On the Versatility of Meshless Kernel Methods

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Hypotheses

- Linear mixed problem

$$L_i u = f_i \text{ in } \Omega_i \subset I\!R^d,\ i = 1, 2, \ldots,$$

- Existence and uniqueness of solution u in RKHS $\mathcal{U}$ with kernel Φ
- $L_i u$ and $L_i^x L_i^y \Phi(x, y)$ continuous on Ω_i

Result

Then the minimum norm solution by meshless symmetric collocation on dense subsets of Ω_i converges towards u.

Linear Problems

Given Problem

$$L_i u = f_i \text{ in } \Omega_i \subset I\!R^d,\ i = 1, 2, \ldots,$$

Discretize:

$$\begin{aligned} ((L_i u)(x_{ji}) &= f_i(x_{ji}) \quad x_{ji} \in \Omega_i \quad \text{for all } j \in I_i \\ (\delta_{x_{ji}} \circ L_i)(u) &= \delta_{x_{ji}} f_i \quad x_{ji} \in \Omega_i \quad \text{for all } j \in I_i \end{aligned}$$

Generally:

$$\lambda(u) = f_\lambda \in I\!R \text{ for all } \lambda \in \Lambda$$

Generalized Interpolation

Given $\Lambda \subset \mathcal{U}^*,\ f\ :\ \Lambda \to I\!R$
Find $u \in \mathcal{U}$ with

$$\lambda(u) = f(\lambda) \in I\!R \text{ for all } \lambda \in \Lambda$$

Minimum Norm Reconstruction

Given $\Lambda \subset \mathcal{U}^*,\ f\ :\ \Lambda \to I\!R$

Assume exact solution $u \in \mathcal{U}$ with

$$\lambda(u) = f(\lambda) \in I\!R \text{ for all } \lambda \in \Lambda$$

Reconstruction $\tilde{u} \in \mathcal{U}$ with

$$\lambda(\tilde{u}) = f(\lambda) \in I\!R \text{ for all } \lambda \in \Lambda$$

with minimal norm $\|\tilde{u}\|_{\mathcal{U}}$

Simplification: Hilbert Space

Closed subspaces

$$\mathcal{U}_\Lambda^\perp := \{u \in \mathcal{U} : \lambda(u) = 0 \text{ for all } \lambda \in \Lambda\}$$
$$\mathcal{U}_\Lambda := \left(\mathcal{U}_\Lambda^\perp\right)^\perp$$

Decompose solution $u = u_\Lambda + u_\Lambda^\perp \in \mathcal{U}_\Lambda + \mathcal{U}_\Lambda^\perp$

- Minimal norm solution is $\tilde{u} = u_\Lambda$
- Unique reconstruction iff $\mathcal{U}_\Lambda^\perp = \{0\}$

Problem: Construct u or $\tilde{u} = u_\Lambda$

Reproducing Kernel Hilbert Spaces

Space $\mathcal{U}$ of functions

Dual space $\mathcal{U}^*$ of functionals

Reproduction Property

$$\lambda(u) = (u, \lambda^x \Phi(x, \cdot))_{\mathcal{U}} \text{ for all } u \in \mathcal{U}, \quad \text{for all } \lambda \in \mathcal{U}^*$$

Minimum Norm Solution

Given $\Lambda_N = \{\lambda_1, \ldots, \lambda_N\} \subset \Lambda \subset \mathcal{U}^*,\ f\ :\ \Lambda \to I\!R$
Find $\tilde{u}_N \in \mathcal{U}$ of minimum norm $\|\tilde{u}_N\|_{\mathcal{U}}$ with

$$\lambda_j(\tilde{u}_N) = f(\lambda_j) \in I\!R \text{ for all } \lambda_j \in \Lambda_N$$

Solution:

- $\tilde{u}_N = \sum_k c_k \lambda_k^x \Phi(x, \cdot)$
- $\lambda_j(\tilde{u}_N) = \sum_k c_k \lambda_k^x \lambda_j^y \Phi(x, y) = f(\lambda_j)$

Used incrementally for Greedy Techniques

Problem: Usually $|\Lambda| = \infty$

Convergence

Given $\Lambda_N = \{\lambda_1, \ldots, \lambda_N\} \subset \mathcal{U}^*$ for $N \to \infty$

Sequence of minimum norm solutions $\tilde{u}_N \in \mathcal{U}$ is a Cauchy sequence, thus convergent.
Reason: $\|\tilde{u}_N\|^2_{\mathcal{U}} \nearrow \|\tilde{u}\|^2_{\mathcal{U}} \leq \|u\|^2_{\mathcal{U}}$ is a Cauchy sequence.
Construction works for $|\Lambda| = |\mathbb{N}|$.

Problem: Usually $|\Lambda| > |\mathbb{N}|$.

Density

Given $\Lambda_{\mathbb{N}} \subset \Lambda \subset \mathcal{U}^*, \ |\Lambda_{\mathbb{N}}| = |\mathbb{N}|$.
Construct minimal norm solution $\tilde{u}_{\mathbb{N}}$ for $\Lambda_{\mathbb{N}}$.
Problem: $\tilde{u}_{\mathbb{N}} = \tilde{u}$?

$$\lambda(\tilde{u}_{\mathbb{N}} - \tilde{u}) = 0 \text{ for all } \lambda \in \Lambda_{\mathbb{N}}$$

Assumption: $\Lambda_{\mathbb{N}}$ dense in Λ, i.e.

$$\begin{aligned} \lambda_j(v) &= 0 \quad \text{for all } \lambda_j \in \Lambda_{\mathbb{N}} \\ \Rightarrow \quad \lambda(v) &= 0 \quad \text{for all } \lambda \in \Lambda \end{aligned}$$

Density and Continuity

Example:

$$(L_i(\tilde{u}_{\mathbb{N}} - \tilde{u}))(x_{ji}) = 0 \text{ for all } x_{ji} \in \Omega_i$$

Goal:

$$L_i(\tilde{u}_{\mathbb{N}}) = L_i(\tilde{u}) \text{ on } \Omega_i$$

if both continuous and $x_{ji} \in \Omega_i$ dense.

Final Result: Hypotheses

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