Non–symmetric coupling of FEM and BEM: 30 years after

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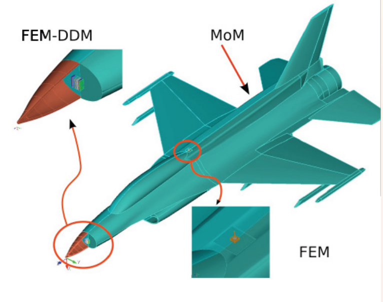
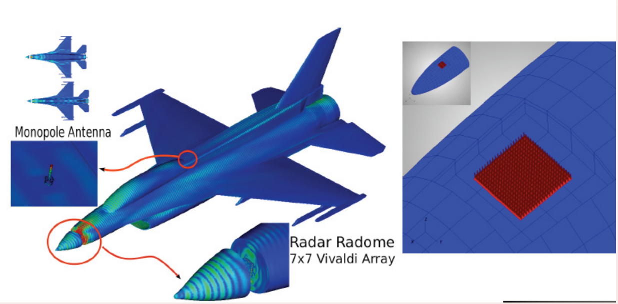
Outline

1. Recalling the problem
2. The original approach
3. The new proof
4. Straightforward extensions
5. Extensions where you have to scratch your head
6. Final words

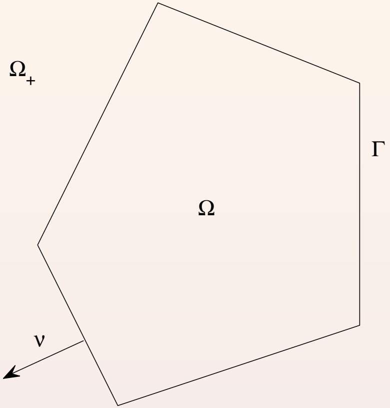
This work collects results in collaboration with Gabriel Gatica (Concepción, Chile), George Hsiao (Delaware), Salim Meddahi (Oviedo, Spain) & Virginia Selgás (La Coruña, Spain)

A introduction to BEM–FEM coupling


FEM-DDM
MoM
FEM


Monopole Antenna
Radar Radome
7x7 Vivaldi Array

Model geometry...



Problem set **in free space** $\mathbb{R}^d$ ($d = 2$ or 3), with an interface separating bounded from unbounded.

Remark. The index $+$ is used for all things exterior. No index is used for interior quantities and limits.

Transmission problem

$u \in H^1(\Omega), \quad u^+ \in H^1(\Omega_+)$

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega, \\ -\Delta u^+ + u^+ &= 0 && \text{in } \Omega_+, \\ \gamma^+ u + g_D &= \gamma u && \text{on } \Gamma, \\ \partial_\nu^+ u + g_N &= \partial_\nu u && \text{on } \Gamma. \end{aligned}$$

Warning. I'm considering the Yukawa equation to forget about conditions at infinity. Wait for Laplace though.

$E(\mathbf{x}, \mathbf{y})$:= fundamental solution (Green's function in free space)

Potentials are defined in $\mathbb{R}^d \setminus \Gamma$:

$$\begin{aligned} \mathcal{S}\psi &:= \int_\Gamma E(\,\cdot\,, \mathbf{y})\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ \mathcal{D}\phi &:= \int_\Gamma \nabla_{\mathbf{y}} E(\,\cdot\,, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\phi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \end{aligned}$$

Operators are defined on Γ:

$$\begin{aligned} \mathsf{V}\psi &:= \int_\Gamma E(\,\cdot\,, \mathbf{y})\psi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ \mathsf{K}\phi &:= \int_\Gamma \nabla_{\mathbf{y}} E(\,\cdot\,, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\phi(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \end{aligned}$$

Green's Integral Representation Theorem

(Part 1)

For any

$$u \in H^1(\Omega_+) \qquad \text{such that} \qquad -\Delta u + u = 0 \quad \text{in } \Omega_+,$$

we can explicitly write u in terms of its Cauchy data on the boundary Γ, using both layer potentials:

$$u = \mathcal{D}\gamma^+ u - \mathcal{S}\partial_\nu^+ u \qquad \text{in } \Omega_+.$$

... aka Green's Third Formula

(Parts 2 & 3)

This was Part 1:

$$u = \mathcal{D}\gamma^+ u - \mathcal{S}\partial_\nu^+ u \qquad \text{in } \Omega_+.$$

If we specialize the right–hand side of this expression to the boundary we obtain an integral identity

$$\tfrac{1}{2}\gamma^+ u = \mathrm{K}\gamma^+ u - \mathrm{V}\partial_\nu^+ u \qquad \text{on } \Gamma,$$

and if we look inside, everything vanishes

$$0 = \mathcal{D}\gamma^+ u - \mathcal{S}\partial_\nu^+ u \qquad \text{in } \Omega.$$

Notation and facts

- Interior bilinear form:

$$((u, v))_{1,\Omega} := \int_\Omega (\nabla u \cdot \nabla v + u\, v)$$

- $\langle \cdot\,, \cdot \rangle$ is the duality product $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$; think of it as the $L^2(\Gamma)$ inner product (when we discretize it'll be just that)
- We can write the integral identity as follows:

$$(\tfrac{1}{2} - \mathrm{K})\gamma^+ u + \mathrm{V}\partial_\nu^+ u = 0 \qquad \text{on } \Gamma$$

or in weak form

$$\langle \mu, (\tfrac{1}{2} - \mathrm{K})\gamma^+ u \rangle + \langle \mu, \mathrm{V}\partial_\nu^+ u \rangle = 0 \qquad \forall \mu \in H^{-1/2}(\Gamma)$$

How to derive the coupling method

- Imagine for a while that you know $\lambda := \partial_\nu^+ u$. Then you can solve the interior Neumann problem

$$\begin{aligned} ((u,v))_{1,\Omega} &= (f,v) + \langle \partial_\nu u, \gamma v\rangle \\ &= (f,v) + \langle g_N, \gamma v\rangle + \langle \lambda, \gamma v\rangle. \end{aligned}$$

- Imagine now that you have solved this problem and you export γu. Then you can use your integral identity as an equation to find $\lambda = \partial_\nu^+ u$

$$\begin{aligned} \langle \mu, \mathrm{V}\lambda\rangle &= -\langle \mu, (\tfrac{1}{2} - \mathrm{K})\gamma^+ u\rangle \\ &= -\langle \mu, (\tfrac{1}{2} - \mathrm{K})\gamma u\rangle + \langle \mu, (\tfrac{1}{2} - \mathrm{K}) g_D\rangle \end{aligned}$$

Instead of iterating, take the two equations as a whole.

The Johnson–Nédélec non–symmetric coupling

Unknowns: $u \in H^1(\Omega)$, $\lambda := \partial_\nu^+ u \in H^{-1/2}(\Gamma)$

Find $u \in H^1(\Omega), \lambda \in H^{-1/2}(\Gamma)$ s.t.

$$\begin{aligned} ((u,v))_{1,\Omega} \quad & -\langle \lambda, \gamma v\rangle && = (f,v) + \langle g_N, \gamma v\rangle \\ \langle \mu, (\tfrac{1}{2} - \mathrm{K})\gamma u\rangle \quad & +\langle \mu, \mathrm{V}\lambda\rangle && = \langle \mu, (\tfrac{1}{2} - \mathrm{K}) g_D\rangle \end{aligned}$$

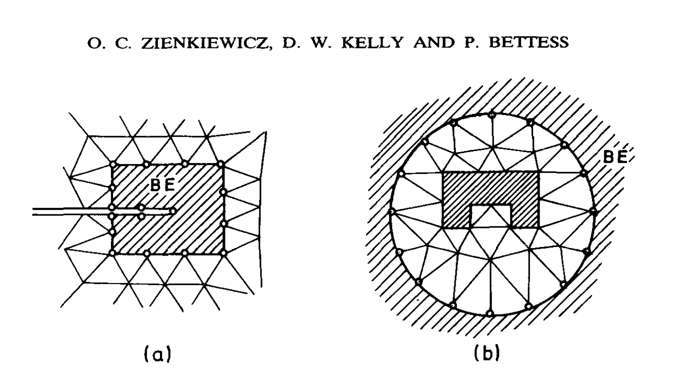
for all $v \in H^1(\Omega)$, $\mu \in H^{-1/2}(\Gamma)$.

Representation formula:

$$u^+ := \mathcal{D}(\gamma u - g_D) - \mathcal{S}\lambda$$

Numerical approximation by Galerkin methods (BEM–FEM)

A picture in 1977



The story of it

- Formulations and experiments in the engineering literature: see for instance Zienkiewicz, Kelly & Bettess (77)
- Analysis by Johnson & Nédélec (80), Brezzi & Johnson (79), using discrete Fredholm theory (see later)
- More numerical evidence in Costabel, Ervin & Stephan (91)
- Generalization by Wendland (86,88) for sufficiently discretized BEM
- Giving up and moving to symmetric methods, by Costabel (87) & Han (90)

The original proof (using Fredholm theory)

Bilinear form

$$
\begin{aligned}
a\big((u,\lambda),(v,\mu)\big) \quad &:= \quad ((u,v))_{1,\Omega} - \langle \lambda, \gamma v\rangle \\
&\quad\;\; \langle \mu, (\tfrac{1}{2} - \mathrm{K})\gamma u\rangle + \langle \mu, \mathrm{V}\lambda\rangle
\end{aligned}
$$

Gårding's inequality

$$
\begin{aligned}
a\big((u,\lambda),(u,2\lambda)\big) \quad &= \quad ((u,u))_{1,\Omega} - \langle \lambda, \gamma u\rangle \\
&\quad\;\; \langle \lambda, \gamma u\rangle - \langle 2\lambda, \mathrm{K}\gamma u\rangle + 2\langle \lambda, \mathrm{V}\lambda\rangle \\
&= \quad ((u,u))_{1,\Omega} + 2\langle \lambda, \mathrm{V}\lambda\rangle - \langle 2\lambda, \mathrm{K}\gamma u\rangle
\end{aligned}
$$

The original proof (using Fredholm theory, cont'd)

Assume that K is compact (as in *Kompakt*) and take the principal part of the quadratic form:

$$a_{\text{ppal}}\big((u,\lambda),(u,2\lambda)\big) = ((u,u))_{1,\Omega} + 2\langle\lambda, \mathrm{V}\lambda\rangle$$

Fortunately, V is elliptic in $H^{-1/2}(\Gamma)$. The key identity is this one[1]:

$$\langle\lambda, \mathrm{V}\lambda\rangle = ((u^*, u^*))_{1,\mathbb{R}^d}, \qquad \text{where } u^* = \mathcal{S}\lambda.$$

From this identity you prove

$$\langle\lambda, \mathrm{V}\lambda\rangle \geq C_\Gamma \|\lambda\|^2_{-1/2,\Gamma}.$$

[1] This was first set this clearly (in the numerical analysis community) by Nédélec & Planchard (1973), but it might be older.

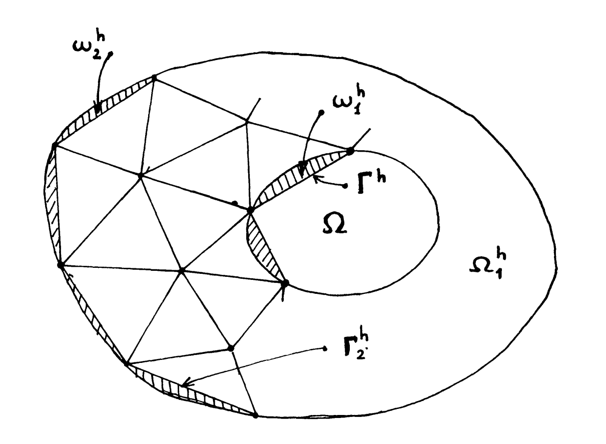
Wrapping it up

- The principal part of the operator is elliptic (after multiplying the second equation by two)...
- ... provided that K is compact.
- Elliptic + compact = Fredholm of index zero (Fredholm/Riesz/Hilbert theory)
- Galerkin methods with the approximation property are convergent for elliptic + compact operators. (Probably first shown, in a very abstract language, by Hildebrandt & Wienholtz, 60-ish.)

So the question is...

... is K ever compact?

A picture in 1980



... and the explanation to the picture

- K is compact for sufficiently smooth Γ (Lyapunov smooth)
- K is not compact for polygons/polyhedra
- K is never compact for the elasticity system

The story of it (cont'd, apologies for omissions)

- The (mathematical) non–symmetric branch:
  - Sequeira for Stokes
  - Gatica & Hsiao for nonlinear problems
  - Bielak & MacCamy for elasticity (with pseudo-stress)
  - Meddahi & cols, using smooth boundaries and curved triangles, etc
  - Rapún & FJS, with nodal BEM–FEM connection
- The pro–symmetrics:
  - Stephan for nonlinear, also Gatica & Hsiao
  - Meddahi et al for mixed
  - Carstensen & Funken for mixed and non–conforming
  - Maischack for variational inequalities
  - Gatica & FJS for DG–BEM with several sequels (Bustinza, Heuer, Cockburn, Guzmán)

The new proof

The story of it (recent past)

- FJS (Somewhere over the Atlantic, May 08, pub 09): every Galerkin method for the Johnson–Nédélec coupling is equivalent to a well posed elliptic problem and hence stable.
- Steinbach (Graz, July 09): with a small modification requiring the equilibrium distribution (not needed for the Yukawa operator), the JN system is elliptic: proof uses the machinery of symmetric BIE representations of Steklov–Poincaré operators.
- Gatica, Hsiao & FJS (Delaware, August 09): the JN system is elliptic: proof uses results and representations that are already present in Nédélec–Planchard (73).

What you need to know for the new proof

If $u^* := \mathcal{S}\lambda \in H^1(\mathbb{R}^d)$, then

- $\langle \lambda, \mathrm{V}\lambda \rangle = ((u^*, u^*))_{1,\mathbb{R}^d}$
- $\partial_\nu^- u^* = \frac{1}{2}\lambda + \mathrm{K}^t\lambda$

Here K^t is the transpose of K, so that you can write

$$\langle \mathrm{K}^t\lambda, \phi \rangle = \langle \lambda, \mathrm{K}\phi \rangle, \qquad \lambda \in H^{-1/2}(\Gamma), \quad \phi \in H^{1/2}(\Gamma).$$

The new ellipticity result

$$
\begin{aligned}
a\big((u,\lambda),(u,\lambda)\big) &= ((u,u))_{1,\Omega} - \langle \lambda, \gamma u\rangle \\
&\quad \langle \lambda, (\tfrac{1}{2} - \mathrm{K})\gamma u\rangle + \langle \lambda, \mathrm{V}\lambda\rangle \\
&= ((u,u))_{1,\Omega} - \langle \lambda, (\tfrac{1}{2} + \mathrm{K})\gamma u\rangle + \langle \lambda, \mathrm{V}\lambda\rangle \\
&= ((u,u))_{1,\Omega} - \langle (\tfrac{1}{2} + \mathrm{K})^t \lambda, \gamma u\rangle + \langle \lambda, \mathrm{V}\lambda\rangle \\
&= ((u,u))_{1,\Omega} - \langle \partial_\nu^- u^*, \gamma u\rangle + ((u^*,u^*))_{1,\mathbb{R}^d} \\
&= ((u,u))_{1,\Omega} - ((u^*,u))_{1,\Omega} + ((u^*,u^*))_{1,\mathbb{R}^d} \\
&\ge \tfrac{1}{2}((u,u))_{1,\Omega} + \tfrac{1}{2}((u^*,u^*))_{1,\mathbb{R}^d} \\
&= \tfrac{1}{2}((u,u))_{1,\Omega} + \tfrac{1}{2}\langle \lambda, \mathrm{V}\lambda\rangle
\end{aligned}
$$

QUOD DEMOSTRANDUM ERAT

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&= \tfrac{1}{2}((u,u))_{1,\Omega} + \tfrac{1}{2}\langle \lambda, \mathrm{V}\lambda\rangle
\end{aligned}
$$

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The key point to keep in mind...

The quadratic form satisfies:

$$a\big((u,\lambda),(u,\lambda)\big) = ((u,u))_{1,\Omega} - ((u^*,u))_{1,\Omega} + ((u^*,u^*))_{1,\mathbb{R}^d}$$

where $u^* = \mathcal{S}\lambda$. In principle we are taking

$$u^* \in H^1(\mathbb{R}^d)$$

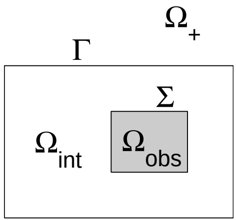
to control the term **in blue** and only then we go back to the integral equation.
Actually,

$$u^* \in \{u^* \in H^1(\mathbb{R}^d) \,:\, -\Delta u^* + u^* = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma\}.$$

We'll come back to this later.

Straightforward generalizations

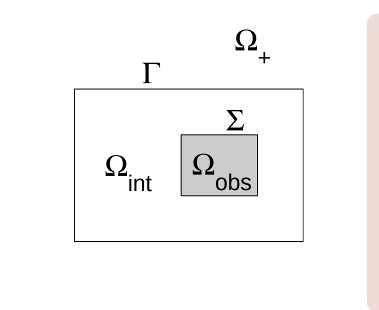
Exterior Dirichlet problem



$$
\begin{aligned}
u = 0 &\quad \text{on } \Sigma \\
-\Delta u + u = f &\quad \text{in } \Omega_{\text{int}} \\
-\Delta u^+ + u^+ = 0 &\quad \text{in } \Omega_+ \\
\text{T.C.} &\quad \text{on } \Gamma
\end{aligned}
$$

Formulation with $u \in H^1_\Sigma(\Omega_{\text{int}})$ and $\lambda = \partial_\nu^+ u \in H^{-1/2}(\Gamma)$.
Step in the analysis:

$$-\langle \partial_\nu^- u^*, \gamma u \rangle = -((u^*, u))_{1,\Omega_{\text{int}}} \geq -\frac{1}{2}\|u\|^2_{1,\Omega_{\text{int}}} - \frac{1}{2}\|u^*\|^2_{1,\mathbb{R}^d}$$



$$u = 0 \quad \text{on } \Sigma$$

$$-\Delta u = f \quad \text{in } \Omega_{\text{int}}$$

$$-\Delta u^+ = 0 \quad \text{in } \Omega_+$$

$$\text{T.C.} \quad \text{on } \Gamma$$

$$u = o(1) \quad \text{at infinity}$$

Novelty: single layer potentials are in a different space

$$u^* \in \{u \,:\, \rho\, u, \nabla u \in L^2(\mathbb{R}^3)\}, \qquad \rho(\mathbf{x}) := (1 + |\mathbf{x}|^2)^{-1/2}$$

(Weighted spaces by Leray, Beppo–Levi, etc)

Modifications:

- $\lambda \in \{\lambda \in H^{-1/2}(\Gamma) \,:\, \langle 1, \lambda \rangle = 0\}$
- New weight in the Sobolev space for the single layer potential.

Remarks:

- Bounded solutions can be obtained by adding a constant
- Logarithmically unbounded solutions (with prescribed behavior) can also be studied
- Problems in free space (2 and 3 dimensions) have energy free solutions that have to be dealt with

Variable coefficients

If we change the interior bilinear form by

$$((u, v))_{\kappa,\Omega} := \int_\Omega \Big(\kappa \nabla u \cdot \nabla v + \rho\, u\, v\Big)$$

we are dealing with the interior non–homogeneous problem

$$-\mathrm{div}\,(\kappa \nabla u) + \rho u = f,$$

and we naturally adapt the second transmission condition to

$$(\kappa \nabla u) \cdot \boldsymbol{\nu} = \partial_\nu^+ u + g_N.$$

The argument requires to think of this quadratic form

$$((u,u))_{\kappa,\Omega} - ((u^*,u))_{1,\Omega} + ((u^*,u^*))_{1,\mathbb{R}^d}$$

and *it looks like we really need*

$$\|u\|_{1,\Omega} \le C_{\mathrm{mat}} \|u\|_{\kappa,\Omega} \qquad \text{with } C_{\mathrm{mat}} < 2.$$

In a way, the boundary integral method is flooding the interior domain with energy and we need the interior material to be able to cope with all the incoming energy. (This is not entirely true.)

Non–straightforward generalizations

Waiting for the upgrade

Remember that actually $u^* = \mathcal{S}\lambda$ and therefore u^* is in the space

$$\{u^* \in H^1(\mathbb{R}^d) \,:\, -\Delta u^* + u^* = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma\}.$$

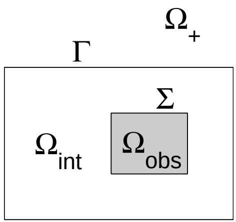
The quadratic form can (and should) be written as follows

$$a((u,\lambda),(u,\lambda)) = ((u,u))_{1,\Omega} - \langle \partial_\nu^+ u^*, \gamma u\rangle + ((u^*,u^*))_{1,\mathbb{R}^d}.$$

We are going to introduce a cut–off function φ:

$$0 \le \varphi \le 1 \qquad \begin{array}{ll} \varphi \equiv 1 & \text{near } \Gamma \\ \varphi \equiv 0 & \text{away from } \Gamma. \end{array}$$

Exterior Neumann problem



$$\begin{aligned} \partial_\nu u = 0 &\quad \text{on } \Sigma \\ -\Delta u + u = f &\quad \text{in } \Omega_{\text{int}} \\ -\Delta u^+ + u^+ = 0 &\quad \text{in } \Omega_+ \\ \text{T.C.} &\quad \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} -\langle \partial_\nu^- u^*, \gamma u \rangle &= -\langle \partial_\nu^- u^*, \gamma(\varphi u) \rangle \\ &= -((u^*, \varphi u))_{1,\Omega_{\text{int}}} \geq -\tfrac{1}{2}\|\varphi u\|^2_{1,\Omega_{\text{int}}} - \tfrac{1}{2}\|u^*\|^2_{1,\mathbb{R}^d} \\ &\geq -\tfrac{1}{2} C_\varphi \|u\|^2_{1,\Omega_{\text{int}}} - \tfrac{1}{2}\|u^*\|^2_{1,\mathbb{R}^d} \end{aligned}$$

With this idea we can...

- Deal with the Neumann problem, at the price of introducing a pre–asymptotic regime (the operator is actually **not elliptic** in many cases)
- Deal with very small coefficients inside the domain, as long as the change happens at a certain distance from the coupling boundary. There will be a pre–asymptotic regime again. Apparently this has been known in the engineering community since the beginning of times.

Conclusions and perspectives

- We have shown how a result that most people believed to be false, but most practitioners took as true, was actually true.
- We are beginning to understand the rule of *put your coupling boundary at some distance of changes of material properties*.
- In many cases, where energy is important (mathematically or practically), it looks like the symmetric coupling (using all four integral operators of Calderón's projector) is going to be the way to go: DG–BEM is one important case.
- We are trying to understand why (or figure out whether) non–symmetric coupling methods are not good for waves in the time–domain.

Thanks for your attention