Convergence rates in expectation for Tikhonov-type regularization of Inverse Problems with Poisson data

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joint work with Thorsten Hohage

CSR 2012
Outline

1. Introduction
2. Results on Poisson processes
3. Deterministic convergence analysis
4. Convergence rates in expectation
5. Conclusion
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1 Introduction

2 Results on Poisson processes

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5 Conclusion
Problem setup

Data: Total number $N$ and positions $x_i \in \mathbb{M}$ of photons distributed due to an unknown photon density $g^\dagger \in L^1(\mathbb{M})$.

Task: Determine the reason $u^\dagger$ of the photon density $g^\dagger$.

Note: The total number $N$ of counted photons depends on the intensity of $g^\dagger$ as well as a parameter $t$ interpreted as exposure time.


Poisson Processes

Mathematical model:

Poisson Process with intensity $tg^\dagger$, i.e.

$$\tilde{G}_t = \sum_{i=1}^{N} \delta_{x_i}$$

with the following properties:
Poisson process - Axiom I

\[ N(A) := \# \{ i \in \{1, \ldots, N\} \mid x_i \in A \} \]

**Independence:**

For any choice of \( A_1, \ldots, A_n \subset \mathbb{M} \) disjoint and measurable, the random variables

\[ N(A_1), \ldots, N(A_n) \]

are independent.
Poisson process - Axiom II

\[ N(A) := \# \{ i \in \{1, \ldots, N\} \mid x_i \in A \} \]

Poisson distribution:

For any measurable \( A \subset \mathbb{M} \) the random variable

\[ N(A) \]

is Poisson distributed with parameter

\[ t \int_{A} g^\dagger \, dx. \]
discretization / binning

- $\mathbb{M} = \bigcup_{j=1}^{J} \mathbb{M}_j$, each $\mathbb{M}_j$ corresponds to one detector

- $g^\dagger \in \mathbf{L}^1(\mathbb{M}) \leadsto S_J : \mathbf{L}^1(\mathbb{M}) \to \mathbb{R}^J$ defined by

  $$(S_Jg)_j := \int_{\mathbb{M}_j} g \, dx \quad \text{and} \quad S_j^*g := \sum_{j=1}^{J} |\mathbb{M}_j|^{-1} g_j, \quad j = 1, \ldots, J$$

- $P_J := S_j^*S_J$ is the $\mathbf{L}^2$-orthogonal projection onto the subspace of functions, which are constant on each $\mathbb{M}_j$.

- natural extension to measures via $(S_J(\tilde{G}_t))_j = \tilde{G}_t(\mathbb{M}_j) = N(\mathbb{M}_j)$.

  measured data: $g^\text{obs}_j = N(\mathbb{M}_j), \quad j = 1, \ldots, J$
Influence of $t$

We expect 20,000 photons per second
Difficulties

Model assumption: The imaging process can be described by an operator equation

\[ F \left( u^\dagger \right) = g^\dagger \]

where \( F : \mathcal{B} \subset \mathcal{X} \to \mathcal{Y} \subset L^1(\mathbb{M}) \) is in general nonlinear and \( \mathcal{X} \) and \( \mathcal{Y} \) are Banach spaces.

The exact right-hand side \( g^\dagger \) is unknown and in general \( F^{-1} \) is not continuous.

\[ \Rightarrow \text{direct reconstruction impossible, regularization necessary!} \]
Difficulties (cont.)

Several applications yield only data for small $t$, i.e.

- positron emission tomography (radiation exposure)
- astronomical imaging (limited observation time, motion artifacts)
- fluorescence microscopy (photobleaching)

⇒ use a negative log-likelihood approach to use the information at hand on the Poisson distribution:

Minimize

$$u \mapsto S \left( \tilde{G}_t; F(u) \right) := - \ln \left( P \left[ \tilde{G}_t \mid \text{the exact photon density is } F(u) \right] \right)$$

over all admissible $u$. 
Approach

Tackle the problem with **Tikhonov**-type regularization:

\[ u_\alpha \in \arg\min_{u \in \mathcal{B}} \left[ S \left( \tilde{G}_t; F(u) \right) + \alpha R(u) \right] \]

where \( R \) is a convex penalty term and \( \alpha > 0 \) a regularization parameter.

- \( R \) allows to incorporate a **priori** information on \( u^\dagger \)
- \( R \) stabilizes the reconstruction procedure
- \( u_\alpha \) can be interpreted as a **MAP estimator** if \( C \exp(-\alpha R(u)) \) models the prior
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Results on Poisson processes

Data fidelity terms

Scaled data $G_t = \frac{1}{t} \sum_{i=1}^{N} \delta_{x_i}$, $tG_t = \tilde{G}_t$ Poisson process.

- Negative log-likelihood:
  \[ S(G_t; g) = \int_{M} g \, dx - \int_{M} \ln(g) \, dG_t, \quad g \geq 0 \ a.e. \]

- It holds $E[S(G_t; g)] = \int_{M} [g - g^\dagger \ln(g)] \, dx$

- $\Rightarrow$ ideal data misfit functional for exact data $g^\dagger$ given by
  \[ E[S(G_t; g)] - E[S(G_t; g^\dagger)] = \int_{M} \left[ g - g^\dagger - g^\dagger \ln \left( \frac{g}{g^\dagger} \right) \right] \, dx \]

  which is the Kullback-Leibler divergence $\text{Kullback-Leibler} (g^\dagger; g)$.

- Error at $g$:
  \[ |S(G_t; g) - E[S(G_t; g^\dagger)] - \text{Kullback-Leibler} (g^\dagger; g)| = \left| \int_{M} \ln(g) \, (dG_t - g^\dagger \, dx) \right|. \]
Controlling the error

• we want to control the error

\[ \text{err}(g) := \left| S(G_t; g) - E\left[ S(G_t; g^\dagger) \right] - \KL(g^\dagger; g) \right| \]

with \( g = F(u) \)

• therefore we need to control the integrals

\[ \int_M g \left( dG_t - g^\dagger \, dx \right) \]

where \( g = \ln(F(u)) \)

\( \Rightarrow \) uniform concentration inequalities!

• well-studied for white noise (e.g. Gaussian), less known for Poisson processes
Uniform concentration inequalities for Poisson processes

Uniform concentration inequality (Reynaud-Bouret 2003)

- \( \{ f_a \}_{a \in A} \) countable family of functions with values in \([-b, b]\)
- \( Z := \sup_{a \in A} \left| \int_M f_a(x) \left( dG_t - g^\dagger dx \right) \right| \)
- \( v_0 := \sup_{a \in A} \int_M f_a^2(x) g^\dagger dx \)

Then for all \( \rho, \varepsilon > 0 \) it holds

\[
P \left[ Z \geq (1 + \varepsilon) E[Z] + \frac{\sqrt{12v_0\rho}}{\sqrt{t}} + \left( \frac{5}{4} + \frac{32}{\varepsilon} \right) \frac{b\rho}{t} \right] \leq \exp(-\rho).
\]

P. Reynaud-Bouret.
Adaptive estimation of the intensity of inhomogeneous Poisson processes via concentration inequalities.

\( \rightsquigarrow \) analogue to Talagrand’s concentration inequalities for empirical processes!
Uniform concentration inequalities for Poisson processes

\[
\mathbb{P} \left[ Z \geq (1 + \varepsilon) \mathbb{E}[Z] + \frac{\sqrt{12v_0}}{\sqrt{t}} + \left( \frac{5}{4} + \frac{32}{\varepsilon} \right) \frac{b\rho}{t} \right] \leq \exp(-\rho).
\]

• Suppose \( \mathbb{M} \subset \mathbb{R}^d \) bounded and Lipschitz

• \( \{ f_a \} \) dense subset of \( B_s(R) := \left\{ g \in H^s(\mathbb{M}) \mid \| g \|_{H^s(\mathbb{M})} \leq R \right\} \) with \( s > \frac{d}{2} \) (Sobolev’s embedding theorem)

• Easy: \( v_0 \leq R^2 C \| g^\dagger \|_{L^1(\mathbb{M})} \), \( C > 0 \) depending only on \( \mathbb{M} \) and \( s \)

• More difficult (uses periodization and Fourier expansion):

\[
\mathbb{E}[Z] \leq \frac{CR}{\sqrt{t}} \| g^\dagger \|_{L^1(\mathbb{M})}
\]

with a constant \( C > 0 \) depending only on \( \mathbb{M} \) and \( s \).
Controlling the error (cont.)

Uniform concentration inequality (W., Hohage 2012)

- $\mathcal{M} \subset \mathbb{R}^d$ bounded and Lipschitz,
- $s > d/2, R > 1$.

Then there exists $C_{\text{conc}} = C_{\text{conc}} (\mathcal{M}, s, g^\dagger) \geq 1$ such that

$$
P \left[ \sup_{g \in B_s(R)} \left| \int_{\mathcal{M}} g \left( dG_t - g^\dagger dx \right) \right| \leq \frac{\rho}{\sqrt{t}} \right] \geq 1 - \exp \left( - \frac{\rho}{RC_{\text{conc}}} \right)
$$

for all $t \geq 1$ and $\rho \geq RC_{\text{conc}}$. 


Controlling the error (cont.)

- Concentration inequality requires \( g \in H^s (\mathbb{M}) \subset L^\infty (\mathbb{M}) \) due to \( s > d/2 \)

- Error at \( g = F(u) \) leads to \( g = \ln (F(u)) \)

\[ \Rightarrow \text{Too strong assumption!} \]

\[ \Downarrow \text{Shift by} \sigma > 0: \]

\[ S (G_t; g) := \int_\mathbb{M} g \, dx - \int_\mathbb{M} \ln (g + \sigma) \, (dG_t + \sigma \, dx) \]

\[ \mathcal{T} (g^\dagger; g) := \text{KL} \left( g^\dagger + \sigma; g + \sigma \right) \]

- Then the error is given by

\[ \left| \int_\mathbb{M} \ln (g + \sigma) \, (dG_t - g^\dagger \, dx) \right|. \]
Controlling the error (cont.)

Corollary (final concentration inequality)

- \( \mathcal{M} \subset \mathbb{R}^d \) bounded and Lipschitz
- \( F(u) \geq 0 \) a.e. for all \( u \in \mathcal{B} \)
- there exists a Sobolev index \( s > \frac{d}{2} \) such that

\[
R := \sup_{u \in \mathcal{B}} \| F(u) \|_{H^s(\mathcal{M})} < \infty
\]

Then there exists \( C_{\text{conc}} = C_{\text{conc}}(\mathcal{M}, s, g^\dagger) \geq 1 \) such that

\[
P \left[ \sup_{u \in \mathcal{B}} \text{err}(F(u)) \leq \frac{\rho}{\sqrt{t}} \right] \geq 1 - \exp \left( -\frac{\rho}{R \max \{\sigma^{-\lfloor s \rfloor - 1}, |\ln(R)|\} C_{\text{conc}}} \right)
\]

for all \( t \geq 1, \rho \geq R \max \{\sigma^{-\lfloor s \rfloor - 1}, |\ln(R)|\} C_{\text{conc}} \).

Proof relies on composition theorems in the Sobolev space \( H^s(\mathcal{M}) \).
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Deterministic noise level

We have two data fidelity terms:

- $S$ w.r.t. the measured data $g^{\text{obs}}$
- $T$ w.r.t. the photon density $g^{\dagger}$

As before: consider the difference between both as noise level!

**Noise level**

There exist constants $\text{err} \geq 0$ and $C_{\text{err}} \geq 1$ such that

$$S \left( g^{\text{obs}}, g \right) - S \left( g^{\text{obs}}, g^{\dagger} \right) \geq \frac{1}{C_{\text{err}}} T \left( g^{\dagger}; g \right) - \text{err}$$

for all $g \in F(\mathcal{B})$. 
Deterministic noise level (cont.)

- **Classical deterministic noise model:**
  If \( S(g; \hat{g}) = \mathcal{T}(g; \hat{g}) = \|g - \hat{g}\|_Y^r \), then \( C_{err} = 2^{r-1} \) and \( err = 2 \|g^\dagger - g^{obs}\|_Y^r \).

- **Poisson data:**
  \( C_{err} = 1 \) and
  \[
  err \geq - \int_M \ln (g^\dagger + \sigma) \left( dG_t - g^\dagger dx \right) + \int_M \ln (F(u) + \sigma) \left( dG_t - g^\dagger dx \right)
  \]
  for all \( u \in \mathcal{B} \).

  Uniform concentration inequality: \( err \leq \frac{2\rho}{\sqrt{t}} \) with probability \( \geq 1 - \exp(-c\rho) \) for some constant \( c > 0 \).
Source condition

- **Bregman distance:**

\[
D_{\mathcal{R}}^{u^*}(u, u^\dagger) := \mathcal{R}(u) - \mathcal{R}(u^\dagger) - \langle u^*, u - u^\dagger \rangle
\]

where \( u^* \in \partial \mathcal{R}(u^\dagger) \subset \mathcal{X}' \).

- **Use a variational inequality** as source condition:

\[
\beta D_{\mathcal{R}}^{u^*}(u, u^\dagger) \leq \mathcal{R}(u) - \mathcal{R}(u^\dagger) + \varphi(\mathcal{T}(g^\dagger; F(u)))
\]

for all \( u \in \mathcal{B} \) with \( \beta > 0 \). \( \varphi \) is assumed to fulfill

- \( \varphi(0) = 0 \),
- \( \varphi \uparrow \),
- \( \varphi \) concave.
Deterministic convergence analysis

Assumptions

**Source condition (cont.)**

\[
\beta D_{\mathcal{R}}^u (u, u^\dagger) \leq \mathcal{R} (u) - \mathcal{R} (u^\dagger) + \varphi \left( \mathcal{T} \left( g^\dagger; F (u) \right) \right)
\]

- does not depend on the structure of \( \mathcal{X} \) and \( \mathcal{Y} \)
- includes structure of \( \mathcal{R} \) and \( \mathcal{T} \), allows for formulation in a general setup
- nonlinear \( F \): combination of source and nonlinearity condition
- connection to conditional stability estimates
Source condition (cont.)

important special case: $\mathcal{X}$, $\mathcal{Y}$ Hilbert spaces, $\mathcal{R}(u) = \|u - u_0\|_{\mathcal{X}}^2$.

- if $\mathcal{I}(\hat{g}; g) = \|g - \hat{g}\|_{\mathcal{Y}}^2$:
  - spectral source condition + nonlinearity condition imply variational inequality
  - provided deterministic convergence analysis is optimal in case of linear $F$!

J. Flemming.
*Generalized Tikhonov regularization - Basic theory and comprehensive results on convergence rates.*

- if $\mathcal{I}$ given as above ($\sim$ negative log-likelihood) and $F(\mathcal{B}) \subset L^\infty$ bounded:
  - it holds $\|F(u) - g^\dagger\|_{L^2}^2 \leq C\mathcal{I}(g^\dagger; F(u))$ for all $u \in \mathcal{B}$

J. M. Borwein and A. S. Lewis.
*Convergence of best entropy estimates.*

- thus spectral source condition + nonlinearity condition imply variational inequality!
Deterministic convergence analysis

Suppose

- the noise assumption is fulfilled with $\text{err} \geq 0$ and
- the variational inequality holds true.

**Theorem (error decomposition)**

Then

$$
\beta \mathcal{D}_R^{u_*} (u_\alpha, u_{\dagger}^*) \leq \frac{\text{err}}{\alpha} + (-\varphi)^* \left( -\frac{1}{C_{\text{err}}} \alpha^\alpha \right)
$$

for all $\alpha > 0$.

Fenchel conjugate:

$$
(-\varphi)^* (s) = \sup_{\tau \geq 0} (s \tau + \varphi (\tau))
$$
Proof I

\[ \beta D^u_{u^*} (u_\alpha, u^\dagger) \leq R(u_\alpha) - R(u^\dagger) + \varphi (T(g^\dagger; F(u_\alpha))) \]

variational inequality
Proof II

\[ \beta \mathcal{D}_{\mathcal{R}}^{u^*} (u_\alpha, u^\dagger) \leq \mathcal{R} (u_\alpha) - \mathcal{R} (u^\dagger) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{1}{\alpha} \left( \mathcal{S} (g^{\text{obs}}, g^\dagger) - \mathcal{S} (g^{\text{obs}}, F (u_\alpha)) \right) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

Definition of \( u_\alpha \):

\[ \mathcal{S} (g^{\text{obs}}, F (u_\alpha)) + \alpha \mathcal{R} (u_\alpha) \leq \mathcal{S} (g^{\text{obs}}, g^\dagger) + \alpha \mathcal{R} (u^\dagger) \]
Proof III

\[ \beta D^{u^*}_{\mathcal{R}} (u_\alpha, u^\dagger) \leq \mathcal{R} (u_\alpha) - \mathcal{R} (u^\dagger) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{1}{\alpha} (S (g^{\text{obs}}; g^\dagger) - S (g^{\text{obs}}; F (u_\alpha))) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{\text{err}}{\alpha} - \frac{1}{C_{\text{err}} \alpha} \mathcal{T} (g^\dagger; F (u_\alpha)) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

Deterministic noise assumption: \( S (g^{\text{obs}}; g) - S (g^{\text{obs}}; g^\dagger) \geq \frac{1}{C_{\text{err}}} \mathcal{T} (g^\dagger; g) - \text{err} \)
Proof IV

\[ \beta D_{\mathcal{R}}^u (u_\alpha, u^\dagger) \leq \mathcal{R} (u_\alpha) - \mathcal{R} (u^\dagger) + \varphi (\mathcal{T} (g^\dagger; F(u_\alpha))) \]

\[ \leq \frac{1}{\alpha} \left( S (g^\text{obs}; g^\dagger) - S (g^\text{obs}; F(u_\alpha)) \right) + \varphi (\mathcal{T} (g^\dagger; F(u_\alpha))) \]

\[ \leq \frac{\text{err}}{\alpha} - \frac{1}{\mathcal{C}_{\text{err}} \alpha} \mathcal{T} (g^\dagger; F(u_\alpha)) + \varphi (\mathcal{T} (g^\dagger; F(u_\alpha))) \]

\[ \leq \frac{\text{err}}{\alpha} + \sup_{\tau \geq 0} \left[ \frac{\tau}{-\mathcal{C}_{\text{err}} \alpha} - (-\varphi) (\tau) \right] \]
Proof V

\[ \beta \mathcal{D}_{\mathcal{R}}^u (u_\alpha, u^\dagger) \leq \mathcal{R} (u_\alpha) - \mathcal{R} (u^\dagger) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{1}{\alpha} (\mathcal{S} (g^{\text{obs}}, g^\dagger) - \mathcal{S} (g^{\text{obs}}, F (u_\alpha))) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{\text{err}}{\alpha} - \frac{1}{C_{\text{err} \alpha}} \mathcal{T} (g^\dagger; F (u_\alpha)) + \varphi (\mathcal{T} (g^\dagger; F (u_\alpha))) \]

\[ \leq \frac{\text{err}}{\alpha} + \sup_{\tau \geq 0} \left[ \frac{\tau}{-C_{\text{err} \alpha}} - (-\varphi) (\tau) \right] \]

\[ = \frac{\text{err}}{\alpha} + (-\varphi)^* \left( -\frac{1}{C_{\text{err} \alpha}} \right). \]

Definition of Fenchel conjugate: \((-\varphi)^* (s) = \sup_{\tau \geq 0} (s \tau + \varphi (\tau))\)
Deterministic convergence analysis (cont.)

\[ \beta D^{u^*} (u_\alpha, u^{\dagger}) \leq \frac{\text{err}}{\alpha} + (\varphi^*) \left( -\frac{1}{C_{\text{err}} \alpha} \right) \]

**Theorem (a priori rates)**

The infimum of the right-hand side is attained at \( \alpha = \bar{\alpha} \) if and only if

\[ \frac{-1}{C_{\text{err}} \bar{\alpha}} \in \partial (-\varphi) (C_{\text{err}} \text{err}) \quad \left[ \hat{\alpha} = \frac{1}{C_{\text{err}} \varphi' (C_{\text{err}} \text{err})} \right] \]

and in that case

\[ \beta D^{u^*} (u_{\bar{\alpha}}, u^{\dagger}) \leq C_{\text{err}} \varphi (\text{err}) . \]
Proof

Young’s inequality:

\[ s\tau \leq f(\tau) + f^*(s) \quad \text{for all} \quad s, \tau \in \mathbb{R}, \]

\[ s\tau = f(\tau) + f^*(s) \quad \iff \quad \tau \in \partial f(s). \]

moreover \( f^{**} = f \) whenever \( f \) is convex, proper and lower-semicontinuous.

\[
\inf_{\alpha > 0} \left[ \frac{\text{err}}{\alpha} + (-\varphi)^* \left( -\frac{1}{C_{\text{err}} \alpha} \right) \right] \quad - \frac{1}{C_{\text{err}} \alpha} \; = \; s \\
= \quad - \sup_{s < 0} \left[ s C_{\text{err}} \text{ err} - (-\varphi)^*(s) \right] \\
= \quad - (-\varphi)^{**} (C_{\text{err}} \text{ err}) \\
= \quad \varphi (C_{\text{err}} \text{ err}) \\
\leq \quad C_{\text{err}} \varphi (\text{err})
\]

supremum is attained at \( \alpha = \overline{\alpha} \) if and only if

\[ \overline{s} \in \partial (-\varphi) (C_{\text{err}} \text{ err}) \quad \iff \quad \frac{-1}{C_{\text{err}} \overline{\alpha}} \in \partial (-\varphi) (C_{\text{err}} \text{ err}) \]
Deterministic convergence analysis (cont.)

Suppose moreover $\mathcal{X}$ Hilbert space, $\mathcal{R}(u) = \|u - u_0\|_\mathcal{X}^2$, $\beta \geq \frac{1}{2}$. Set

- $r > 1$
- $\alpha_j := \text{err} r^{2j-2}$ for $j = 2, ..., m$ such that $\alpha_{m-1} < 1 \leq \alpha_m$
- $j_{\text{bal}} := \max \left\{ j \leq m \mid \|u_{\alpha_i} - u_{\alpha_j}\|_{\mathcal{X}} \leq 4\sqrt{2}r^{1-i} \text{ for all } i < j \right\}$

Theorem (a posteriori rates)

Then for $\text{err} > 0$ sufficiently small:

$$\left\| u_{\alpha_{j_{\text{bal}}}} - u^\dagger \right\|_{\mathcal{X}}^2 \leq 6r \min_{j=1,\ldots,m} \left[ \frac{\text{err}}{\alpha_j} + (-\varphi)^* \left( -\frac{1}{C_{\text{err}}\alpha_j} \right) \right].$$

If $\varphi^{1+\varepsilon}$ is additionally concave ($\varepsilon > 0$), then

$$\left\| u_{\alpha_{j_{\text{bal}}}} - u^\dagger \right\|_{\mathcal{X}}^2 \leq 6r^{1+\frac{1}{\varepsilon}} C_{\text{err}} \varphi(\text{err})$$

as $\text{err} \searrow 0$. 

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Convergence rates for known $\varphi$

Suppose

- variational inequality holds true
- $\mathcal{X}$ Banach space, $u^\dagger \in \mathcal{B} \subset \mathcal{X}$ bounded, closed and convex
- $\mathcal{M} \subset \mathbb{R}^d$ bounded and Lipschitz
- $F(u) \geq 0$ a.e. for all $u \in \mathcal{B}$
- there exists a Sobolev index $s > \frac{d}{2}$ such that $F(\mathcal{B})$ is a bounded subset of $H^s(\mathcal{M})$

A priori convergence rates (W., Hohage 2012)

Then for $\alpha = \alpha(t)$ such that $\frac{1}{\alpha} \in -\partial (-\varphi) \left( \frac{1}{\sqrt{t}} \right)$ we obtain the convergence rate

$$E \left[ \mathcal{D} u^*_R \left( u_\alpha, u^\dagger \right) \right] = O \left( \varphi \left( \frac{1}{\sqrt{t}} \right) \right), \quad t \to \infty.$$
Sketch of proof

- let $E_k := \left\{ \sup_{u \in \mathcal{B}} \text{err} (F (u)) \leq \frac{\rho_k}{\sqrt{t}} \right\}$, $\rho_k = c^{-1} k$
  where $c \triangleq$ constant from concentration inequality

  $\leadsto P [E_k^c] \leq \exp (-c\rho_k) = \exp (-k)$

- on $E_k$: $C_{err} = 1$ and $\text{err} = 2 \sup_{u \in \mathcal{B}} \text{err} (F (u)) \leq 2\rho_k/\sqrt{t}$,
  i.e. deterministic convergence analysis is applicable

\[
\mathbb{E} \left[ \mathcal{D}_{\mathcal{R}}^{u^*} \left( u_{n^*}, u^\dagger \right) \right] \leq \sum_{k=1}^{\infty} P [E_k \setminus E_{k-1}] \max_{E_k} \mathcal{D}_{\mathcal{R}}^{u^*} \left( u_{n^*}, u^\dagger \right)
\]

\[
\leq C \varphi \left( \frac{1}{\sqrt{t}} \right) \sum_{k=1}^{\infty} P [E_k \setminus E_{k-1}] k^\frac{1}{\varepsilon}
\]

sum converges due to $P [E_k \setminus E_{k-1}] \leq P [E_{k-1}^c] \leq \exp (- (k - 1))$
Convergence rates in expectation

Main results

Convergence rates for unknown $\varphi$

Suppose moreover $\mathcal{X}$ Hilbert space, $\mathcal{R}(u) = \|u - u_0\|^2_{\mathcal{X}}$, $\beta \geq \frac{1}{2}$, $\varphi^{1+\varepsilon}$ concave ($\varepsilon > 0$). Set

- $r > 1$, $\tau > 0$ sufficiently large
- $\alpha_j := \frac{\tau \ln(t)}{\sqrt{t}} r^{2j-2}$ for $j = 2, \ldots, m$ such that $\alpha_{m-1} < 1 \leq \alpha_m$
- $j_{\text{bal}} := \max \left\{ j \leq m \mid \|u_{\alpha_i} - u_{\alpha_j}\|_{\mathcal{X}} \leq 4\sqrt{2}r^{1-i} \text{ for all } i < j \right\}$

A posteriori convergence rates (W., Hohage 2012)

Then we obtain

$$E \left[ \|u_{\alpha_{j_{\text{bal}}}} - u^\dagger\|_{\mathcal{X}}^2 \right] = O \left( \varphi \left( \frac{\ln(t)}{\sqrt{t}} \right) \right) \quad \text{as} \quad t \to \infty.$$

Adaptivity causes a loss of a logarithmic factor!

A. Tsybakov.
On the best rate of adaptive estimation in some inverse problems.

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Presented results

- proper setup for inverse problems with Poisson data:
  - Poisson processes
  - uniform concentration inequality

- improvements in theory:
  - convergence and convergence rates
  - generalized source conditions
  - a priori and a posteriori parameter choice

- regularization theory with general data fidelity terms

F. Werner and T. Hohage.
Convergence rates in expectation for Tikhonov-type regularization of Inverse Problems with Poisson data.

Thank you for your attention!