Inversion of Reynolds stresses in the solar interior

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Abstract

Local helioseismology aims at recovering the motions (flow velocities) in the solar interior from observations of solar oscillations on the surface of the Sun. In time-distance helioseismology, the basic input data are travel times of waves between pairs of points on the surface. These quantities are linked to the internal properties of the Sun via an integral operator. In previous publications the reconstruction of flow velocities from travel times has been studied by solving an inverse problem. The aim of this paper is to recover directly the Reynolds stresses instead of first recovering the velocities then computing the correlations. This paper is a first attempt in this direction and all the necessary ingredients to perform directly the inversion are presented.

Introduction

Time-distance helioseismology [1] aims at recovering subsurface structure and dynamics of the Sun by the measurement and analysis of travel-times for wave packets moving between two points on the solar surface. Travel-times are obtained thanks to high-resolution Doppler images of the Sun surface given by space and ground-based networks. Once these quantities are known, a forward model has to be derived to link them to internal properties of the Sun. In the upper layers of the Sun, the convective motions are described by a flow field with velocity $\mathbf{v}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, which we would like to image using helioseismology. The relation between the travel time τ^a (between two surface points $\mathbf{r}_1$ and $\mathbf{r}_2$) and the flow velocity $\mathbf{v}(\mathbf{x})$ can be expressed as:

$$\tau_a(\mathbf{r}_1, \mathbf{r}_2) = \int_V \mathbf{K}^a(\mathbf{r} - \mathbf{r}_1, \mathbf{r}_2, z) \cdot \mathbf{v}(\mathbf{r}, z) d^2\mathbf{r} dz + n^a(\mathbf{r}_1, \mathbf{r}_2) \qquad (1)$$

where the integration is performed over a volume $V = S \times [0, z_{min}]$ formed by the product of a surface S in the $(x, y)-$plane (supposed planar) by a small depth interval $[0, z_{min}]$ under the Sun's surface. The superscript a denotes the type of travel time, $\mathbf{K}^a$ is the sensitivity kernel and n^a the noise generated by the stochastic excitation of the waves by the smallest scales of convection (granulation). The position vector $\mathbf{x}$ is written as $\mathbf{x} = (\mathbf{r}, z)$ where $\mathbf{r} = (x, y)$ are the horizontal coordinates and z points up.

To recover internal properties of the Sun, the inverse problem corresponding to (1) has to be solved. A good knowledge of the sensitivity kernel and of the noise model is required to perform the inversion reliably. A methodology to construct the kernels is presented in [2], [3] and in [4] for the noise model. If one wants to recover another quantity related to velocities like the Reynolds stresses one could first compute the velocities and then deduce the Reynolds stresses. However, this method is time-consuming and not very accurate. This paper presents a first attempt to compute directly the Reynolds stresses from the travel times spatial correlations. In a first part, inversion methods for travel times are presented, then we show that the velocity correlations can also be linked to travel times via an integral operator and so deduced by inversion methods.

1 Inversion for velocities

Two methods are traditionally used to invert (1): the Regularized Least Square (RLS) method (equivalent to the Tikhonov method in the mathematical literature) and the Optimally Localized Averages (OLA) (equivalent to the approximate inverse). A modified version of the latter has been recently used [5] in order to invert (1) in the Fourier space instead of the real space. It is of great interest as the different modes are not correlated in the Fourier space so a lot of small matrices ($\approx 300 \times 300$) have to be inverted instead of a large one. The small size of the matrices makes possible the calculation of its singular value and thus, to perform the inversion by singular value decomposition (SVD). This method is particularly efficient for our problem. As the problem is severely ill-posed, a lot of eigenvalues are close to 0 so only few ones are kept for most of the modes.

2 Inversion for velocity correlations

Using Reynolds's decomposition, the i^{th} composant of the flow velocity vector can be written as $v_i = V_i + v_i'$ where V_i is the deterministic mean part of the flow velocity and v_i' the small scale (random) part. The Reynolds stresses R_{ij} are then defined as:

$$\mathbf{R}_{ij}(\mathbf{x}, \mathbf{x}') = \langle v_i'(\mathbf{x}), v_j'(\mathbf{x}') \rangle \tag{2}$$

If we assume that the correlations of the fluctuating part of the flow are horizontally homogeneous, $\mathbf{R}_{ij}$ depends only on the distance $\boldsymbol{\delta} = \mathbf{r}' - \mathbf{r}$, z and z':

$$\mathbf{R}_{ij}(\boldsymbol{\delta}, z, z') = \langle v_i'(\mathbf{r}, z), v_j'(\mathbf{r} + \boldsymbol{\delta}, z') \rangle \tag{3}$$

Multiplying $\tau^a(\mathbf{r_1}, \mathbf{r_2})$ and $\tau^b(\mathbf{r_1}', \mathbf{r_2}')$ (using (1)) and supposing for the sake of clarity that $V_i = 0$, one can link Reynolds stresses and measured travel times:

$$\langle \tau^a(\mathbf{r_1}, \mathbf{r_2}), \tau^b(\mathbf{r_1}', \mathbf{r_2}') \rangle = \boldsymbol{\Lambda}^{ab}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_1}', \mathbf{x_2}') + \int_{\odot'} \int_z K_{ij}^{ab}(\mathbf{r}', z, z'; \mathbf{r_1}, \mathbf{r_2}, \mathbf{r_1}', \mathbf{r_2}') \times \mathbf{R}_{ij}(\mathbf{r}', z, z') dz d^2\mathbf{r}' dz' \tag{4}$$

where $K_{ij}^{ab}(\mathbf{r}', z, z'; \mathbf{r_1}, \mathbf{r_2}, \mathbf{r_1}', \mathbf{r_2}') =$

$$\int_{\mathbf{r}} \mathbf{K}_i^a(\mathbf{r} - \mathbf{r_1}, \mathbf{r_2}, z) \mathbf{K}_j^b(\mathbf{r} + \mathbf{r}' - \mathbf{r_1}', \mathbf{r_2}', z') d^2\mathbf{r} \tag{5}$$

with $\mathbf{K}_i^a$ the i^{th} component of $\mathbf{K}^a$ and $\boldsymbol{\Lambda}^{ab}$ defined in [4] the noise covariance matrix for travel times.

Eq. (4) gives a relation between the Reynolds stresses and the travel times correlations via an integral operator. It turns out that the kernels are known as they are a correlation between two kernels for travel times. They can even be computed efficiently by Fast Fourier Transform (FFT):

$$K_{ij}^{ab} = \mathcal{F}^{-1} \left\{ \mathcal{F}\left(\overline{\mathbf{K}_i^a}\right) \mathcal{F}\left(\mathbf{K}_j^b\right) \right\} \tag{6}$$

where $\overline{\mathbf{K}_i^a}(\mathbf{r}) = \mathbf{K}_i^a(-\mathbf{r})$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the Fourier and inverse Fourier transform.

A 2D case used for kernel computations is presented in Figure 1. The kernels are computed between a one-way wave packet traveling west-east from the point $(-10\text{Mm}, 0)$ to $(10\text{Mm}, 0)$ and a south-north one traveling from $(0, -10\text{Mm})$ to $(0, 10\text{Mm})$. The kernel representing the cross-correlation for the Reynolds stresses is shown in Figure 1. Once the kernels are known, a noise model for Reynolds stresses is required. This can be done following the ideas of [4]. Computations are unfortunately way more complicated. In [4] it was necessary to compute the expected value of a product of four complex random Gaussian variables. Here, the moments of order six and eight are required which lead to about one hundred terms to estimate. However an exact formula can be derived for this purpose. Knowing the kernels and a noise model, (4) can be inverted for example by using a singular value decomposition.

This approach is significantly more efficient both concerning memory storage and computation time. In this direct approach, computations are made with mean values instead of maps on the whole domain if one wants to first compute the velocities then deduce the Reynolds stresses. So the size of the matrix to invert is much smaller. This approach will be validated using a model for flow velocities and travel times.

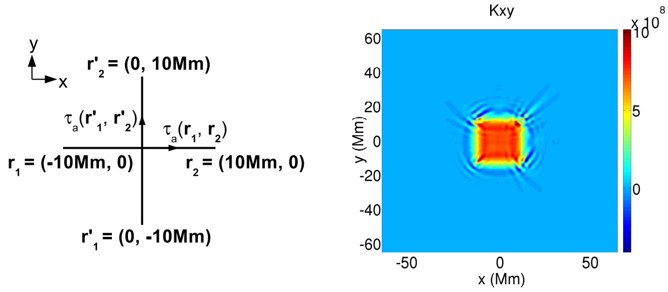

Figure 1: Test case for kernels computations (left); $K_{xy}^{aa}(x, y, \mathbf{r_1}, \mathbf{r_2}, \mathbf{r_1'}, \mathbf{r_2}')$, a refers to a f-mode measure between pairs separated by 20Mm (right)

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