Convergence Rates for Inverse Problems with Impulsive Noise

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Outline

1. Impulsive Noise
2. Analysis of Tikhonov regularization
3. Application to Impulsive Noise
4. Numerical simulations
5. Conclusion
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1 Impulsive Noise

2 Analysis of Tikhonov regularization

3 Application to Impulsive Noise

4 Numerical simulations

5 Conclusion
What is Impulsive Noise?

- noise $\xi$ is small in large parts of the domain $\mathbb{M}$, but large on small parts of the domain
- occurs e.g. in digital image acquisition
- caused by faulty memory locations, malfunctioning pixels etc.
- popular example: salt-and-pepper noise
A continuous model for impulsive noise

Suppose $\xi \in L^1(M)$, $\mathcal{B}(M) \doteq Borel \ \sigma$-algebra of $M$.

**Noise model**

There exist two parameters $\varepsilon, \eta \geq 0$ such that

$$\exists \ P \in \mathcal{B}(M) : \|\xi\|_{L^1(M\setminus P)} \leq \varepsilon, \quad |P| \leq \eta.$$
Inverse Problems with Impulsive Noise

- we want to reconstruct $f^\dagger$ from

$$g^{\text{obs}} = F \left( f^\dagger \right) + \xi =: g^\dagger + \xi$$

where $\xi$ is impulsive noise
- natural setup: $F : D(F) \subset \mathcal{X} \rightarrow L^1(M) \subseteq \mathcal{Y}$, possibly nonlinear
- Favorable method: Tikhonov regularization

$$\hat{f}_\alpha \in \arg\min_{f \in D(F)} \left[ \frac{1}{\alpha r} \left\| F(f) - g^{\text{obs}} \right\|_Y^r + \mathcal{R}(f) \right]$$

- Minimizer $\hat{f}_\alpha$ exists under reasonable assumptions.
Impulsive Noise

How to choose $\mathcal{Y}$ and $r$

here: $F = \text{linear integral operator (two times smoothing)}$ on $M = [0, 1]$

$$
f^r_\alpha = \arg\min_{f \in L^2(M)} \left[ \frac{1}{r\alpha} \left\| F(f) - g^{\text{obs}} \right\|^{r}_{L^r(M)} + \left\| f \right\|^2_{L^2(M)} \right], \quad r = 1, 2$$

computation of $f^1_\alpha$ via dual formulation, see e.g.

C. Clason, B. Jin, K. Kunisch.
A semismooth Newton method for $L^1$ data fitting with automatic choice of regularization parameters and noise calibration.

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Theoretical state of the art

- known theory provides rates of convergence as \( \|\xi\|_Y \) tends to 0
- this does not fully explain the remarkable quality of the \( L^1 \)-reconstruction!

Example: 'Most impulsive' noise. \( Y = M(\mathcal{M}) \) (space of all signed measures) and

\[
\xi = \sum_{j=1}^{N} c_j \delta_{x_j}
\]

with \( N \in \mathbb{N}, c_j \in \mathbb{R} \) and \( x_j \in \mathcal{M} \) for \( 1 \leq j \leq N \).

Then \( \|\xi\|_{\mathcal{M}(\mathcal{M})} = \sum_{j=1}^{N} |c_j| \) might be large! However

\[
\|g - g^{\text{obs}}\|_{\mathcal{M}(\mathcal{M})} = \|g - g^\dagger\|_{L^1(\mathcal{M})} + \sum_{j=1}^{N} |c_j| = \|g - g^\dagger\|_{L^1(\mathcal{M})} + \|\xi\|_{\mathcal{M}(\mathcal{M})}.
\]

So \( \xi \) does not influence the minimizer \( \hat{f}_\alpha \)!
Improving the noise level

'Most impulsive' noise $\xi$ influences $g \mapsto \|g - g^{\text{obs}}\|_{\mathcal{M}(\mathcal{M})}$ only as an additive constant, no influence on $\hat{f}_\alpha$!
Idea: For general $\xi$ study the influence of $\xi$ on the data fidelity term $\|g - g^{\text{obs}}\|_\gamma$ for all $g$.

Variational noise assumption

Suppose there exist $C_{\text{err}} > 0$ and a noise level function $\text{err} : F(D(F)) \to [0, \infty]$ such that
\[
\|g - g^{\text{obs}}\|_\gamma^r - \|\xi\|_\gamma^r \geq \frac{1}{C_{\text{err}}} \|g - g^\dagger\|_\gamma^r - \text{err}(g), \quad g \in F(D(F)).
\]
Examples for the noise function $\text{err}$

$$\| g - g^{\text{obs}} \|_Y^r - \| \xi \|_Y^r \geq \frac{1}{C_{\text{err}}} \| g - g^\dagger \|_Y^r - \text{err}(g), \quad g \in F(D(F)).$$

1. It follows from the triangle inequality that the Assumption is always fulfilled with

$$C_{\text{err}} = 2^{r-1} \quad \text{and} \quad \text{err} \equiv 2 \| \xi \|_Y^r.$$

2. In the Example of 'most impulsive' noise ($\mathcal{Y} = \mathcal{M}(\mathbb{M}), r = 1$) the Assumption holds true with the optimal parameters

$$C_{\text{err}} = 1 \quad \text{and} \quad \text{err} \equiv 0.$$
Convergence analysis under the variational noise assumption

- **Bregman distance:**

\[
D_{\mathcal{R}}^{f^*} \left( f, f^\dagger \right) := \mathcal{R}(f) - \mathcal{R}(f^\dagger) - \langle f^*, f - f^\dagger \rangle
\]

where \( f^* \in \partial \mathcal{R}(f^\dagger) \subset \mathcal{X}' \).

- **use a variational inequality as source condition:**

\[
\beta D_{\mathcal{R}}^{f^*} \left( f, f^\dagger \right) \leq \mathcal{R}(f) - \mathcal{R}(f^\dagger) + \varphi \left( \| F(f) - g^\dagger \|_y \right)
\]

for all \( f \in D(F) \) with \( \beta > 0 \). \( \varphi \) is assumed to fulfill

- \( \varphi(0) = 0 \),
- \( \varphi \uparrow \),
- \( \varphi \) concave.
Convergence rates

suppose

- the noise assumption is fulfilled with a function $\text{err} \geq 0$ and
- the variational inequality holds true.

**Theorem (error decomposition)**

\[
\beta D^f_R (\hat{f}_\alpha, f^\dagger) \leq \frac{\text{err} \left( F \left( \hat{f}_\alpha \right) \right)}{r \alpha} + \left( -\varphi \right)^* \left( -\frac{1}{r C_{\text{err}} \alpha} \right),
\]

\[
\left\| F \left( \hat{f}_\alpha \right) - g^\dagger \right\|_Y^r \leq \frac{C_{\text{err}}}{\lambda} \text{err} \left( F \left( \hat{f}_\alpha \right) \right) + \frac{r C_{\text{err}} \alpha}{\lambda} \left( -\varphi \right)^* \left( -\frac{1 - \lambda}{r C_{\text{err}} \alpha} \right)
\]

for all $\alpha > 0$ and $\lambda \in (0, 1)$.

Fenchel conjugate:

\[
\left( -\varphi \right)^* (s) = \sup_{\tau \geq 0} (s \tau + \varphi (\tau)).
\]
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Working schedule

• consider Tikhonov regularization for Inverse Problems with Impulsive Noise ($\mathcal{Y} = \mathbb{L}^{-1}(\mathbb{M})$, $r = 1$):

$$\hat{f}_\alpha \in \underset{f \in D(F)}{\text{argmin}} \left[ \frac{1}{\alpha} \left\| F(f) - g^{\text{obs}} \right\|_{\mathbb{L}^{-1}(\mathbb{M})} + \mathcal{R}(f) \right]$$

• recall: noise $\xi$ fulfills

$$\exists \mathcal{P} \in \mathcal{B}(\mathbb{M}) : \left\| \xi \right\|_{\mathbb{L}^{-1}(\mathbb{M} \setminus \mathcal{P})} \leq \varepsilon, \quad |\mathcal{P}| \leq \eta$$

$\overset{\sim}{\Rightarrow}$ need to estimate $\text{err}(g)$ with $g = F(\hat{f}_\alpha)$ defined by

$$\left\| g - g^{\text{obs}} \right\|_{\mathbb{L}^{-1}(\mathbb{M})} - \left\| \xi \right\|_{\mathbb{L}^{-1}(\mathbb{M})} \geq \frac{1}{C_{\text{err}}} \left\| g - g^{\dagger} \right\|_{\mathbb{L}^{-1}(\mathbb{M})} - \text{err}(g)$$
First step: triangle inequalities

\[
\|g - g^{\text{obs}}\|_{L^1(M)} - \|\xi\|_{L^1(M)} \geq \frac{1}{C_{\text{err}}} \|g - g^\dagger\|_{L^1(M)} - \text{err}(g)
\]

\[
\|g - g^{\text{obs}}\|_{L^1(M)} - \|\xi\|_{L^1(M)} = \int_{M\setminus P} \left[|g^{\text{obs}} - g| - |\xi|\right] \, dx + \int_{P} \left[|g^{\text{obs}} - g| - |\xi|\right] \, dx
\]

\[
\quad \geq \|g - g^\dagger\|_{L^1(M\setminus P)} - 2\varepsilon - |P| \|g - g^\dagger\|_{L^\infty(P)}
\]

\[
\quad \geq \|g - g^\dagger\|_{L^1(M)} - 2\varepsilon - 2\eta \|g - g^\dagger\|_{L^\infty(P)}
\]

Here we used

- the first triangle inequality on \(M \setminus P\) and
- the second triangle inequality on \(P\).
Second step: improving the bound

\[ \| g - g^{\text{obs}} \|_{L^1(M)} - \| \xi \|_{L^1(M)} \geq \| g - g^\dagger \|_{L^1(M)} - 2\varepsilon - 2\eta \| g - g^\dagger \|_{L^\infty(P)} \]

If \( F \) is smoothing and \( g = F(f) \), then \( \| g - g^\dagger \|_{L^\infty(P)} \) also decays with \( \eta \)!

**Theorem (Hohage, W.)**

If \( k > d/p \), then for all \( C_{\text{err}} > 1 \) there exist \( C > 0 \) and \( \eta_0 > 0 \) such that

\[ \| v \|_{L^\infty(M)} \leq C\eta^{d-1/p} \| v \|_{W^{k,p}(M)} + \frac{C_{\text{err}} - 1}{2C_{\text{err}}\eta} \| v \|_{L^1(M)} \]

for all \( v \in W^{k,p}(M) \) and \( \eta \in (0, \eta_0] \).

Follows from techniques used in approximation theory / FEM analysis (Ehrling’s lemma and Sobolev’s embedding theorem).
Second step: improving the bound (cont’)

Smoothing assumption on $F$

$\mathbb{M} \subset \mathbb{R}^d$ bounded & Lipschitz, $\exists \ k \in \mathbb{N}_0$, $p \in [1, \infty]$, $k > d/p$ and $q \in (1, \infty)$ such that

$$F(D(F)) \subset W^{k,p}(\mathbb{M}) \quad \text{and} \quad \left| F(f) - g^\dagger \right|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} D^f_R \left( f, f^\dagger \right)^{\frac{1}{q}}$$

for all $f \in D(F)$ with some $C_{F,k,p} > 0$.

This allows us to use $\nu = F(f) - g^\dagger$, e.g. it follows

$$\left\| F(f) - g^\dagger \right\|_{L^\infty(\mathbb{M})} \leq C\eta^{\frac{k}{d} - \frac{1}{p}} \left| F(f) - g^\dagger \right|_{W^{k,p}(\mathbb{M})} + \frac{C_{\text{err}} - 1}{2C_{\text{err}} \eta} \left\| F(f) - g^\dagger \right\|_{L^1(\mathbb{M})}$$

whenever $\eta$ is sufficiently small.
Second step: improving the bound (cont’)

\[
\| F(f) - g^{\text{obs}} \|_{L^1(M)} - \| \xi \|_{L^1(M)} \\
\geq \| F(f) - g^\dagger \|_{L^1(M)} - 2\varepsilon - 2\eta \| F(f) - g^\dagger \|_{L^\infty(P)} \\
\geq \left(1 - \frac{C_{\text{err}} - 1}{C_{\text{err}}} \right) \| F(f) - g^\dagger \|_{L^1(M)} - 2\varepsilon - 2C\eta^{\frac{k}{d} - \frac{1}{p} + 1} \| F(f) - g^\dagger \|_{W^{k,p}(M)} \\
\geq \frac{1}{C_{\text{err}}} \| F(f) - g^\dagger \|_{L^1(M)} - 2\varepsilon - 2C C_{F,k,p} \eta^{\frac{k}{d} - \frac{1}{p} + 1} D^*_R (f, f^\dagger)^{\frac{1}{q}} \\
\geq \frac{1}{C_{\text{err}}} \| F(f) - g^\dagger \|_{L^1(M)} - \text{err} (F(f))
\]

\[
\| F(f) - g^\dagger \|_{L^\infty(M)} \leq C\eta^{\frac{k}{d} - \frac{1}{p}} \| F(f) - g^\dagger \|_{W^{k,p}(M)} + \frac{C_{\text{err}} - 1}{2C_{\text{err}}\eta} \| F(f) - g^\dagger \|_{L^1(M)}
\]

\[
\| F(f) - g^\dagger \|_{W^{k,p}(M)} \leq C_{F,k,p} D^*_R (f, f^\dagger)^{\frac{1}{q}}
\]

Thus for any $C_{\text{err}} > 1$ we can choose

\[
\text{err} (F(f)) = 2\varepsilon + 2C_{F,k,p} C\eta^{\frac{k}{d} - \frac{1}{p} + 1} D^*_R (f, f^\dagger)^{\frac{1}{q}}
\]
Third step: final estimate for $\text{err} \left( F \left( \hat{f}_\alpha \right) \right)$

Calculation above:

$$\text{err} \left( F \left( \hat{f}_\alpha \right) \right) = 2\varepsilon + 2C_{F,k,p}C\eta^{\frac{k}{d} - \frac{1}{p}} + 1D_{R}^{f^*} \left( \hat{f}_\alpha, f^\dagger \right)^{\frac{1}{q}}$$

General convergence analysis:

$$\beta D_{R}^{f^*} \left( \hat{f}_\alpha, f^\dagger \right) \leq \frac{\text{err} \left( F \left( \hat{f}_\alpha \right) \right)}{\alpha} + (-\varphi)^* \left( -\frac{1}{C_{\text{err} \alpha}} \right)$$

This implies using Young’s inequality and $(a + b)\frac{1}{q} \leq a\frac{1}{q} + b\frac{1}{q}$ that

$$\text{err} \left( F \left( \hat{f}_\alpha \right) \right) \leq 2q'\varepsilon + (q' - 1)\frac{q'^k + q'(p-1)}{\alpha q'^{-1}} + C' (-\varphi)^* \left( -\frac{1}{C_{\text{err} \alpha}} \right)$$

where $1/q + 1/q' = 1$ and $C' > 0$ whenever $\alpha > 0$ and $\eta \geq 0$ is sufficiently small.
Error bound for Tikhonov regularization

Insert the estimate for $\text{err} \left( F \left( \hat{f}_\alpha \right) \right)$ into the general error decomposition to obtain

**Theorem (Hohage, W.)**

Suppose the variational inequality is fulfilled and $F$ obeys the smoothing assumption. Then we have for arbitrary $C_{\text{err}} > 1$ and all $\alpha > 0$ and $\eta > 0$ sufficiently small

$$
\beta D_{\mathcal{R}}^f \left( \hat{f}_\alpha, f^\dagger \right) \leq 2q' \frac{\varepsilon}{\alpha} + (q' - 1) \frac{\eta \frac{q'k}{d} + \frac{q'(p-1)}{p}}{\alpha q'} + C' (-\varphi)^* \left( -\frac{1}{C_{\text{err}} \alpha} \right)
$$

$$
\left\| F \left( \hat{f}_\alpha \right) - g^\dagger \right\|_{L^1(\mathcal{M})} \leq 4q' \varepsilon + 2(q' - 1) \frac{\eta \frac{q'k}{d} + \frac{q'(p-1)}{p}}{\alpha q' - 1} + 2C' C_{\text{err}} \alpha (-\varphi)^* \left( -\frac{1}{C_{\text{err}} \alpha} \right)
$$

For simplicity we study only $q = 2$ and $\varphi (\tau) = c \tau^\kappa$ with $c > 0$ and $\kappa \in (0, 1)$ in the following.
An optimal a priori parameter choice

\[
\beta D^*_{\mathcal{R}} \left( \hat{f}_\alpha, f^\dagger \right) \leq 4 \frac{\varepsilon}{\alpha} + \frac{\eta}{\alpha^2} d + \frac{2(p-1)}{p} \quad + C' \left( -\varphi \right)^* \left( -\frac{1}{C_{\text{err}} \alpha} \right)
\]

If \( \varphi(t) = c \cdot t^\kappa \) with \( c > 0 \) and \( \kappa \in (0, 1) \), then \( (-\varphi)^* \left(-\frac{1}{\alpha}\right) = C \cdot \alpha^{\kappa/(1-\kappa)} \).

So for \( \alpha \sim \max \left\{ \varepsilon^{1-\kappa}, \eta^{\left(\frac{1-\kappa}{2-\kappa}\right)} \left(\frac{2k}{d} + \frac{2(p-1)}{p}\right) \right\} \) we obtain

\[
D^*_{\mathcal{R}} \left( \hat{f}_\alpha, f^\dagger \right) = O \left( \max \left\{ \varepsilon^{\kappa}, \eta^{\frac{\kappa \gamma}{2-\kappa}} \right\} \right)
\]

with \( \gamma := \frac{2k}{d} + \frac{2(p-1)}{p} \) as \( \max \{ \varepsilon, \eta \} \searrow 0 \).
Functional dependence of $\varepsilon$ and $\eta$

$$\exists \ P \in \mathcal{B}(M) : \quad \|\xi\|_{L^1(M\setminus P)} \leq \varepsilon, \quad |P| \leq \eta \quad \text{(1)}$$

- model allows for different choices of $\varepsilon$ and $\eta$ which depend on each other
- study the dependence function

$$\varepsilon_\xi(\eta) := \inf \left\{ \|\xi\|_{L^1(M\setminus P)} \mid P \in \mathcal{B}(M), |P| \leq \eta \right\} .$$

- then for any $\eta \geq 0$ eq. (1) is fulfilled with $\varepsilon = \varepsilon_\xi(\eta)$
- for $\xi \in L^1(M)$ the following holds true:
  1. $\varepsilon_\xi(0) = \|\xi\|_{L^1(M)}$, $\varepsilon_\xi(|M|) = 0$
  2. $\varepsilon_\xi$ is continuous, decreasing, and convex
Examples for $\varepsilon_\xi$
Convergence rates in terms of an optimal \( \eta \)

- Recall: \( \mathcal{D}_{f^*}^R \left( \hat{f}_\alpha, f^\dagger \right) = \mathcal{O} \left( \max \{ \varepsilon^\kappa, \eta^{2-\kappa} \} \right) \)

- Substituting \( \varepsilon \) by \( \varepsilon_\xi (\eta) \) yields

\[
\mathcal{D}_{f^*}^R \left( \hat{f}_\alpha, f^\dagger \right) \leq C \inf_{0 \leq \eta \leq |\mathcal{M}|} \left[ \varepsilon_\xi (\eta)^\kappa + \eta^{2-\kappa} \gamma \right] \quad \text{as} \quad \xi \to 0
\]

- Note that \( \xi \) and \( \varepsilon_\xi \) are unknown in general, but possibly an upper bound for \( \varepsilon_\xi \) can be calculated

- As \( \varepsilon_\xi \downarrow \) and \( \eta^{2-\kappa} \uparrow \) in \( \eta \), there exists an intersecting point \( \bar{\eta} > 0 \)

- Thus we have

\[
\mathcal{D}_{f^*}^R \left( \hat{f}_\alpha, f^\dagger \right) \leq 2C \varepsilon_\xi (\bar{\eta})^\kappa \quad \text{as} \quad \xi \to 0
\]

- The state-of-the-art analysis yields (\( \eta = 0 \))

\[
\mathcal{D}_{f^*}^R \left( \hat{f}_\alpha, f^\dagger \right) \leq \bar{C} \varepsilon_\xi (0)^\kappa \quad \text{as} \quad \xi \to 0.
\]

\( \sim \) improvement measured by the factor \( (\varepsilon_\xi (0) / \varepsilon_\xi (\bar{\eta}))^{\kappa} \), which is arbitrary large for impulsive noise
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Considered operator

- $\mathbb{M} = [0, 1]$ and $T : L^2(\mathbb{M}) \rightarrow L^2(\mathbb{M})$ defined by
  \[
  (Tf)(x) = \int_{0}^{1} k(x, y) f(y) \, dy, \quad x \in \mathbb{M}
  \]
  with kernel $k(x, y) = \min\{x \cdot (1 - y), y \cdot (1 - x)\}, \ x, y \in \mathbb{M}$.

- then $(Tf)'' = f$ for any $f \in L^2(\mathbb{M})$ and $T$ is 2 times smoothing ($k = 2$ and $p = 2$).

- the smoothing Assumption is valid with exponent
  $\gamma = \frac{2k}{d} + 2(p - 1)/p = 5$ and $q = 2$.

- discretization: equidistant points $x_1 = \frac{1}{2n}, x_2 = \frac{3}{2n}, \ldots, x_n = \frac{2n-1}{2n}$ and composite midpoint rule
  \[
  (Tf)(x) = \int_{0}^{1} k(x, y) f(y) \, dy \approx \frac{1}{n} \sum_{i=1}^{n} k(x, x_i) f(x_i).
  \]
Simulations

- $f^\dagger$ and $g^\dagger$ are calculated analytically to avoid an inverse crime
- we consider 'purely impulsive noise' ($\varepsilon = 0$) for different values of $\eta$
- generation of $\xi$:
  - given $\eta$, choose randomly $\lceil\eta \cdot n\rceil$ grid points forming $\mathbb{P}$
  - simulate $\xi$ such that $\xi|_{\mathcal{M}\setminus\mathbb{P}} = 0$ and $\xi|_{\mathbb{P}} = \pm 1/\eta$ with probability $1/2$ respectively for each $x_i \in \mathbb{P}$
- for each $\eta_j = (4/5)^j, j = 1, \ldots$ we perform 10 experiments
- in each experiment $\alpha$ is chosen optimally by trial and error
- following plots show $\eta$ vs. empirical mean of $D^f_{\mathcal{R}}(\hat{f}_\alpha, f^\dagger)$
Example 1

(a) Exact solution $f^\dagger$

(b) Estimated index function $\varphi$

(c) mean convergence in $\mathcal{X}$
Example 2

(d) Exact solution $f^\dagger$

(e) Estimated index function $\varphi$

(f) mean convergence in $\mathcal{X}$

$\varphi \quad t \mapsto c \cdot t^{0.35}$
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Presented results and future work

- Inverse Problems with Impulsive noise
  - continuous model for Impulsive noise
  - improved convergence rates
- numerical examples suggest order optimality
- future work: infinitely smoothing operators!

T. Hohage and F. Werner
Convergence rates for Inverse Problems with Impulsive Noise.

Thank you for your attention!