

Mathematical Results Concerning Kernel Techniques

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Overview

Overview

Black-Box systems

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Black-Box systems
Learning Machines

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- Identification of Black-Box systems
Learning Machines

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- Identification of Black-Box systems is Training of Learning Machines

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Kernel Techniques

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If Kernel Techniques are used:

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- Kernel selection allows optimal Model selection

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Connection to Support Vector Machines



Systems



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Input \mapsto *Output*

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$$\begin{array}{ccc} \textit{Input} & \mapsto & \textit{Output} \\ \mathbb{R}^M & \xrightarrow{F} & \mathbb{R}^N \end{array}$$

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Reproduction $y_j = F(x_j)$

Black-Box-Models

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Input \mapsto *Output*

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- Learning:** same



Learning



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Known Training samples $(x_j, y_j) \in \mathbb{R}^M \times \mathbb{R}^N$

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Consequence

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- Identification in black-box systems is Training in Learning

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Both problems:

Consequence

- Identification in black-box systems is Training in Learning

Both problems:

- Reconstruction of an unknown multivariate function from scattered data



Feature Map



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$$\mathbb{R}^M \xrightarrow{\Phi} \mathcal{F}$$

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Input/Stimulus \mapsto features



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K : Kernel $\mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}$

Kernel Trick

$\mathbb{R}^M \xrightarrow{\Phi} \mathcal{F}$: Reproducing Kernel Hilbert Space

$\mathcal{F} \subset \{v : \mathbb{R}^M \rightarrow \mathbb{R}\}$ (scalar case)

K : Kernel $\mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}$

$$\Phi(x) = K(x, \cdot)$$

$$(\Phi(x), \Phi(y))_{\mathcal{F}} = (K(x, \cdot), K(y, \cdot))_{\mathcal{F}}$$

$$= K(x, y)$$

$$(\Phi(x), v)_{\mathcal{F}} = (K(x, \cdot), v)_{\mathcal{F}}$$

$$= v(x) \text{ (Reproduction)}$$

Optimal Models from Kernels

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is solved by models of the form

$$F_{\alpha}(x) = \sum_j \alpha_j K(x_j, x)$$

with coefficients $\alpha_j \in \mathbb{R}^N$.

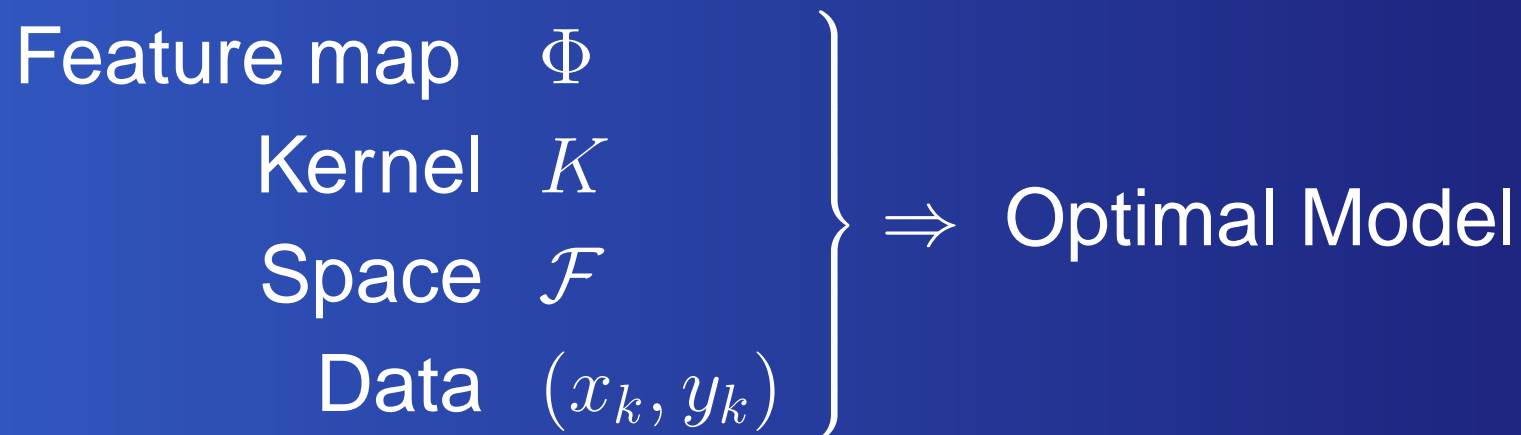
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- Features (determine Φ and K)

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New discipline: Kernel Engineering

Identification or Training

Identification or Training

Model $F_{\alpha}(x) = \sum_j \alpha_j K(x_j, x)$

Identification or Training

$$\begin{array}{l} \text{Model} \quad F_{\alpha}(x) = \sum_j \alpha_j K(x_j, x) \\ \text{Data} \quad \quad y_k = F_{\alpha}(x_k) \end{array}$$

Identification or Training

Model $F_{\alpha}(x) = \sum_j \alpha_j K(x_j, x)$

Data $y_k = F_{\alpha}(x_k)$

System $y_k = \sum_j \alpha_j K(x_j, x_k)$

Identification or Training

$$\text{Model} \quad F_{\alpha}(x) = \sum_j \alpha_j K(x_j, x)$$

$$\text{Data} \quad y_k = F_{\alpha}(x_k)$$

$$\text{System} \quad y_k = \sum_j \alpha_j K(x_j, x_k)$$

Large positive definite symmetric system

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Given $\epsilon \geq 0$.

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Support points (vectors) :

$$(x_k, y_k) \text{ with } |F(x_k) - y_k| = \epsilon$$

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In the Learning Machines context:

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Consequence of Optimization Theory

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No Statistics

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<http://www.num.math.uni-goettingen.de/schaback>