

Solving the 3D Laplace Equation by Meshless Collocation via Harmonic Kernels

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Abstract

This paper solves the Laplace equation $\Delta u = 0$ on domains $\Omega \subset \mathbb{R}^3$ by meshless collocation on scattered points of the boundary $\partial\Omega$. In contrast to the Method of Fundamental Solutions, there are no singularities and no artificial boundaries, since we use new singularity-free positive definite kernels which are harmonic in both arguments. In contrast to many other techniques, e.g. the Boundary Point Method or the Method of Fundamental Solutions, we provide a solid mathematical foundation which includes error bounds and works for general star-shaped domains. The convergence rates depend only on the smoothness of the domain and the boundary data. Some numerical examples are included.

1 Introduction

For solving the Laplace equation $\Delta u = 0$ in a domain Ω for given data $u(\partial\Omega)$ on the boundary $\partial\Omega$ there are various techniques, including

- finite differences,
- finite elements,
- integral equations,
- the Method of Fundamental Solutions [1],
- Boundary Point or Knot Methods [2].

Due to the Maximum Principle, it is tempting to use harmonic trial functions s for constructing a good approximation to $u(\partial\Omega)$ on the boundary $\partial\Omega$, and then apply the error bound

$$\|u - s\|_{\infty, \bar{\Omega}} \leq \|u - s\|_{\infty, \partial\Omega}$$

for the true solution u . This does not require a domain discretization as used for finite element or finite difference methods, and it does

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not require boundary integration like the methods based on integral equations or boundary elements. Any good approximation s of $u(\partial\Omega)$ will suffice, but it is debatable how to set up a useful trial space S of harmonic functions in $\overline{\Omega}$ such that approximation or interpolation of u on the boundary can be carried out reliably and efficiently.

Spaces of singularity-free harmonic trial functions like solid harmonics are useful in practice and have a long history, but there is no satisfying theory of interpolation or approximation on the boundary by those functions. Fundamental Solutions have been used widely [1], but since they have singularities, these must be placed outside the domain on an “artificial” boundary. Convergence analysis, so far, was provided only for simple geometric configurations [4] like concentric circles in \mathbb{R}^2 .

This paper, like its 2D predecessor [8], uses trial spaces of harmonic functions, but makes sure that the functions

- are singularity-free,
- lead to well-posed interpolation problems on the boundary, and
- allow a general mathematical analysis including convergence rates.

This is done by construction of kernels $K : \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}$ which are symmetric and harmonic in both arguments, positive definite and scalable. We focus on the examples

$$P_c(x, y) = (1 - 2x^T y c^{-2} + c^{-4} \|x\|_2^2 \|y\|_2^2)^{-1/2},$$

$$B_c(x, y) = e^{x^T y c^{-2}} J_0 \left(c^{-2} \sqrt{\|x\|_2^2 \|y\|_2^2 - (x^T y)^2} \right)$$

for $x, y \in \mathbb{R}^3$ and a scale $c > 0$, but we shall provide a general recipe to construct other harmonic kernels in \mathbb{R}^3 .

The calculation of a good approximation s to the boundary data $u(\partial\Omega)$ proceeds by interpolation of u in boundary points x_1, \dots, x_n by functions

$$s(x) := \sum_{j=1}^n \alpha_j K(x, x_j), \quad x \in \overline{\Omega}$$

via solving the linear system

$$\sum_{j=1}^n \alpha_j K(x_k, x_j) = u(x_k), \quad 1 \leq k \leq n$$

with the *kernel matrix* $(K(x_k, x_j))_{k,j}$ well-known from Machine Learning. By inspecting the error on the boundary, users can add new points where necessary, and thus get an adaptive technique like [5] that uses update formulae for matrix decompositions. We shall provide numerical examples below.

So far, our theoretical analysis works on domains which are star-shaped with respect to the origin. These can be written in spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}_{\geq 0} \times [0, 2\pi] \times [0, \pi]$ as

$$\overline{\Omega} = \{(r, \theta, \varphi) : 0 \leq r \leq R(\theta, \varphi)\}$$

by a function R describing the boundary. This allows to map $\overline{\Omega}$ to a ball and its boundary $\partial\Omega$ to a sphere \mathbb{S} , and we use this fact to do our analysis there instead on $\partial\Omega$. We get a new kernel L which is positive definite and zonal on \mathbb{S} , and the boundary data $u(\partial\Omega)$ are transformed into data $v(\mathbb{S})$. We do not need harmonicity on the sphere, because we only look at the approximation quality there, but the smoothness of R enters crucially into what follows. We can use the standard literature on kernel-based methods on spheres [7, 6] to derive error bounds on \mathbb{S} for functions with Sobolev smoothness, and this provides convergence rates which are dominated by the smoothness of R and the boundary data, as is to be expected.

The outline of the paper is as follows. To provide the basics for our kernel construction and the convergence analysis on spheres, we start with standard notation for spherical and solid harmonics. We then construct our kernels and give examples for their application. The rest of the paper focuses on the mathematical analysis, proving positive definiteness of the kernels and deriving error bounds for the interpolation of the boundary data.

2 Spherical Harmonics and Kernels

Let $Y_{\ell m}$, $\ell \geq 0$, $-\ell \leq m \leq \ell$ be the real-valued spherical harmonics which are normalized to be orthonormal on the sphere \mathbb{S}^2 . When on the sphere \mathbb{S}^2 , we use both the Cartesian and spherical arguments x and (θ, ϕ) interchangeably. We shall denote L_2 expansions of functions f on the sphere by

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{f}_{\ell m} Y_{\ell m}(\theta, \phi). \quad (1)$$

Sobolev space $H_{\tau}(\mathbb{S}^2)$ of order τ is then defined as the span of all functions of the above form with

$$\|f\|_{\tau}^2 := \sum_{\ell=0}^{\infty} (1 + \ell(\ell + 1))^{\tau} \sum_{m=-\ell}^{\ell} |\hat{f}_{\ell m}|^2 < \infty, \quad (2)$$

and an inner product defined similarly.

If x and y are two Cartesian vectors on the sphere \mathbb{S}^2 with spherical coordinates (θ, ϕ) and (θ', ϕ') , respectively, then

$$P_{\ell}(x^T y) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta, \phi) Y_{\ell m}(\theta', \phi')$$

where P_ℓ is the ℓ -th Legendre polynomial.

Zonal kernels on the sphere \mathbb{S}^2 are kernels of the form

$$\begin{aligned} K(x^T y) &= \sum_{\ell=0}^{\infty} \hat{K}(\ell) \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) Y_{\ell m}(y) \\ &= \sum_{\ell=0}^{\infty} \hat{K}(\ell) \frac{2\ell+1}{4\pi} P_\ell(x^T y) \text{ for all } x, y \in \mathbb{S}^2. \end{aligned} \quad (3)$$

By the usual argument for expansion kernels, the native space for such a kernel is the span of all functions f of the form (1) with

$$\|f\|_K^2 := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{|\hat{f}_{\ell m}|^2}{\hat{K}(\ell)} < \infty, \quad (4)$$

and the inner product is defined analogously. Clearly, the kernel reproducing Sobolev space will be

$$\begin{aligned} K_\tau(x^T y) &= \sum_{\ell=0}^{\infty} (1 + \ell(\ell+1))^{-\tau} \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) Y_{\ell m}(y) \\ &= \sum_{\ell=0}^{\infty} (1 + \ell(\ell+1))^{-\tau} \frac{2\ell+1}{4\pi} P_\ell(x^T y) \text{ for all } x, y \in \mathbb{S}^2 \end{aligned} \quad (5)$$

which does not seem to be available in closed form.

But we do not work on spheres alone, since we want to work with trivariate harmonic functions on balls and more general domains. The easiest case is the transition from the sphere into balls around the origin, using *regular solid harmonics* defined as $r^\ell Y_{\ell m}(\theta, \phi)$ for $\ell \geq 0$. They are orthogonal on the ball $B_R(0)$ of Radius R around the origin, and their normalization there comes via

$$\begin{aligned} &\int_0^R \int_0^{2\pi} \int_0^\pi r^{2\ell} Y_{\ell m}^2(\theta, \phi) r^2 \sin(\theta) d\theta d\phi dr \\ &= \int_0^R r^{2\ell+2} dr = \frac{R^{2\ell+3}}{2\ell+3}. \end{aligned}$$

Here, we used the normalization of the spherical harmonics to

$$1 = \int_0^{2\pi} \int_0^\pi Y_{\ell m}^2(\theta, \phi) \sin(\theta) d\theta d\phi.$$

Now any function of the form (1) on the sphere has a harmonic extension

$$f(x) = \sum_{\ell=0}^{\infty} r^\ell \sum_{m=-\ell}^{\ell} \hat{f}_{\ell m} Y_{\ell m}(\theta, \phi).$$

into the unit ball where now $x \in \mathbb{R}^3$ has to be rewritten in spherical coordinates (r, θ, ϕ) .

3 Harmonic Functions in the Unit Ball

By transition from spherical to solid harmonics it is easy to prove that any approximation of functions on the sphere by spherical harmonics can be extended to a harmonic function in the interior of the sphere having the same boundary values:

Theorem 3.1. *Assume that a function*

$$g(\theta, \phi) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} g_{\ell m} Y_{\ell m}(\theta, \phi)$$

is well-defined by an absolutely convergent series on the sphere S^2 , and that it satisfies

$$\|f - g\|_{S^2, \infty} \leq \epsilon$$

for some function $f \in C(S^2)$. Let u be the solution of the problem $\Delta u = 0$ posed on S^2 with the boundary values of f . Then the function

$$v(r, \theta, \phi) := \sum_{\ell=0}^{\infty} r^{\ell} \sum_{m=-\ell}^{\ell} g_{\ell m} Y_{\ell m}(\theta, \phi)$$

is harmonic in the unit ball B , and the error bound

$$\|u - v\|_{B, \infty} \leq \epsilon$$

holds by the maximum principle. □

Thus, fitting spherical data by spherical harmonics on spheres is a quite useful strategy for getting approximations to harmonic functions in the ball. There is quite some literature on how to find good approximations to data on the sphere by using interpolation by zonal kernels, see e.g. [7, 6].

But there are some drawbacks here.

1. This strategy is bound to the sphere only and does not easily generalize to other closed smooth manifolds in \mathbb{R}^3 .
2. It is easy to take a well-known kernel on \mathbb{R}^3 to solve interpolation problems on S^2 , but in most cases there is no expansion into spherical harmonics that allows to write down the harmonic extension as a simple formula.
3. On the other hand, if one does the interpolation on the sphere by using an expansion into spherical harmonics, one gets an easy harmonic extension, but one has to sum up series all the time.

This is why we want to derive simple formulas that provide *harmonic* kernels. They can be used on every kind of boundary to set up a boundary interpolation problem, and the extension will be immediately available without series expansions.

4 Harmonic Kernels

We go back to zonal kernels (3) and extend them radially to

$$\begin{aligned} K(x, y) &= \sum_{\ell=0}^{\infty} \hat{K}(\ell) r^\ell r'^\ell \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \phi) Y_{\ell m}(\theta', \phi') \\ &= \sum_{\ell=0}^{\infty} \hat{K}(\ell) \frac{2\ell+1}{4\pi} r^\ell r'^\ell P_\ell\left(\frac{x^T y}{rr'}\right) \end{aligned}$$

for all $x, y \in B_1(0)$ written in spherical coordinates (r, θ, ϕ) and (r', θ', ϕ') . To write them in a convenient closed form, any formula of the type

$$f(t, s) = \sum_{\ell=0}^{\infty} \mu_\ell s^\ell P_\ell(t) \quad (6)$$

will do and lead to a kernel

$$K(x, y) = \sum_{\ell=0}^{\infty} \mu_\ell (rr')^\ell P_\ell\left(\frac{x^T y}{rr'}\right) \quad (7)$$

with

$$\hat{K}(\ell) = \frac{4\pi}{2\ell+1} \mu_\ell, \quad \ell \geq 0.$$

Here are two well-known cases

$$\begin{aligned} \frac{1}{\sqrt{1-2st+s^2}} &= \sum_{n=0}^{\infty} P_n(t) s^n \\ e^{st} J_0\left(s\sqrt{1-t^2}\right) &= \sum_{n=0}^{\infty} \frac{P_n(t)}{n!} s^n \end{aligned}$$

of (6), but there may be others. After introduction of an additional scaling constant $c > 0$ into (7) to yield

$$K_c(x, y) = \sum_{\ell=0}^{\infty} \frac{\mu_\ell}{c^{2\ell}} (\|x\|_2 \|y\|_2)^\ell P_\ell\left(\frac{x^T y}{\|x\|_2 \|y\|_2}\right) \quad (8)$$

they lead to kernels

$$\begin{aligned} P_c(x, y) &= (1 - 2x^T y c^{-2} + c^{-4} \|x\|_2^2 \|y\|_2^2)^{-1/2} \\ B_c(x, y) &= e^{x^T y c^{-2}} J_0\left(c^{-2} \sqrt{\|x\|_2^2 \|y\|_2^2 - (x^T y)^2}\right) \end{aligned}$$

we shall call the *harmonic Poisson and Bessel kernel*, respectively. On the unit sphere, they are zonal and can be plotted as functions of $t = x^T y$ on $[-1, +1]$. In the notation (8) they have the coefficients $\mu_\ell = 1$ and $\mu_\ell = 1/\ell!$, respectively. Note that due to the degree ℓ of P_ℓ , the denominators in the P_ℓ argument will cancel against the factor coming with it.

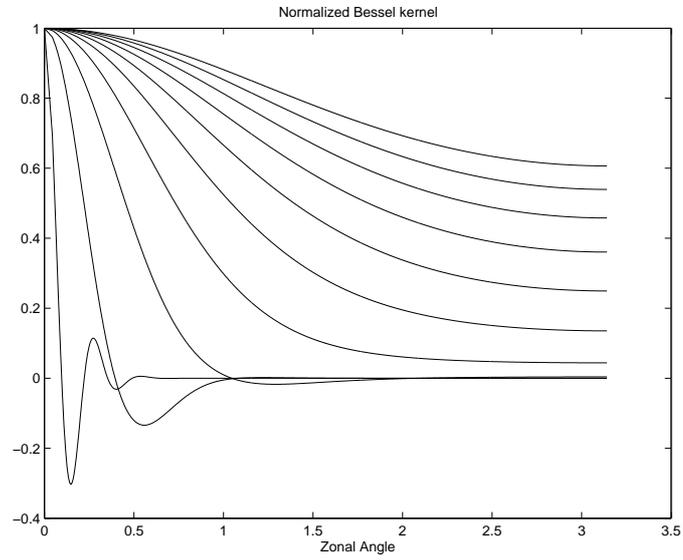


Figure 1: The Bessel kernels

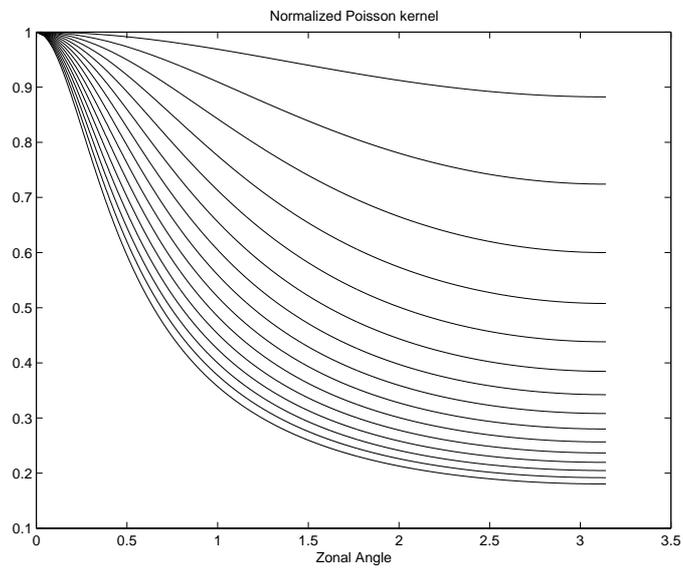


Figure 2: The Poisson kernels

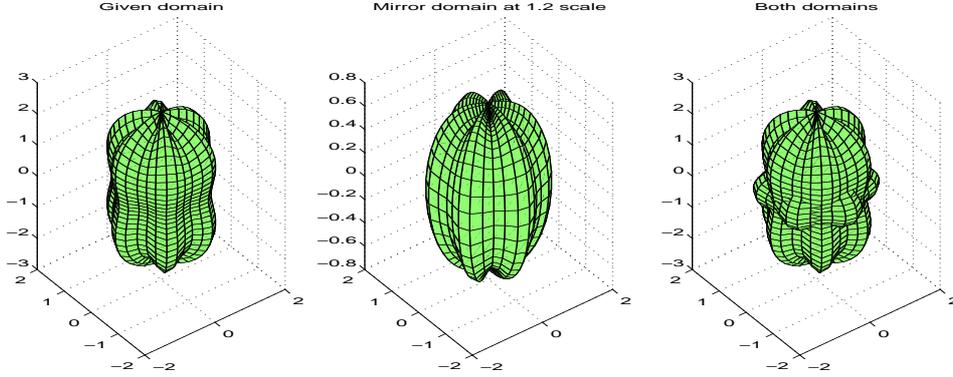


Figure 3: Domain and mirror domain intersecting

5 Mirror Points for Poisson Kernels

Note that the Poisson kernel has a singularity. In fact, the expression under the square root can be written as

$$\begin{aligned} & 1 - 2x^T y c^{-2} + c^{-4} \|x\|_2^2 \|y\|_2^2 \\ &= (1 - c^{-2} \|x\|_2 \|y\|_2)^2 + 2c^{-2} (\|x\|_2 \|y\|_2 - x^T y) \\ &\geq 0 \end{aligned}$$

and the two terms can vanish simultaneously if and only if x and y are proportional and satisfy $1 = c^{-2} \|x\|_2 \|y\|_2$. Thus for fixed $x \neq 0$, the Poisson kernel at scale has an isolated singularity at the “mirror point”

$$y(x, c) := \frac{c^2}{\|x\|_2} \frac{x}{\|x\|_2}$$

arising in constructions of Green’s functions. For $c = 1$ and x inside the unit ball, it will be outside. For practical work with translates $P_c(\cdot, x_j)$ of the Poisson kernel for points x_j on the unit sphere, we have to take a scaling factor $c > 1$ to move the mirror points safely away. When placing x on the surface of a general domain with zero as interior point, the mirror points $y(x, c)$ will lie on another “mirror” surface that should not intersect the original domain, but this has to be checked in practice. See Figure 3 for an example. It uses the domain obtained by radial transformation of the unit sphere into

$$r(\theta, \phi) = (2 - \cos^2(\phi)) \cdot (1 + 0.2 \cos^2(3\theta))$$

and the scale 1.2 for mirroring. Much larger or much smaller scales will lead to mirror domains completely outside or completely inside the given domain. The same phenomenon already occurs for the 2D Poisson kernel, see [8]. But keep in mind that the placement of mirror domains is irrelevant for the Bessel kernel.

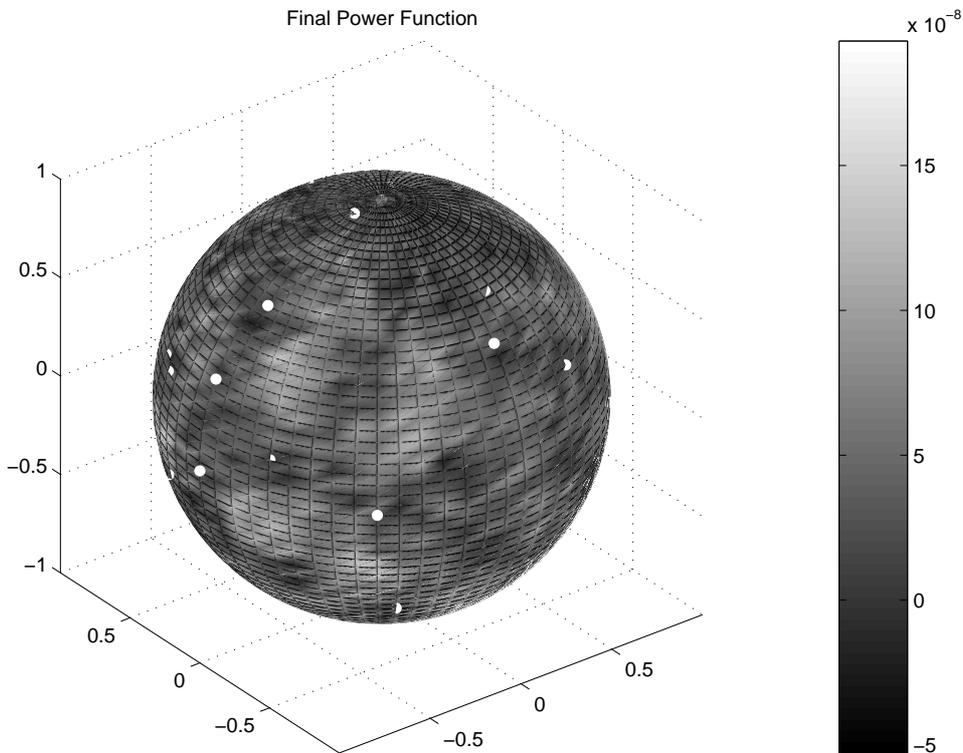


Figure 4: Power Function

6 Examples

Our first example works for the unit ball by finding good interpolation points on the sphere. We use the greedy method of [3] to find the points. After having found n points, it evaluates the *power function* giving the pointwise norm of the error functional for interpolation in these points, and then it adds the point where the power function is maximal, i.e. it adds the point where the worst-case error is maximal. Figure 4 shows the resulting power function in grayscale coding. The maximum value of the power function is $1.9\text{e-}7$ for 55 points given by white crosses on the sphere. This means that all functions from the unit ball of the native space will have an interpolation error of at most $1.9\text{e-}7$. The harmonic kernel was of Bessel type with scale 3.

A second example uses an UFO-like domain (see Figure 5) parametrized by

$$r = (2 + 2 \cos^2(2\phi)) \cdot (1 + 0.2 \cos^2(3\theta) \cos^2(\phi))$$

on the sphere and smooth non-harmonic data from $f(x, y, z) = x^2 - 2y^2$ on its boundary, normalized to be of maximal absolute value 1. Again, a greedy method was used to adaptively find interpolation points, this time by adding the point where the L_∞ error takes its maximum absolute value. The domain and the grayscale-coded data function are

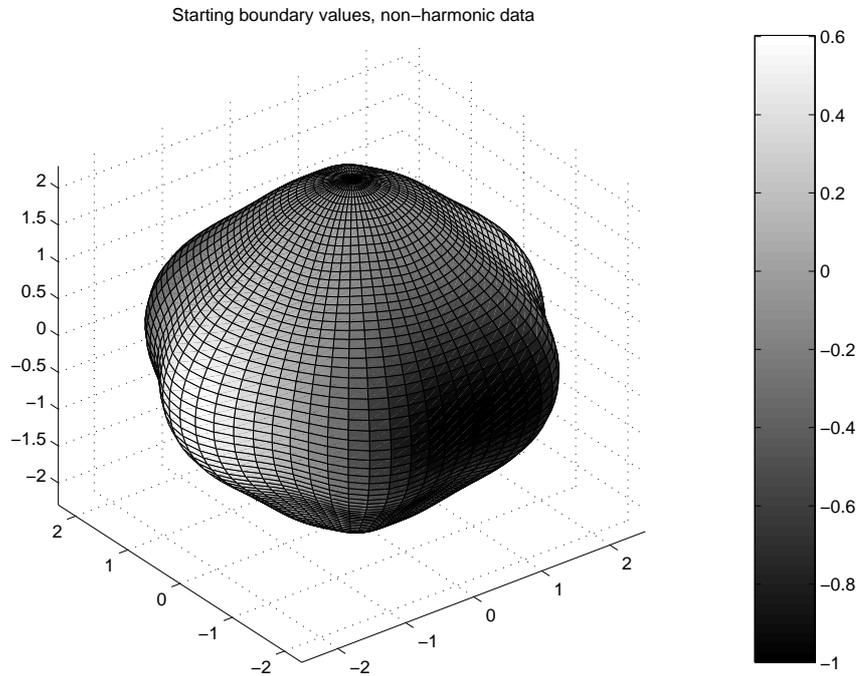


Figure 5: Domain and Boundary Values, non-harmonic data

given in Figure 5. The overall boundary error using the Bessel kernel at scale 5 was 0.0199 using 68 selected points (see Figure 6).

The numerical behavior is much better when the boundary data are from harmonic functions. If we use the boundary data of $f(x, y, z) = x^2 - 2y^2 + z^2$ (normalized to maximal absolute value 1), we start from Figure 7 and end with Figure 8. The final error is 5.e-10 for 68 points used.

7 Native Hilbert Spaces

The Poisson and Bessel kernels are reproducing in their “native” Hilbert spaces of harmonic functions. But these spaces will depend on where we work. To this end, we fix a bounded domain Ω in \mathbb{R}^3 having the origin as an interior point, and we assume its boundary Γ to be a smooth compact manifold. If K is one of the harmonic kernels, the native space for it will be the Hilbert space closure of

$$K_0 := \text{span} \{K(\cdot, x) : x \in \Gamma\}$$

under the inner product

$$(K(\cdot, x), K(\cdot, y))_K := K(x, y) \text{ for all } x, y \in \Gamma.$$

This is a standard argument in kernel-based methods, but it requires

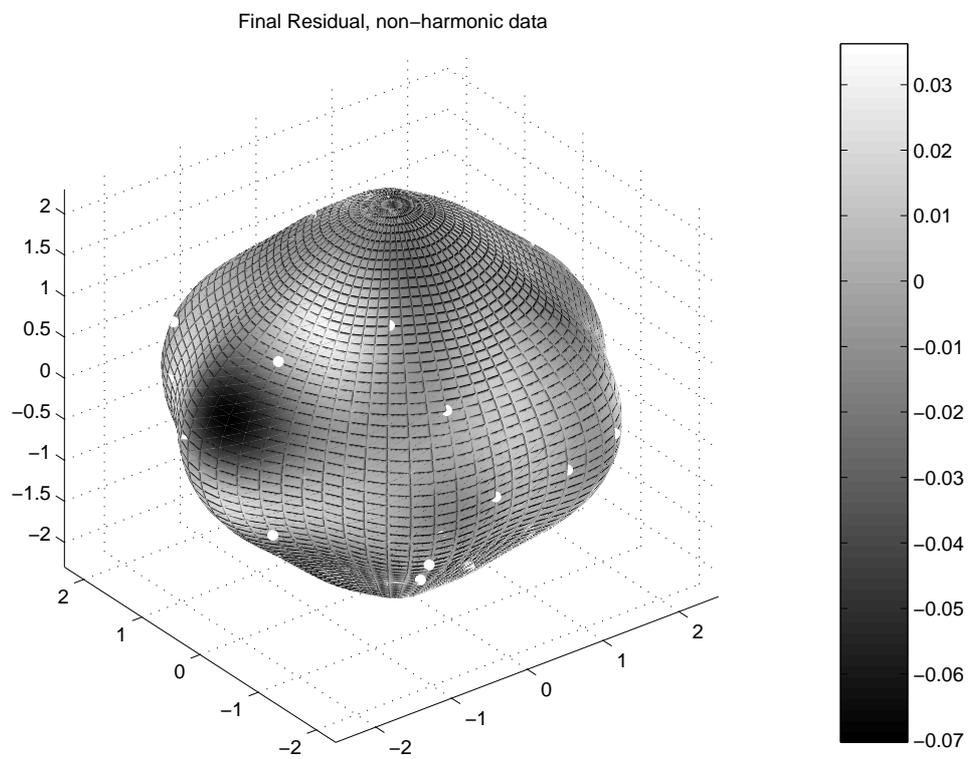


Figure 6: Bessel Solution at Scale 5, non-harmonic data

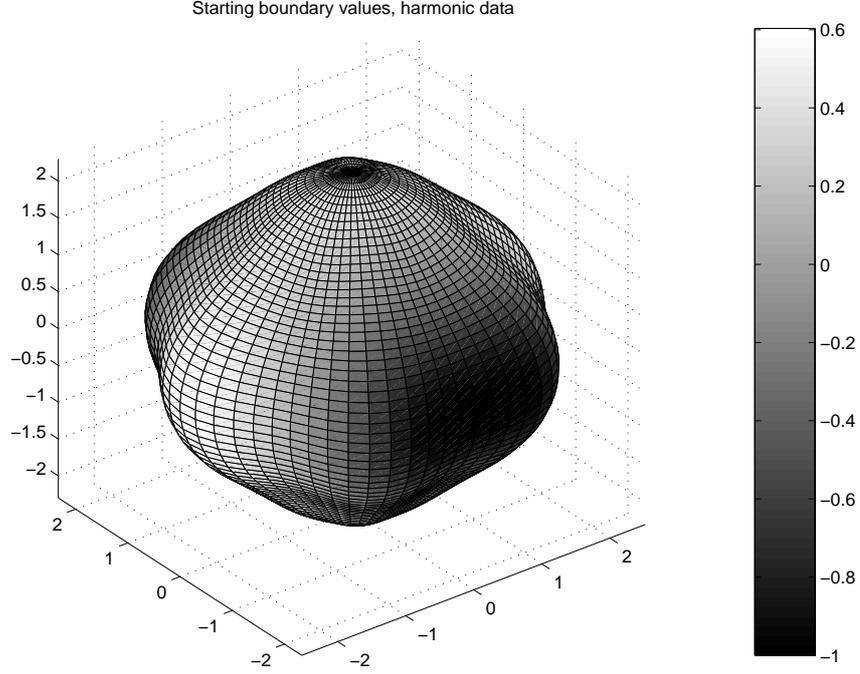


Figure 7: Domain and Boundary Values, harmonic data

Theorem 7.1. *The harmonic kernels of Bessel or Poisson type are positive definite on each subset in \mathbb{R}^3 on which they are defined.*

Proof. We rewrite (8) in terms of spherical coordinates as

$$K_c(x, y) = \sum_{\ell=0}^{\infty} \frac{\mu_{\ell}}{c^{2\ell}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} r^{\ell} Y_{\ell m}(\theta, \phi) r'^{\ell} Y_{\ell m}(\theta', \phi')$$

and use points x_0, \dots, x_n with similarly defined spherical coordinates. Then we have to consider the quadratic form

$$\begin{aligned} & \sum_{j,k=0}^n a_j a_k K_c(x_j, x_k) \\ &= \sum_{\ell=0}^{\infty} \frac{\mu_{\ell}}{c^{2\ell}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \sum_{j,k=0}^n a_j a_k r_j^{\ell} Y_{\ell m}(\theta_j, \phi_j) r_k^{\ell} Y_{\ell m}(\theta_k, \phi_k) \\ &= \sum_{\ell=0}^{\infty} \frac{\mu_{\ell}}{c^{2\ell}} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \left(\sum_{j=0}^n a_j r_j^{\ell} Y_{\ell m}(\theta_j, \phi_j) \right)^2 \end{aligned}$$

to see that it is nonnegative. Thus K_c is positive semidefinite.

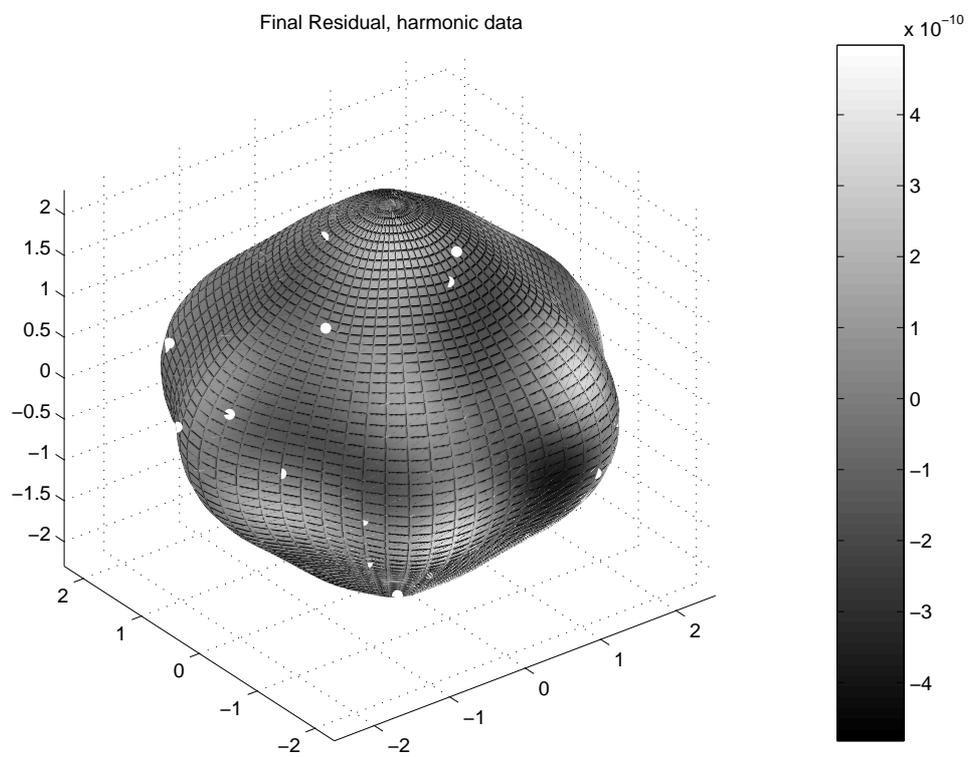


Figure 8: Bessel Solution at Scale 5, harmonic data

Positive definiteness needs that every functional

$$f \mapsto \sum_{j=0}^n a_j f(x_j)$$

which vanishes for all solid harmonics $f_{\ell m}(x) = r^\ell Y_{\ell m}(\theta, \phi)$ must be zero. Since the spherical harmonics are dense in L_2 and in C on \mathbb{S}^2 , the assertion is correct if all points lie on \mathbb{S}^2 . If the points are in general position such that their radial projections to \mathbb{S}^2 are different, the radii can be merged with the coefficients and the assertion follows as well. For the general case, we sort the points into J groups with the same radial projection onto \mathbb{S}^2 . This means to have J different locations (θ_j, ϕ_j) on the sphere and for each j , $1 \leq j \leq J$ some points x_{j1}, \dots, x_{jn_j} with norms r_{j1}, \dots, r_{jn_j} that are different. Then we assume that all quantities

$$\sum_{j=1}^J Y_{\ell m}(\theta_j, \phi_j) \sum_{k=1}^{n_j} a_{jk} r_{jk}^\ell, \quad \ell \geq 0, \quad -\ell \leq m \leq \ell$$

vanish. By the argument on \mathbb{S}^2 , all sums

$$\sum_{k=1}^{n_j} a_{jk} r_{jk}^\ell, \quad \ell \geq 0, \quad 1 \leq j \leq J$$

must be zero, and usual arguments concerning univariate polynomial interpolation and Vandermonde matrices prove the rest. \square

But this makes it not easier to characterize the native space of the kernels unless Γ is a sphere itself. However it turns out that this is no serious problem, since we have to go back to spheres for parametrization of general closed smooth surfaces anyway.

8 Error Bounds

We assume a domain Ω in \mathbb{R}^3 to be given such that

1. it is star-shaped with respect to the origin, and
2. its boundary surface Γ can be parametrized smoothly from spherical coordinates on \mathbb{S}^2 .

We shall pose our boundary interpolation problem on Γ , but we shall analyze it by going back to \mathbb{S}^2 . More specifically, we assume

$$F : \mathbb{S}^2 \rightarrow \Gamma$$

to be a smooth and regular bijection. We assume interpolation points y_1, \dots, y_n on Γ be given, and we shall also use the points $x_j = F^{-1}(y_j)$ on the sphere. The boundary data are prescribed by a function $f :$

$\Gamma \rightarrow \mathbb{R}$, but we shall analyze $g : \mathbb{S}^2 \rightarrow \mathbb{R}$ with $f(F(x)) = g(x)$ for $x \in \mathbb{S}^2$. Using a kernel K on Γ is the same as using a kernel

$$L(x, y) := K(F(x), F(y)) \text{ for all } x, y \in \mathbb{S}^2.$$

Instead of analyzing the error of interpolation to f on Γ by translates of K , we can analyze the error of interpolation of g on \mathbb{S}^2 by translates of L .

For this analysis, we can rely on well-established convergence results on spheres. Following [7, 6] we define Sobolev spaces H_τ on the sphere by (2) for orthonormal expansions (1). If considered on \mathbb{S}^2 they contain continuous functions and allow continuous point evaluations if $\tau > 3/2$. The kernel K_τ of (5) reproduces H_τ .

To proceed towards a citation of Proposition 2.1 of [7], we consider a Sobolev space H_τ with $\tau > 3/2$ and work with functions g from H_τ and kernels L on \mathbb{S}^2 whose native space is contained in H_τ . Interpolation of data of g is done in scattered point sets $X \subset \mathbb{S}^2$ with *fill distance*

$$h_X := \sup_{y \in \mathbb{S}^2} \min_{x \in X} d(x, y)$$

where d is the geodesic distance on \mathbb{S}^2 .

Theorem 8.1. [7]

There is a constant C , dependent on $\tau > 3/2$ and L only, such that for all sets X of scattered points on the sphere \mathbb{S}^2 and all functions $g \in H_\tau$ the interpolant $I_{X,L,g}$ to g on X using X -translates of L satisfies

$$\|g - I_{X,L,g}\|_{\infty, \mathbb{S}^2} \leq Ch_X^{\tau-3/2} \|g\|_\tau.$$

We now shift this to general smooth closed bounded manifolds Γ in the way we described above. We pick one of our harmonic kernels K with a fixed scale and such that the mirror boundary stays outside of our general domain Ω at a minimum positive distance. Then we assume that the boundary map $F : \mathbb{S}^2 \rightarrow \Gamma$ is smooth enough to guarantee that the kernel $L(x, y) = K(F(x), F(y))$ is smooth enough on \mathbb{S}^2 to have a native space contained in H_τ .

Theorem 8.2. *Then there is a constant C , dependent on $\tau > 3/2$, K , F , and Ω , such that for all sets $Y = F(X)$ of scattered points on Γ as images of sets X on the sphere \mathbb{S}^2 and all functions f on Γ such that $g := f \circ F$ is in H_τ , the interpolant $I_{Y,K,f}$ to f on Γ using Y -translates of K satisfies*

$$\|f - I_{Y,K,f}\|_{\infty, \Gamma} = \|g - I_{X,L,g}\|_{\infty, \mathbb{S}^2} \leq Ch_X^{\tau-3/2} \|g\|_\tau$$

holds on the boundary, By the maximum principle, this error bound extends to Ω for the harmonic function u having boundary values of f , namely

$$\|u - I_{Y,K,f}\|_{\infty, \Omega} \leq \|f - I_{Y,K,f}\|_{\infty, \Gamma} \leq Ch_X^{\tau-3/2} \|g\|_\tau.$$

Note how the smoothness of the boundary map F and the boundary function f are combined into the smoothness of $f \circ F$ determining the approximation order.

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