Reconstruction of Multivariate Signals via Prony's Method

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***Abstract*—The problem of recovering translates and corresponding amplitudes of sparse sums of Gaussians out of sampling values as well as reconstructing sparse sums of exponentials are nonlinear inverse problems that can be solved for example by Prony's method. Here, we want to demonstrate a new extension to multivariate input data.**

I. Introduction

Assume we only know equidistant measurements $f(k)$, $k = 0, \dots, N$ of the signal

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{2\pi \mathrm{i} t_j x}, \quad t_j \in [0,1], c_j \in \mathbf{C}. \tag{1}$$

The task is to recover the parameters t_j and the corresponding coefficients c_j. Methods that accomplish this include Super-Resolution [1], Prony's Method [4], ESPRIT [9], MUSIC [10], Matrix-Pencil-Method [2] or the Annihilating Filter Method for signals with finite rate of innovation [11], where all but the first method can be seen as Prony-like methods [7].

The Prony method in one dimension works as follows. We define a Prony polynomial

$$P(z) = \prod_{j=1}^{M} (z - \mathrm{e}^{2\pi \mathrm{i} t_j}) = \sum_{k=0}^{M} p_k z^k$$

and observe

$$\begin{aligned} \sum_{k=0}^{M} p_k f(k+m) &= \sum_{j=1}^{M} c_j \mathrm{e}^{2\pi \mathrm{i} t_j m} \sum_{k=0}^{M} p_k (\mathrm{e}^{2\pi \mathrm{i} t_j})^k \\ &= \sum_{j=1}^{M} c_j \mathrm{e}^{2\pi \mathrm{i} t_j m} \underbrace{P(\mathrm{e}^{2\pi \mathrm{i} t_j})}_{=0} = 0 \end{aligned} \tag{2}$$

for arbitrary shifts m. Thus, we have to solve the linear problem $Hp = (f(k+m))_{m,k=0}^{M} (p_k)_{k=0}^{M} = 0$ in order to find the coefficients p_k of the monomial representation of the Prony polynomial $P(z)$. The roots $\mathrm{e}^{2\pi \mathrm{i} t_j}$, $j = 1, \dots, M$ of this polynomial carry the knowledge of the unknown parameters t_j. After extracting the parameters t_j of the found roots, we determine the coefficients c_j as least square solution of the Vandermonde-type system

$$\left(\mathrm{e}^{2\pi \mathrm{i} t_j (k+m)}\right)_{\ell=0, j=1}^{2M+1, M} (c_j)_{j=1}^{M} = (f(\ell))_{\ell=0}^{2M+1}.$$

If the 1-dimensional kernel $K_{1,b}(x) := \mathrm{e}^{-bx^2}$, $b > 0$ is known beforehand, all above mentioned algorithms can also be used to recover translates t_j and corresponding coefficients c_j of a signal $s(x)$, $x \in \mathbf{R}$ of the form

$$\begin{aligned} s(x) &= \sum_{j=1}^{M} c_j \mathrm{e}^{-b(x-t_j)^2} = \sum_{j=1}^{M} c_j K_{1,b}(x - t_j), \\ & x \in \mathbf{R}, t_j \in [0,1], c_j \in \mathbf{C}, \end{aligned} \tag{3}$$

from sufficiently many sampling values $s(k), k = 0, \dots, N$, if the data is transferred to the Fourier-domain [5].

In [1], it is demonstrated that Super-Resolution is also applicable for multivariate signals, whereas for Prony-like methods this can be done by reducing the problem to a number of one-dimensional problems [8]. Thus, sums of multivariate translates of a given function $\phi(x)$ can be recovered from sampling points in the Fourier domain. By suitable projections to lines through the origin this reconstruction problem can be transferred to several separated one-dimensional reconstruction problems of the form (1), see [8], [6]. In contrast, we established a new, fully multidimensional Prony method that operates completely in the spatial domain for recovering multivariate translates $t_j \in [0,1]^d$ of

$$\begin{aligned} s(x) &= \sum_{j=1}^{M} c_j \mathrm{e}^{-b(x-t_j)^{\mathrm{T}}(x-t_j)}, \quad x \in \mathbf{R}^d, t_j \in [0,1]^d, c_j \in \mathbf{C} \\ &= \sum_{j=1}^{M} c_j K_{d,b}(x - t_j), \quad K_{d,b} := \mathrm{e}^{-bx^{\mathrm{T}}x} \end{aligned}$$

for the multivariate analogon of (3). Note that this approach is restriction to Gaussians $K_{d,b}(x)$, but this new approach circumvents inherent problems of established multivariate Prony methods that are caused by data-projections to one-dimensional subspaces.

II. Reconstruction of Translations of Multivariate Gaussians

We define a multivariate Prony polynomial

$$P : \mathbf{C}^d \to \mathbf{R},$$
$$P(z) := \sum_{k=0}^{N} p_k z^{n_k}, \quad n_k \in \Omega \subset \mathbf{N}^d. \tag{4}$$

with roots e^{2bt_j}, i.e. $P(\mathrm{e}^{2bt_j}) = 0$, $j = 1, ..., M$. The set $\Omega := \{n_k \,|\, k = 0, \ldots, N\}$ contains all exponents n_k of the multivariate monomials $z_1^{n_{k,1}} \cdots z_d^{n_{k,d}}$ that are active in $P(z)$. For $\alpha(m, n_k) := \mathrm{e}^{2bn_k^{\mathrm{T}} m}$ and $q_k := p_k \mathrm{e}^{bn_k^{\mathrm{T}} n_k} = K_{d,b}(n_k)$, with $n_k \in \Omega \subset \mathbf{N}^d$ and shifts $m \in \mathbf{N}^d$ we get

$$\begin{aligned}
&\sum_{k=0}^{N} q_k s(n_k + m)\alpha(m, n_k) \\
=&\sum_{j=1}^{M} c_j \underbrace{\mathrm{e}^{-b(m-t_j)^{\mathrm{T}}(m-t_j)}}_{=K_{d,b}(m-t_j)} \cdot \\
&\cdot \sum_{k=0}^{N} \underbrace{q_k \mathrm{e}^{-bn_k^{\mathrm{T}} n_k}}_{=p_k} \mathrm{e}^{-2b(n_k^{\mathrm{T}} m - n_k^{\mathrm{T}} t_j)} \underbrace{\mathrm{e}^{2bn_k^{\mathrm{T}} m}}_{=\alpha(m,n_k)} \\
=&\sum_{j=1}^{M} c_j K_{d,b}(m - t_j) \underbrace{P(\mathrm{e}^{2bt_j})}_{=0} = 0.
\end{aligned}$$

in analogy to the calculation in (2). Choosing the number of sampling points N, the exponents n_k in the Prony polynomial and sampling point shifts m_ℓ, $\ell = 0, \ldots, N$ suitably, we have to compute the kernel

$$Hq = 0$$
$$\left(s(n_k + m_\ell)\mathrm{e}^{2bm_\ell^{\mathrm{T}} n_k}\right)_{\ell,k=0}^{N} \underbrace{(p_k K_{d,b}(n_k)^{-1})}_{=q_k}{}_{k=0}^{N} = 0.$$

Once, we calculated the coefficients q_k, we can evaluate the coefficients p_k of the Prony polynomial $P(z)$. By construction, the translates t_j are contained in the $(d-1)$-dimensional zero set of $P(z)$. Thus, we need to ensure by increasing N that the dimension of the kernel of H is large enough, such that we have sufficiently many different polynomials $P(z)$ in order to uniquely determine t_j, $j = 1, \ldots, M$ by intersecting the corresponding zero sets.

Afterwards, the coefficients c_j can be determined as a least square solution of

$$(K_{d,b}(n_k - t_j))_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (s(n_k))_{k=0}^{N}.$$

III. Reconstruction of Sums of Multivariate Exponentials

The method for translations of Gaussians applies directly to sparse sums of multivariate exponentials, too. We now consider a sparse sum of d-variate exponentials

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{\langle x, t_j \rangle}, \quad x, t_j \in \mathbf{C}^d, c_j \in \mathbf{C}$$

and redefine the d-variate Prony polynomial P in (4) such that $P(\mathrm{e}^{t_j}) = 0$ for $j = 1, \ldots, M$ with

$$P(\mathrm{e}^{t_j}) := \sum_{k=0}^{N} p_k \prod_{\ell=1}^{d} (\mathrm{e}^{t_{j,\ell}})^{n_{k,\ell}} = \sum_{k=0}^{N} p_k \mathrm{e}^{\langle n_k, t_j \rangle}.$$

Applying the ideas of the previous section we observe that the coefficients of the Prony polynomial satisfy the relation

$$\sum_{k=0}^{N} p_k f(n_k + m_k) = \sum_{j=1}^{M} c_j \mathrm{e}^{\langle m_k, t_j \rangle} \underbrace{\sum_{k=0}^{N} p_k \mathrm{e}^{\langle n_k, t_j \rangle}}_{=P(\mathrm{e}^{t_j})=0} = 0.$$

Again, we have to choose N and $n_k, m_k \in \mathbf{C}^d$ properly and compute the kernel $Hp = 0$, where

$$H := (f(n_k + m_\ell))_{\ell,k=0}^{N}, \quad p := (p_k)_{k=0}^{N}.$$

Afterwards we construct the intersection of the zero-sets of all polynomials P, whose coefficient vectors p are elements of the kernel of H. When the parameters t_j, $j = 1, \ldots M$, are known, we calculate the corresponding coefficients as a least square solution of

$$\left(\mathrm{e}^{\langle n_k, t_j \rangle}\right)_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (f(n_k))_{k=0}^{N}.$$

Note that n_k defines the sampling pattern. This is also the case in the one-dimensional setting, although it is less obvious there. In the one-dimensional case we expand the Prony polynomial in the (full) monomial basis with exponents $k = 0, \ldots, N$, which leads to equidistant sampling points $f(k)$, as we have seen at the one dimensional case in the introduction. The set $\{n_k \,|\, k = 0, \ldots N\}$ defines the active monomials in the multivariate setting and therefor the sampling points here. The same holds for the shifts $m_k \in \mathbf{C}^d$. In the one-dimensional case these are numbers $m = 0, \ldots, N$, so that we can reuse some sampling points for the m-th row of H. The more involved variant of reusing sampling points in the multidimensional case is addressed in [3].

IV. Algorithmic Considerations

Here, we have assumed that a multivariate polynomial $P(z)$ in (4) can be constructed if sampling points of the above mentioned form are given. In [3], the authors establish sufficient and necessary conditions for the number of input data in order to ensure enough linearly independent polynomial defining vectors in the kernel of H for recovering the translates t_j, $j = 1, \ldots, M$ uniquely. Note that we have some freedom in choosing the monomial exponents n_k and the shifts m_ℓ. For the purpose of understanding we give the following example in the two-dimensional case $d = 2$, were we choose $n_k = (\lfloor k/N \rfloor, k \mod N)$, $m_k = (0, k)$, $k = 0, \ldots, M$.

Algorithm for multivariate exponentials
Input: $f(n_k + m_\ell)$, n_k, m_ℓ, $k, \ell = 0, \ldots, 2N$

1) Calculate all vectors p in the kernel of $H = (f(m_\ell + n_k))_{\ell,k=0}^{N}$ and construct the Polynomials $P(z) = \sum_{k=0}^{N} p_k z^{n_k}$.

2) Find the common zeroes e^{t_j}, $j = 1, \ldots, M$, of at least $d + 1$ Polynomials $P(z)$.
3) Find a least square solution of the linear system $(\mathrm{e}^{\langle n_k, t_j \rangle})_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (f(n_k))_{k=0}^{N}$.

Output: M, t_j, c_j.

Algorithm for multivariate Gaussians
Input: $s(n_k + m_\ell)$, n_k, m_ℓ, $k, \ell = 0, \ldots, 2N$, $b > 0$

1) Calculate all vectors q in the kernel of $H = (s(n_k + m_l)\mathrm{e}^{2bm_\ell^{\mathrm{T}} n_k})_{\ell,k=0}^{N}$ and construct the polynomials $P(z) = \sum_{k=0}^{N} (q_k \mathrm{e}^{-bn_k^{\mathrm{T}} n_k}) z^{n_k}$.
2) Find the common zeroes t_j, $j = 1, \ldots, M$, of at least $d + 1$ polynomials $P(z)$.
3) Find a least square solution of the linear system $(\mathrm{e}^{-b(n_k - t_j)^{\mathrm{T}}(n_k - t_j)})_{k=0,j=1}^{N,M} (c_j)_{j=1}^{M} = (s(n_k))_{k=0}^{N}$.

Output: M, t_j, c_j.

V. Numerical Example for Translates of Gaussians

We set $d = 2$, $M = 3$, $b = c_1 = c_2 = c_3 = 1$, and

$$t_1 = \mathrm{e}^{(1,0)^{\mathrm{T}}} \approx (2.718, 1)^{\mathrm{T}},$$
$$t_2 = \mathrm{e}^{(1,2)^{\mathrm{T}}} \approx (2.718, 7.389)^{\mathrm{T}},$$
$$t_3 = \mathrm{e}^{(-1,3)^{\mathrm{T}}} \approx (0.368, 20.09)^{\mathrm{T}},$$

and consider the function

$$s(x) = \sum_{j=1}^{3} \mathrm{e}^{-(x-t_j)^{\mathrm{T}}(x-t_j)} = \sum_{j=1}^{3} K_{2,1}(x - t_j)$$

Let the Prony polynomial be defined via the exponents $n_0 = (0,0)^{\mathrm{T}}, n_1 = (0,1)^{\mathrm{T}}, n_2 = (0,2)^{\mathrm{T}}, n_3 = (1,0)^{\mathrm{T}}, n_4 = (1,1)^{\mathrm{T}}, n_0 = (2,0)^{\mathrm{T}}$ and $m_\ell = (\ell, 0)^{\mathrm{T}}$, $\ell = 0, \ldots, 5$. Now, we construct the matrix

$$H = \left(s(n_k + m_\ell)\mathrm{e}^{2m_\ell^{\mathrm{T}} n_k} \right)_{\ell,k=0}^{5}$$

and find three linearly independent vectors q_1, q_2, q_3 in the kernel of H. In figure 1 we see the zeros sets of the Prony polynomials whose coefficients are given by the vectors q_1, q_2, q_3 depicted in blue, green, and black. We observe, that the zero sets intersect just at the desired points t_1, t_2, t_3, depicted as red circles.

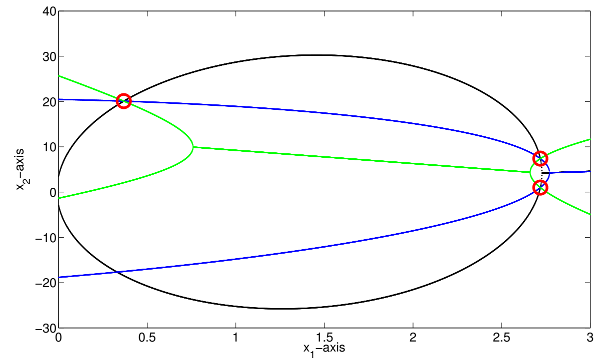

Fig. 1. The absolute values of the roots of three different two dimensional Prony polynomials depicted in blue, green and black. The red circles indicate the location of the shifts t_1, t_2, t_3.

VI. Conclusion

For a linear combination of translates of a multivariate Gaussian, we have presented a new algorithm that can recover the multivariate translates $t_j \in [0,1]^d$ and the corresponding coefficients $c_j \in \mathbf{C}$ out of d-dimensional sampling points. With a second algorithm we are able to recover parameters $t_j \in \mathbf{C}^d$ and corresponding coefficients $c_j \in \mathbf{C}$ of a multivariate exponentials sum. In contrast to already existing multivariate Prony algorithms we present here an approach that resolves the issue of required equally signed coefficients ($c_j \in \mathbf{R}_+$). Furthermore, this new approach is capable of recovering all parameters for an arbitrary (satisfying a minimal-separation-condition) set of translates, whereas the methods in [8], [6] need adaptive further sampling after a preprocessing step, based on an initial sampling process. These advantages come at the price of calculating common zeros of multivariate polynomials, which is a challenging task itself.

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