolant by piecewise quadratics cannot have a zero of curvature. Thus the interpolation can work for convex data only.

This can be characterized in terms of the angles $\gamma_i$ between the lines $y_{i-1}\ y_i$ and $y_i\ y_{i+1}$ for $1 \le i \le N-2$ plus $\gamma_0$ and $\gamma_{N-1}$ being defined via the boundary conditions: the lines with angles $\gamma_0$ at $y_0$ and $\gamma_{N-1}$ at $y_{N-1}$ should be tangent to the solution at $y_0$ and $y_{N-1}$, respectively.

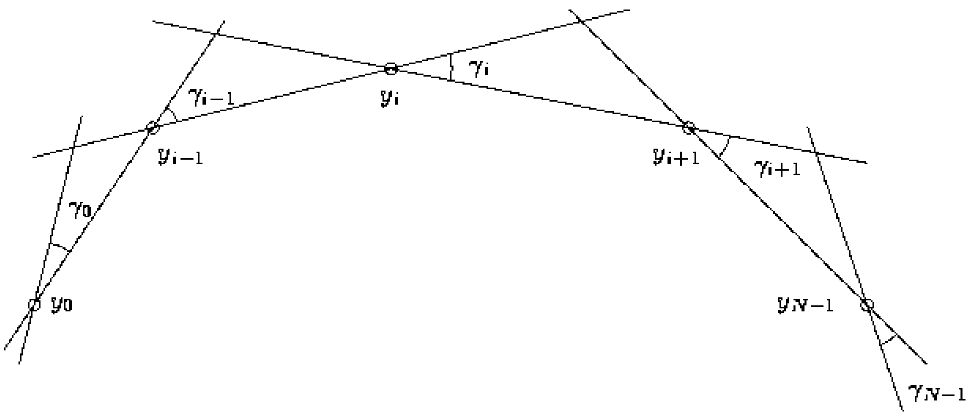


**Figure 5.** Quadratic Lagrange $GC^2$ Interpolation Data.

In [19] the following is proven:

**Theorem 6.** *For*

$$\begin{array}{ccccccc} 0 & < & \gamma_i & < & \pi, & & 0 \le i \le N-1 \\ 0 & < & \gamma_{i-1}+\gamma_i & < & \pi, & & 1 \le i \le N \end{array}$$

*there exists a $GC^2$ interpolant. The interpolant is unique, if*

$$\begin{array}{ccccccc} 0 & < & \gamma_i & < & \pi/2 & & 0 \le i \le N-1, \\ 0 & < & \gamma_{i-1}+\gamma_i & < & \pi/2, & & 1 \le i \le N. \end{array} .$$

*If the data are a sufficiently dense sample from a smooth curve with nonvanishing curvature, there is a unique interpolant with $\mathcal{O}(h^4)$ accuracy.* ■

The solution can be calculated by solving a nonlinear system of equations with a tridiagonal and diagonally dominant Jacobian, using a stepsize-controlled Newton–Raphson iteration.

*Rational* quadratics introduce additional $N-1$ degrees of freedom, because they have one free weight per interval. Consequently, they possibly can interpolate an additional point per interval. As a variation of the situation of Figure 5, there now are additional angles $\delta_i^+$ and $\delta_i^-$ as shown in Figure 6, which now requires an odd number of data points $y_0, y_1, \ldots, y_{2N}$.

Again, the data must be "convex", i.e.: there should be a smooth and regular curve with nonvanishing curvature which interpolates the data. Then [24] proves

**Theorem 7.** *For angles $\gamma_i, \delta_i^-$ and $\delta_i^+$ as in Figure 6, being positive and satisfying*

$$0 < \delta_i^+ + \delta_i^- < \gamma_i < \pi/2,$$

*there is a unique $GC^2$ interpolant by piecewise conics of varying type. The interpolation has accuracy $\mathcal{O}(h^5)$ if the data are a dense sample from a smooth and regular curve with nonvanishing curvature.* ■

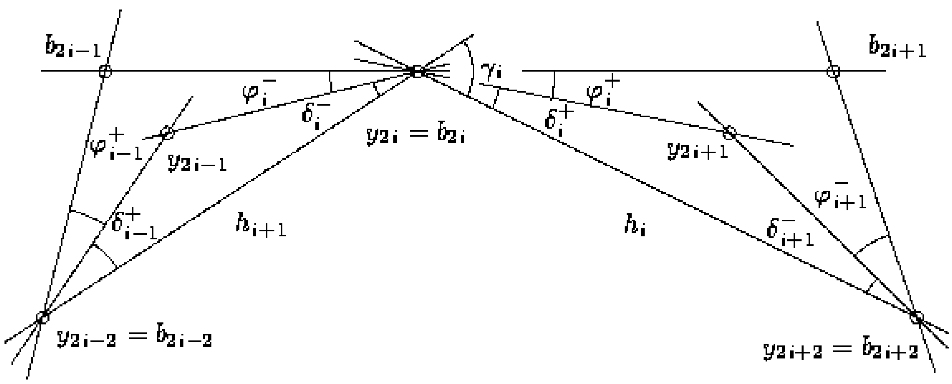


**Figure 6.** Quadratic Rational Lagrange $GC^2$ Interpolation Data.

Both methods of this section have a serious drawback: they cannot handle data with inflection points. For the $GC^2$ piecewise quadratic interpolant, the paper [20] shows how to add cubic pieces to cope with zeros of curvature or collinear data. Some remarks in [24] do the same for general conics, but a proof of the accuracy of a "mixed" method still is missing (see [18] for a "mixed" nonparametric interpolation which also lacks a convergence analysis).

## §7. Examples

This section shows some experimental results for the following methods:

- *H*3 Höllig's method of piecewise rational cubic $GC^2$ Hermite interpolation of $\mathbb{R}^3$ data [11].
- *h*/3 The $h/3$ method for piecewise cubic $GC^1$ Hermite interpolation of $\mathbb{R}^3$ data.
- *BHS* DeBoor, Höllig, and Sabin's method of piecewise cubic $GC^2$ Hermite interpolation of $\mathbb{R}^2$ data [187].
- *GU*2 Goodman's and Unsworth's method of convexity–preserving piecewise rational cubic $GC^2$ Hermite interpolation of $\mathbb{R}^2$ data [5], [6], [9].
- *Sk* Piecewise polynomial $GC^{k-1}$ Lagrange interpolation of degree $\leq 2k$ to $\mathbb{R}^3$ data [22].
- *SH*2 Convexity–preserving piecewise rational quadratic $GC^1$ Hermite interpolation of $\mathbb{R}^2$ data [24].
- *SL*2 Convexity–preserving piecewise rational quadratic $GC^2$ Lagrange interpolation of $\mathbb{R}^2$ data [24].

The algorithms of [17] and [7] were not included here, because they systematically produce much too small values of curvature.

We start with 9 equidistant points on a semi–circle with radius one, and generate (exact) tangent and curvature information from local five–point interpolation by conics. Then the semi–circle will be reproduced by methods $GU2$, $SH2$, and $SL2$. All methods nicely reproduce the circular arc within plot precision, and some even produce a constant–looking curvature plot. Therefore we only include plots that contain nontrivial information (see figures 7 to 9).



**Figure 7.** Curvature of $h/3$ Interpolant.

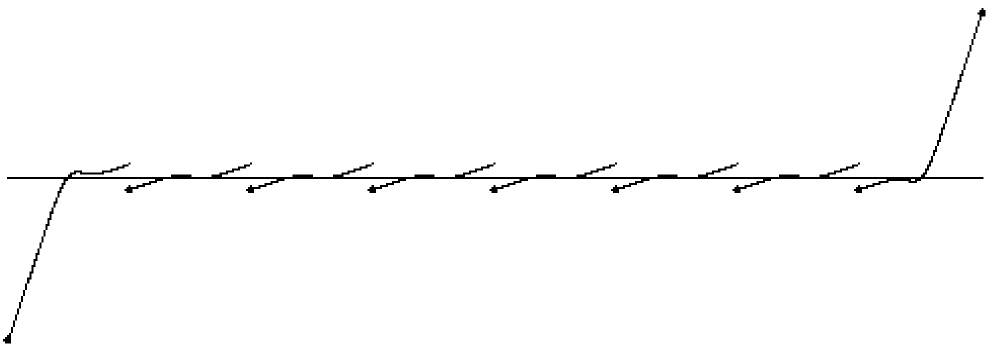

**Figure 8.** Derivative of curvature of $S3$ interpolant of order 6.
Hermite data generated as in [22]

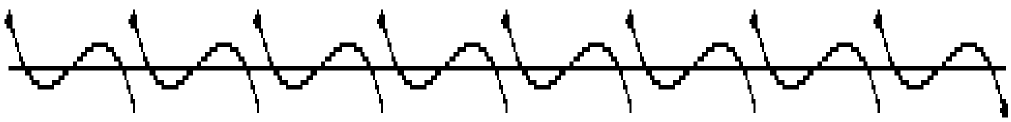

**Figure 9.** Derivative of curvature of $BHS$ interpolant of order 6.

The second example in taken from [22] and comprises 33 points on a twisted logarithmic spiral in $\mathbb{R}^3$ (see Figure 10) with nonvanishing curvature and torsion, as required for Höllig's method. Plots for the method $Sk$ are in the cited paper, while the behavior of Höllig's method is shown in figures 11 and 12.

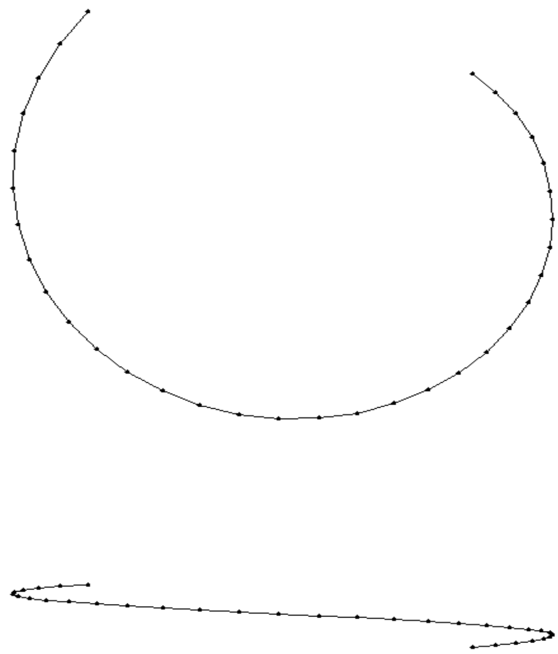

**Figure 10.** Logarithmic spiral, seen from above and from the side.

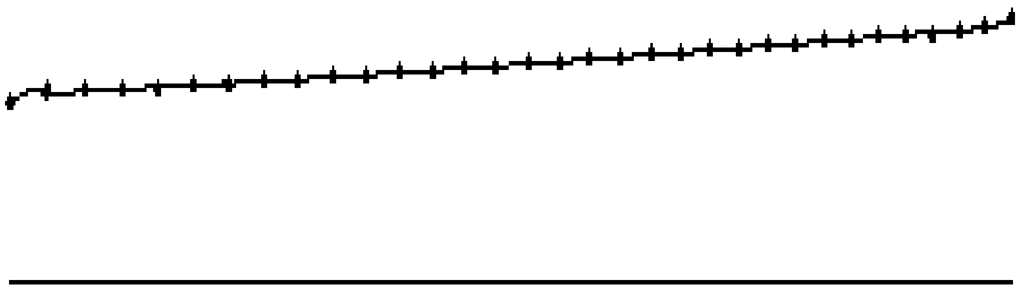

**Figure 11.** Derivative of curvature of Höllig's cubic rational interpolant.

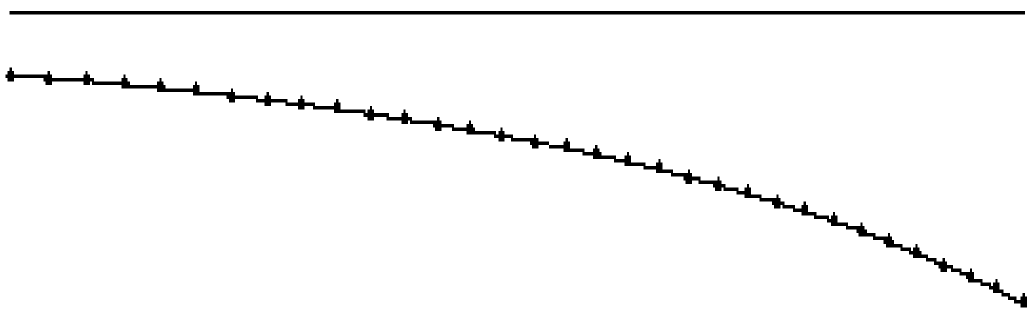

**Figure 12.** Torsion of Höllig's cubic rational interpolant.

The last example shows a manually entered coarse data set in $\mathbb{R}^2$, its form being inspired by the keel of the Oseberg ship. All methods give essentially the same shape of the curve, but the curvature plots show some of the differences.

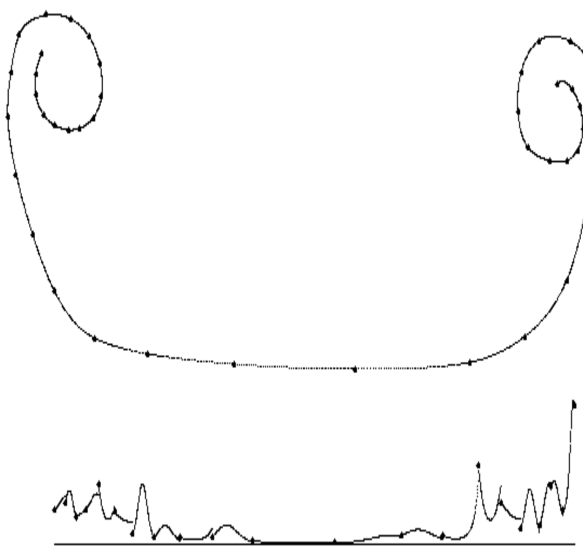

**Figure 13.** $SH2$ interpolant, plus its curvature.

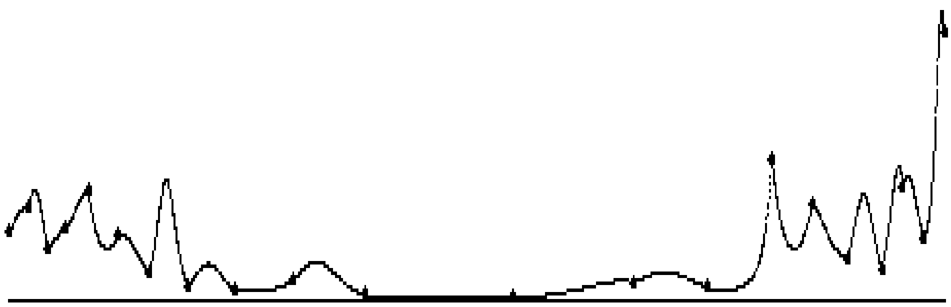

**Figure 14.** Curvature of $SL2$ interpolant.

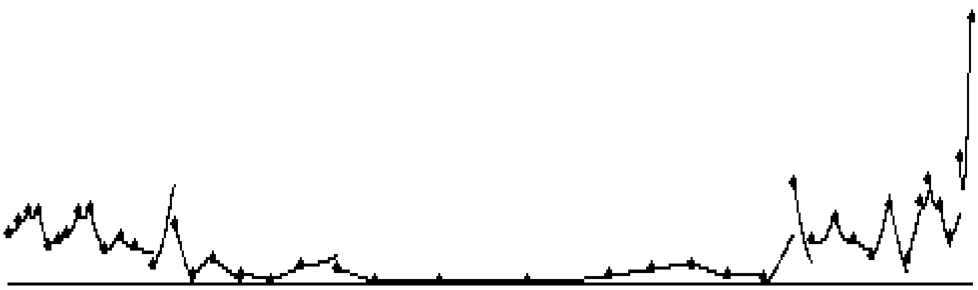

**Figure 15.** Curvature of $h/3$–rule interpolant.

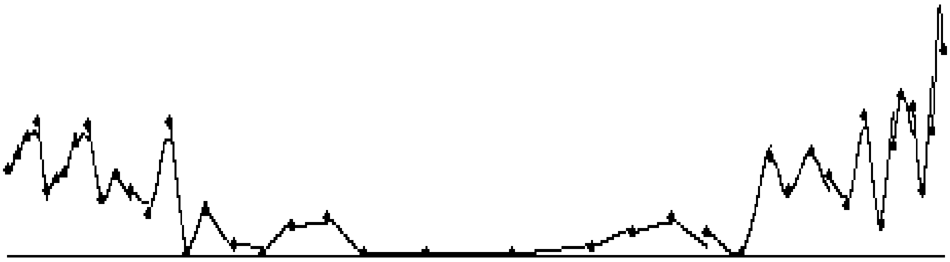

**Figure 16.** Curvature of fourth–order $S2$ interpolant.

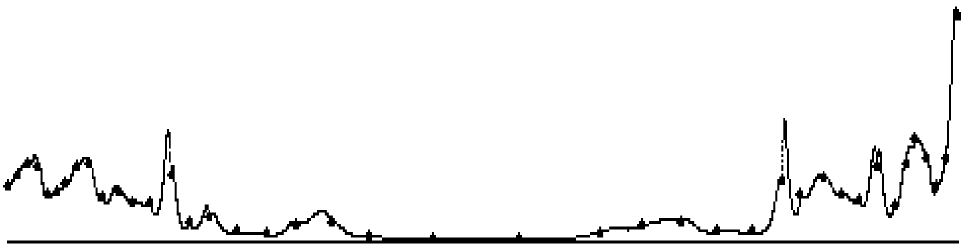

**Figure 17.** Curvature of $GU2$ interpolant.

In all cases the necessary Hermite data were generated by local five–point conic interpolation, the derivatives being cleared from parametrization effects using the methods of [22]. This is why the discontinuous curvature plots still look reasonable: the order of approximation is still high because of the good quality of data estimates. The method of deBoor, Höllig, and Sabin also produced a nice–looking plot of the interpolant, but the curvature values were bad, because in several intervals there was no $GC^2$ solution, and we had to insert pieces using the $h/3$ rule. Note that the quadratic rational methods have only about half the number of breakpoints.

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R. Schaback
Georg–August–Universität Göttingen
Lotzestrasse 16–18
D–3400 Göttingen, Germany
U0881@ DGOGWDG5.Bitnet