The Path Player Game

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**Motivation**

- In routing games: Flow consists of players that choose a path → minimization of cost
- Different point of view: position of path owner, player suggests amount of flow → maximization of income
- Application: Situation where suppliers of infrastructure share a network, e.g. public transport, energy networks, information networks
Notations

- Directed Network $G = (V, E)$
- Single source $s$ and single sink $t$
- $\mathcal{P}$: set of all paths $P$ from $s$ to $t$
- Cost function $c_e(\cdot)$, depends on load on $e$
Notations

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- $\mathcal{P}$: set of all paths $P$ from $s$ to $t$.
- Cost function $c_e(\cdot)$, depends on load on $e$
- Flow: $f : \mathcal{P} \rightarrow \mathbb{R}^+$
- Flow on an edge: $f_e = \sum_{P : e \in P} f_P$
- Cost on path: $c_P = \sum_{e \in P} c_e(f_e)$
Components of the path player game

- Players $\leftrightarrow$ paths $P$

Compare: Components of routing games

- Players $\leftrightarrow$ (atomic/nonatomic) fractions of flow $f$
Components of the path player game

- Players $\Leftrightarrow$ paths $P$
- Strategies $\Leftrightarrow$ $P$ chooses flow $f_P$

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- Players $\Leftrightarrow$ (atomic/nonatomic) fractions of flow $f$
- Strategies $\Leftrightarrow$ choose a path $P$
Components of the path player game

- Players ⇔ paths $P$
- Strategies ⇔ $P$ chooses flow $f_P$
- Strategy sets ⇔ $\mathbb{R}^+$

Compare: Components of routing games

- Players ⇔ (atomic/nonatomic) fractions of flow $f$
- Strategies ⇔ choose a path $P$
- Strategy sets ⇔ $\mathcal{P}$
Components of the path player game

- Players $\leftrightarrow$ paths $P$
- Strategies $\leftrightarrow$ $P$ chooses flow $f_P$
- Strategy sets $\leftrightarrow \mathbb{R}^+$
- Payoff for $P$ $\leftrightarrow$ benefit function depending on $f$

Compare: Components of routing games

- Players $\leftrightarrow$ (atomic/nonatomic) fractions of flow $f$
- Strategies $\leftrightarrow$ choose a path $P$
- Strategy sets $\leftrightarrow \mathcal{P}$
- Payoff for $P$ $\leftrightarrow$ latency/cost function depending on $f$
General Benefit Function

- $r$: flow rate
- $-M$: punishment for infeasible flow
- $\omega_P$: security limit
- $\kappa_P$: security payment

Definition (Benefit function of player $P$)

$$b_P(f) = \begin{cases} 
  c_P(f) & \text{if} & \sum_{P \in \mathcal{P}} f_P \leq r \land f_P \geq \omega_P \\
  \kappa_P & \text{if} & \sum_{P \in \mathcal{P}} f_P \leq r \land f_P < \omega_P \\
  -M & \text{if} & \sum_{P \in \mathcal{P}} f_P > r 
\end{cases}$$
One-dimensional benefit function

To analyze behavior of benefit $b_P(f)$ of player $P$, fix strategies of remaining players: $f_{\neg P}$

One-dimensional benefit function:

$$\tilde{b}_P(f_P) = b_P(f_{\neg P}, f_P)$$
Best reaction set for $P$ and given $f_{-P}$

$$f_{P}^{\text{max}} = \{ f_P \geq 0 : f_P \text{ maximizes } \tilde{b}_P(f_P) \}.$$ 

Lemma

The best reaction sets $f_{P}^{\text{max}}$ are nonempty for all $P \in \mathcal{P}$ if the cost functions $c_e(f_e)$ are continuous for all $e \in E$.

- Important for existence of equilibria
- For non-continuous costs $f_{P}^{\text{max}}$ may be empty
Definition (Nash equilibrium in path player game)
A flow \( f^* \) is a Nash equilibrium if and only if \( \forall P \in \mathcal{P} \) and \( \forall f_P \geq 0 \) holds that

\[
b_P(f^*_P, f^*_P) \geq b_P(f^*_P, f_P)
\]

Corollary
A flow \( f^* \) is a Nash equilibrium if and only if for all \( P \in \mathcal{P} \) holds that

\[
f_P^* \in f_P^{max}
\]
Existence of Equilibria for infinite games [Debreu 1952, Glicksberg 1952]:

- nonempty, compact, convex strategy sets; continuous and quasi-concave payoffs $\rightarrow$ equilibrium in pure strategies
- nonempty, compact strategy sets; continuous payoffs $\rightarrow$ equilibrium in mixed strategies

Path Player Game:
nonempty, compact, convex strategy set and non-continuous payoff $\rightarrow$ equilibrium in pure strategies
$\rightarrow$ use: best response exists for continuous $c_e(\cdot)$
Theorem (Existence of feasible Nash equilibria)

In a path player game with continuous cost functions $c_e(f_e)$, a feasible Nash equilibrium in pure strategies exists.

Proof:

- $\mathbb{F}$: set of feasible strategies $f$, closed, bounded and convex
- construct continuous mapping $T : \mathbb{F} \rightarrow \mathbb{F}$
- Kakutani’s fixed point theorem: $\Rightarrow$ Fixed point of $T$ exists
- Each fixed point of $T$ is a Nash equilibrium in pure strategies
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\[
\begin{align*}
    f'_P &= f_P + \left\{ \begin{array}{ll}
        \min \left\{ f^m_P - f_P; \sum_{P_k \in \mathcal{P}: \mathcal{P}_k < f^m_k} \left( f^m_k - f_k \right) \cdot d \right\} & \text{if } f_P < f^m_P \\
        f^m_P - f_P & \text{if } f_P \geq f^m_P
    \end{array} \right.
\end{align*}
\]
Strictly increasing costs

Strictly increasing cost functions and $\omega_P = 0$ (no-security-limit)

**Lemma**

*Strictly increasing cost functions $c_e(f_e)$, security limit $\omega_P = 0$: A flow $f$ is a feasible Nash equilibrium if and only if*

$$\sum_{P \in \mathcal{P}} f_P = r$$
Strictly increasing cost functions and general $\omega_P \geq 0$

**Lemma**

Strictly increasing cost functions $c_e(f_e)$: If a flow $f$ is a feasible Nash equilibrium then at least one of the following cases holds:

(i) $\sum_{P \in \mathcal{P}} f_P = r$

(ii) $f_P < \omega_P \ \forall \ P \in \mathcal{P}$. 
Question:
When does necessary and sufficient condition:

\[ \text{A flow } f \text{ is a feasible Nash equilibrium } \iff \sum_{P \in \mathcal{P}} f_P = r \]

also hold for problems with general security limit?
Non-compensative-security (NCS) property

Definition (Game with NCS property)
\[ \forall P \in \mathcal{P}, \forall f_P \text{ with } d_P \geq \omega_P \text{ it holds that} \]
\[ \exists f_P \geq \omega_P \text{ such that } \tilde{b}_P(f_P) > \kappa_P . \]

A player always prefers to choose \( f_P \geq \omega_P \) if possible.
Non-compensative-security (NCS) property

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A player always prefers to choose \( f_P \geq \omega_P \) if possible.

**Lemma**

*Strictly increasing cost functions* \( c_e(f_e) \), *security limit* \( \omega_P \geq 0 \) and *NCS game*:

A flow \( f \) is a feasible Nash equilibrium if and only if
\[ \sum_{P \in \mathcal{P}} f_P = r \]
How to identify games with NCS property?

\[ \kappa_P < c_P(0, \ldots, 0, \omega_P, 0, \ldots, 0) =: c_P(\vec{0}_{|P| - 1}, \omega_P) \quad (*) \]

If competitors are routing nothing \( \rightarrow P \) choose \( f_P \geq \omega_P \).
How to identify games with NCS property?

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**Lemma**

A game where each benefit functions has property (*) is a game with NCS property, if

- the graph is path-disjoint
  - or
- the costs are monotonically increasing
  - or
- each path contains exclusive edge that mimics property (*)
Further Results for Special Benefit Functions

- Differentiable costs $\rightarrow$ necessary condition
- Differentiable and concave costs $\rightarrow$ necessary and sufficient condition
- Convex costs $\rightarrow$ dominating strategy set
Future work

- Multiple equilibria, dominance structures
- Optimal flow for a PPG system
- Potential function for the PPG
- Edge-dependent security limits
- Application: Traffic Optimization: Line Planning, including delay minimization
Routing games Flow consists of independent players that choose a path through a network
- Roughgarden, Tardos: How bad is selfish routing?, 2002
- Koutsoupias, Papadimitriou: Worst-case equilibria, 1999
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Bandwidth allocation Capacitated links are used by several players
Player bid for bandwidth, central manager assigns bandwidth and price
- Kelly: Charging and rate control for elastic traffic, 1997
- Johari, Tsitsiklis: Efficiency loss in a network resource allocation game, 2004
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Path auctions Edges are owned by the players, central manager has to buy a path through the network
- Sahai, Elkind, Steiglitz: Frugality in path auctions, 2004
- Archer, Tardos: Frugal path mechanisms, 2002