Shift-invariant approximation

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1 Synonyms
Approximation by integer translates

2 Mathematics Subject Classification
41A25, 42C15, 42C25, 42C40

3 Short Definition
Shift–invariant approximation deals with functions $f$ on the whole real line, e.g. time series and signals. It approximates $f$ by shifted copies of a single generator $\varphi$, i.e.

$$f(x) \approx S_{f,h,\varphi}(x) := \sum_{k \in \mathbb{Z}} c_{k,h}(f) \varphi \left( \frac{x}{h} - k \right), \quad x \in \mathbb{R}. \quad (1)$$

The functions $\varphi \left( \frac{x}{h} - k \right)$ for $k \in \mathbb{Z}$ span a space that is shift–invariant wrt. integer multiples of $h$. Extensions [de Boor et al(1994a)de Boor, DeVore, and Ron, de Boor et al(1994b)de Boor, DeVore, and Ron] allow multiple generators and multivariate functions. Shift–invariant approximation uses only a single scale $h$, while wavelets use multiple scales and refinable generators.

4 Description
Nyquist–Shannon–Whittaker–Kotelnikov sampling provides the formula

$$f(x) = \sum_{k \in \mathbb{Z}} f(kh) \text{sinc} \left( \frac{x}{h} - k \right)$$

for band–limited functions with frequencies in $[-\pi/h, +\pi/h]$. It is basic in Electrical Engineering for AD/DA conversion of signals after low–pass filtering. Another simple example arises from the hat function or order two B–spline
$B_2(x) := 1 - |x|$ for $-1 \leq x \leq 1$ and zero elsewhere. Then the “connect–the–
dots” formula

$$f(x) \approx \sum_{k \in \mathbb{Z}} f(kh)B_2 \left( \frac{x}{h} - k \right)$$

is a piecewise linear approximation of $f$ by connecting the values $f(kh)$ by
straight lines. These two examples arise from a generator $\varphi$ satisfying the car-
dinal interpolation conditions $\varphi(k) = \delta_{0k}$, $k \in \mathbb{Z}$, and then the right–hand
side of the above formulas interpolates $f$ at all integers. If the generator is a
higher–order $B$–spline $B_m$, the approximation

$$f(x) \approx \sum_{k \in \mathbb{Z}} f(kh)B_m \left( \frac{x}{h} - k \right)$$

goes back to I.J. Schoenberg and is not interpolatory in general.

So far, these examples of (1) have very special coefficients $c_{k,h}(f) = f(kh)$
arising from sampling the function $f$ at data locations $h\mathbb{Z}$. This connects shift–
invariant approximation to sampling theory. If the shifts of the generator are
orthonormal in $L_2(\mathbb{R})$, the coefficients in (1) should be obtained instead as
$c_{k,h}(f) = (f, \varphi(\cdot - k))_2$ for any $f \in L_2(\mathbb{R})$ to turn the approximation into an
optimal $L_2$ projection. Surprisingly, these two approaches coincide for the sinc
case.

Analysis of shift–invariant approximation focuses on the error in (1) for
various generators $\varphi$, and for different ways of calculating useful coefficients
$c_{k,h}(f)$. Under special technical conditions, e.g. if the generator $\varphi$ is compactly
supported, the Strang–Fix conditions [Strang and Fix(1973)]

$$\hat{\varphi}^{(j)}(2\pi k) = \delta_{0k}, \ k \in \mathbb{Z}, \ 0 \leq j < m$$

imply that the error of (1) is $O(h^m)$ for $h \to 0$ in Sobolev space $W_m^2(\mathbb{R})$ if the
coefficients are given via $L_2$ projection. This holds for $B$–spline generators of
order $m$.

The basic tool for analysis of shift–invariant $L_2$ approximation is the bracket
product

$$[\varphi, \psi](\omega) := \sum_{k \in \mathbb{Z}} \hat{\varphi}(\omega + 2k\pi)\overline{\hat{\psi}(\omega + 2k\pi)}, \ \omega \in \mathbb{R}$$

which is a $2\pi$–periodic function. It should exist pointwise, be in $L_2[-\pi, \pi]$ and
satisfy a stability property

$$0 < A \leq [\varphi, \varphi](\omega) \leq B, \ \omega \in \mathbb{R}.$$ 

Then the $L_2$ projector for $h = 1$ has the convenient Fourier transform

$$\hat{S}_{f,1,\varphi}(\omega) = \frac{[f, \varphi](\omega)}{[\varphi, \varphi](\omega)} \hat{\varphi}(\omega), \ \omega \in \mathbb{R},$$

and if $[\varphi, \varphi](\omega) = 1/2\pi$ for all $\omega$, the integer shifts $\varphi(\cdot - k)$ for $k \in \mathbb{Z}$ are
orthonormal in $L_2(\mathbb{R})$.

Fundamental results on shift–invariant approximation are in [de Boor et al(1994a), de Boor, DeVore, and Ron,
de Boor et al(1994b), de Boor, DeVore, and Ron], and the survey [Jetter and Plonka(2001)]
gives a comprehensive account of the theory and the historical background.
References


