Minicourse - PDE Techniques for Image Inpainting
Part III

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Outline - Numerical Computation of the Inpainted Image

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1. Unconditionally Stable Schemes
2. A Dual Approach for TV-H^{-1} Minimization

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2. A Dual Approach for TV-H^{-1} Minimization
3. Domain Decomposition for TV-H^{-1} Inpainting

Outline

Convexity Splitting

A minimizer u of an energy $\mathcal{J}(u)$ is formally computed as a stationary solution of

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\begin{aligned}
u_t &= -\nabla \mathcal{J}(u) \\
u(0) &= u_0.
\end{aligned}
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Under certain assumptions on $\mathcal{J}$ this is called a gradient system.

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If $\mathcal{J}(u)$ is convex then only a single equilibrium for the gradient system exists.

If $\mathcal{J}(u)$ is not convex multiple minimizers may exist and the gradient flow can expand $u(t)$. An explicit iterative algorithm, i.e. $u_{k+1} = u_k - \Delta t \nabla \mathcal{J}(u_k)$ in this case may require extremely small time steps, depending of course on $\mathcal{J}$. For the higher order equations $\mathcal{J}(u_k)$ contains second order derivatives resulting in a restriction of $\mathbf{\Delta t}$ up to order $(\mathbf{\Delta x})^{\mathbf{4}}$.

Convexity splitting (cont.)

The idea of convexity splitting is to derive a semi-impicit iterative scheme that is unconditionally stable.

Eyre (1998): Let

$$\mathcal{J}(u) = \mathcal{J}_c(u) - \mathcal{J}_e(u)$$

where $\mathcal{J}_c, \mathcal{J}_e$ are strictly convex. Under certain assumptions on the functionals, the numerical scheme

$$u_{k+1} = u_k - \Delta t(\nabla \mathcal{J}_c(u_{k+1}) - \nabla \mathcal{J}_e(u_k))$$

is gradient stable for every initial condition $u_0 \in \mathbb{R}$ and all $\Delta t > 0$, and possesses a unique solution for each iteration step.

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Although our inpainting models do not obey a variational principle (the are not gradient flows!), we can apply the convexity splitting method in a modified form . . .

Convexity splitting for Cahn-Hilliard inpainting

$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u),$$

where g is a given binary image.

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$$u_t = \Delta(-\epsilon\Delta u + \frac{1}{\epsilon}F'(u)) + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u),$$

where g is a given binary image.
Then the evolution of u can be written as the sum of two gradients, i.e.,

$$u_t = -\nabla_{H^{-1}}\mathcal{J}^1(u) + \nabla_{L^2}\mathcal{J}^2(u),$$

where

$$\mathcal{J}^1(u) = \int_\Omega \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}F(u)\,dx,$$

and

$$\mathcal{J}^2(u) = \frac{1}{2\lambda}\int_\Omega \chi_{\Omega\setminus D}(g-u)^2\,dx.$$

Convexity splitting for Cahn-Hilliard inpainting (cont.)

$$\mathcal{J}^1(u) = \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} F(u)\, dx,$$

$\mathcal{J}^1 = \mathcal{J}^1_c - \mathcal{J}^1_e$ with

$$\mathcal{J}^1_c(u) = \int_\Omega \frac{\epsilon}{2} |\nabla u|^2 + \frac{C_1}{2} |u|^2 \; dx,$$

and

$$\mathcal{J}^1_e(u) = \int_\Omega -\frac{1}{\epsilon} F(u) + \frac{C_1}{2} |u|^2 \; dx.$$

Convexity splitting for Cahn-Hilliard inpainting (cont.)

$$\mathcal{J}^2(u) = \frac{1}{2\lambda}\int_\Omega \chi_{\Omega\setminus D}(g-u)^2\,dx.$$

$\mathcal{J}^2 = \mathcal{J}_c^2 - \mathcal{J}_e^2$ with

$$\mathcal{J}_c^2(u) = \int_\Omega \frac{C_2}{2}\,|u|^2\,\,dx,$$

and

$$\mathcal{J}_e^2 = \frac{1}{2\lambda}\int_\Omega -\chi_{\Omega\setminus D}(g-u)^2\,dx + \int_\Omega \frac{C_2}{2}\,|u|^2\,\,dx.$$

Convexity splitting for Cahn-Hilliard inpainting (cont.)

The resulting time-stepping scheme is

$$\frac{u_{k+1}-u_k}{\tau} = -\nabla_{H^{-1}}(\mathcal{J}_c^1(u^{k+1}) - \mathcal{J}_e^1(u^k)) - \nabla_{L^2}(\mathcal{J}_c^2(u^{k+1}) - \mathcal{J}_e^2(u^k)),$$

where $\nabla_{H^{-1}}$ and ∇_{L^2} represent the Fréchet derivative with respect to the H^{-1} inner product and the L^2 inner product respectively. This translates to a numerical scheme of the form

$$\begin{aligned} & \frac{u_{k+1}-u_k}{\tau} + \epsilon\Delta\Delta u_{k+1} - C_1\Delta u_{k+1} + C_2 u_{k+1} \\ = \quad & \frac{1}{\epsilon}\Delta F'(u_k) - C_1\Delta u_k + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u_k) + C_2 u_k. \end{aligned}$$

To make sure that $\mathcal{J}_c^i, \mathcal{J}_e^i$, $i=1,2$, are convex the constants $C_1 > \frac{1}{\epsilon}$, $C_2 > 1/\lambda$.

Convexity splitting for TV-H^{-1} inpainting

A similar technique can be applied to TV-H^{-1} inpainting:

$$\begin{aligned}\frac{u^{k+1}-u^k}{\tau} + C_1\Delta^2 u^{k+1} + C_2 u^{k+1} &= C_1\Delta^2 u^k - \Delta(\nabla\cdot(\frac{\nabla u^k}{|\nabla u^k|})) \\ &\quad + C_2 u^k + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u^k),\end{aligned}$$

with constants $C_1 > \frac{1}{\epsilon}$ (where here ϵ comes from the regularization of the total variation), $C_2 > 1/\lambda$.

Rigorous results for the schemes

Cahn-Hilliard

- Consistency

TV-H^{-1}

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- Consistency
- Boundedness, i.e., unconditional stability

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Reference:

- C.-B. Schönlieb, A. Bertozzi, *Unconditionally stable schemes for higher order inpainting*, UCLA-CAM report num. 09-78.

Unconditionally stable . . . but not fast!

Lets consider again the numerical scheme for TV-H^{-1} inpainting:

$$\begin{aligned}\frac{u^{k+1}-u^k}{\tau} + C_1\Delta^2 u^{k+1} + C_2 u^{k+1} &= C_1\Delta^2 u^k - \Delta(\nabla\cdot(\frac{\nabla u^k}{|\nabla u^k|})) \\ &\quad + C_2 u^k + \frac{1}{\lambda}\chi_{\Omega\setminus D}(g-u^k),\end{aligned}$$

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The constant C_2 has to be chosen such that $C_2 > 1/\lambda$ . . .

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The constant C_2 has to be chosen such that $C_2 > 1/\lambda$ ...

... since usually in inpainting tasks λ is chosen comparatively small, e.g., $\lambda = 10^{-3}$, the condition on C_2 damps the convergence of this method $\Rightarrow$ converging slow!

Outline

TV-H^{-1} Minimization

For a given function $g \in L^2(\Omega)$ we are interested in the numerical realization of the following minimization problem

$$\min_{u \in BV(\Omega)} \mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2\lambda} \|Tu - g\|_{-1}^2 ,$$

where $T \in \mathcal{L}(L^2(\Omega))$ is a bounded linear operator and $\lambda > 0$ is a tuning parameter. The function $|Du|(\Omega)$ is the total variation of u and $\|.\|_{\mathbf{-1}}$ is the norm in $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$.

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This results in a fourth order optimality condition, e.g., if $T = Id$:

$$\Delta p + \frac{1}{\lambda}(g - u) = 0, \quad p \in \partial |Du|(\Omega),$$

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We want to numerically solve the minimization problem

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Usually: the numerical solution of TV-H^{-1} minimization depends on the specific problem at hand.

The Approach of Lieu & Vese for Denoising/Decomposition

TV-H^{-1} denoising/decomposition is solved by using the Fourier representation of the H^{-1} norm on the whole $\mathbb{R}^d$, $d \geq 1$. Thereby the space $H^{-1}(\mathbb{R}^d)$ is defined as a Hilbert space equipped with the inner product

$$\langle f, g \rangle_{-1} = \int \left(1 + |\xi|^2\right)^{-1} \hat{f}\bar{\hat{g}}\, d\xi$$

and associated norm $\|f\|_{-1} = \sqrt{\langle f, f \rangle_{-1}}$.

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$\Rightarrow$

only have to solve a 2nd order PDE

$$\begin{array}{ll} \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \left[2\, Re \left\{ \widetilde{\frac{\bar{\hat{g}} - \bar{\hat{u}}}{\left(1+|\xi|^2\right)^{-1}}} \right\} \right] = 0 & \text{in } \Omega \\ \frac{\nabla u}{|\nabla u|} \cdot \vec{n} = 0 & \text{on } \partial\Omega \\ u = 0 & \text{outside } \bar{\Omega}, \end{array}$$

Convexity splitting for TV-H^{-1} inpainting[1]

Convexity splitting: iterative scheme for TV-H^{-1} inpainting that is unconditionally stable:

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Apply convexity splitting to the two energies

$$\mathcal{J}^1(u) = \int_\Omega |\nabla u| \, dx, \quad \mathcal{J}^2(u) = \frac{1}{2\lambda} \int_\Omega \chi_{\Omega \setminus D}(u-g)^2$$

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$\Rightarrow$

$$\begin{aligned} \frac{u^{k+1}-u^k}{\tau} + C_1 \Delta^2 u^{k+1} + C_2 u^{k+1} &= C_1 \Delta^2 u^k - \Delta(\nabla \cdot (\frac{\nabla u^k}{|\nabla u^k|})) \\ &\quad + C_2 u^k + \frac{1}{\lambda}\chi_{\Omega \setminus D}(g - u^k), \end{aligned}$$

with constants $C_1 > \frac{1}{\epsilon}$ (where here ϵ comes from the regularization of the total variation), $C_2 > 1/\lambda$.

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A dual method

- Since usually in inpainting tasks λ is chosen comparatively small, e.g., $\lambda = 10^{-3}$, the condition on C_2 damps the convergence of this method $\Rightarrow$ although unconditionally stable, **converging slow**! ...
- ... this will be similar for the new approach, but with the new approach we will be able to apply **domain decomposition** to solve the minimization problem $\Rightarrow$ Parallelize the numerical computation $\Rightarrow$ **Shorten the computational time**.
- The new approach will give us a **"unified" algorithm** to solve TV-H^{-1} minimization.

A dual method (cont.)

Chambolle (04): A dual method to numerically compute a minimizer of

$$\mathcal{J}(u) = \frac{1}{2\lambda} \|u - g\|^2_{L^2(\Omega)} + |Du|(\Omega).$$

A dual method (cont.)

Chambolle (04): A dual method to numerically compute a minimizer of

$$\mathcal{J}(u) = \frac{1}{2\lambda} \|u - g\|_{L^2(\Omega)}^2 + |Du|(\Omega).$$

It amounts to compute the minimizer u of $\mathcal{J}$ as

$$u = g - \mathbb{P}_{\lambda K}(g),$$

where $\mathbb{P}_{\lambda K}$ denotes the orthogonal projection over $L^2(\Omega)$ on the convex set K which is the closure of the set

$$\left\{\nabla \cdot \xi : \xi \in C_c^1(\Omega; \mathbb{R}^2),\ |\xi(x)| \leq 1\ \forall x \in \mathbb{R}^2\right\}.$$

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$$\left\{\nabla \cdot \xi : \xi \in C^1_c(\Omega; \mathbb{R}^2),\ |\xi(x)| \leq 1\ \forall x \in \mathbb{R}^2\right\}.$$

To numerically compute the projection $\mathbb{P}_{\lambda K}(g)$ he uses a fixed point algorithm.

A dual method (cont.)

...now we want to do something similar for TV-H^{-1} minimization, i.e., we want to numerically solve

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$$\min_{u \in BV(\Omega)} \mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2\lambda} \|Tu - g\|_{-1}^2 .$$

- **First Step:** Solve the simplified problem when $T = Id$
- **Second Step:** Use the solution for $T = Id$ in order to solve the general case with the method of surrogate functionals.

A dual method (cont.)

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$$\min_u \{\mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2\lambda}\|u-g\|_{-1}^2\},$$

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With

$$s \in \partial f(x) \iff x \in \partial f^*(s),$$

this can be rewritten as

$$u \in \partial |D\cdot|(\Omega)^* \left(\frac{\Delta^{-1}(g-u)}{\lambda}\right).$$

where

$$|D\cdot|(\Omega)^*(v) = \chi_K(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{otherwise,} \end{cases}$$

and K is the closure of the set

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A dual method (cont.)

First Step:

With $w = \Delta^{-1}(g-u)/\lambda$ it reads

$$\begin{array}{ll} 0 \in (-\Delta w - g/\lambda) + \frac{1}{\lambda}\partial \left|D\cdot\right|(\Omega)^*(w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{array}$$

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In other words w is a minimizer of

$$\frac{\left\|w - \Delta^{-1}g/\lambda\right\|^2_{H_0^1(\Omega)}}{2} + \frac{1}{\lambda}\left|D\cdot\right|(\Omega)^*(w),$$

where $H_0^1(\Omega) = \left\{v \in H^1(\Omega) : \ v = 0 \text{ on } \partial\Omega\right\}$ and $\|v\|_{H_0^1(\Omega)} = \|\nabla v\|$.

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A minimizer w fulfills

$$w = \mathbb{P}_K^1(\Delta^{-1}g/\lambda),$$

where $\mathbb{P}_K^1$ is the orthogonal projection on K over $H_0^1(\Omega)$, i.e.,

$$\mathbb{P}_K^1(u) = \text{argmin}_{v\in K}\,\|u - v\|_{H_0^1(\Omega)}\,.$$

A dual method (cont.)

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In other words w is a minimizer of

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where $\mathbb{P}^1_K$ is the orthogonal projection on K over $H^1_0(\Omega)$, i.e.,

$$\mathbb{P}^1_K(u) = \operatorname{argmin}_{v\in K} \|u - v\|_{H^1_0(\Omega)}.$$

Hence the solution u of the problem is given by

$$u = g + \Delta\left(\mathbb{P}^1_{\lambda K}(\Delta^{-1} g)\right).$$

A dual method (cont.)

First Step:
Computing the nonlinear projection $\mathbb{P}^1_{\lambda K}(\Delta^{-1} g)$ amounts to solving the following problem:

$$\min \left\{ \left\| \left(\nabla \left(\lambda \nabla \cdot p - \Delta^{-1} g\right)\right)_{i,j} \right\|^2 : \; p \in Y, \; |p_{i,j}| \leq 1 \; \forall i = 1, \ldots, N; \; j = 1, \ldots, M \right\}.$$

Using the Karush-Kuhn-Tucker conditions for the above constrained minimization one can propose the following gradient descent algorithm: for an initial $p^0 = 0$, iterate for $n \geq 0$

$$p^{n+1}_{i,j} = \frac{p^n_{i,j} - \tau \left(\nabla \Delta \left(\nabla \cdot p^n - \Delta^{-1} g/\lambda\right)\right)_{i,j}}{1 + \tau \left| \left(\nabla \Delta \left(\nabla \cdot p^n - \Delta^{-1} g/\lambda\right)\right)_{i,j} \right|}.$$

Redoing the convergence proof from the paper of Chambolle we end up with a similar result:

Theorem

Let $\tau \leq 1/64$. Then, $\lambda \nabla \cdot p^n$ converges to $\mathbb{P}^1_{\lambda K}(\Delta^{-1} g)$ as $n \to \infty$.

A dual method (cont.)

Second Step:
The second step is to use the presented algorithm in order to solve

$$\min_u \{\mathcal{J}(u) = |Du|(\Omega) + \frac{1}{2\lambda} \|Tu - g\|_{-1}^2\}.$$

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Approximate a minimizer iteratively by a sequence of minimizers of, what we call, surrogate functionals $\mathcal{J}^s$. Let $\tau > 0$ be a fixed stepsize. Starting with an initial condition $u^0 = g$, we solve for $k \geq 0$

$$u^{k+1} = \mathrm{argmin}_u \mathcal{J}^s(u, u^k) = |Du|(\Omega) + \frac{1}{2\tau} \left\|u - u^k\right\|_{-1}^2 + \frac{1}{2\lambda} \left\|u - \left(g + (Id - T)u^k\right)\right\|_{-1}^2.$$

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Note that:

- A rigorous derivation of convergence properties is still missing!
- For image inpainting the surrogate functionals have a fourth order optimality conditions!

A dual method (cont.)

Second Step:
Now, the corresponding optimality condition reads

$$0 \in \partial \left|Du\right|(\Omega) + \frac{1}{\tau}\Delta^{-1}(u - u^k) + \frac{1}{\lambda}\Delta^{-1}\left(u - \left(g + (Id - T)u^k\right)\right).$$

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This can be rewritten as

$$\Delta^{-1}\left(\frac{g_1 - u}{\tau} + \frac{g_2 - u}{\lambda}\right) \in \partial |Du|(\Omega),$$

where $g_1 = u^k$, $g_2 = g + (Id - T)u^k$.

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where $g_1 = u^k$, $g_2 = g + (Id - T)u^k$.
Setting

$$g = \frac{g_1\lambda + g_2\tau}{\lambda + \tau}$$
$$\mu = \frac{\lambda\tau}{\lambda + \tau},$$

we end up with the same inclusion as before, i.e.,

$$\frac{\Delta^{-1}(g - u)}{\mu} \in \partial |Du|(\Omega).$$

A dual method (cont.)

A "unified" algorithm to solve TV-H^{-1} Minimization:

- In the case $T = Id$ directly compute a minimizer with

$$u = g + \Delta\left(\mathbb{P}^1_{\lambda K}(\Delta^{-1} g)\right).$$

- In the case $T \neq Id$ iteratively minimize the surrogate functionals by solving

$$u^k = g + \Delta\left(\mathbb{P}^1_{\lambda K}(\Delta^{-1} g)\right).$$

  in every iteration step until the two subsequent iterates u^k and u^{k+1} are sufficiently close.

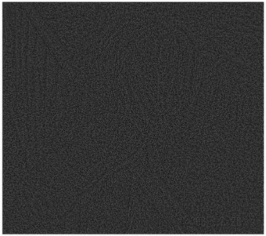
Denoising examples

(a) $g = u + v$

Figure: Noisy image with $SNR = 25.4$

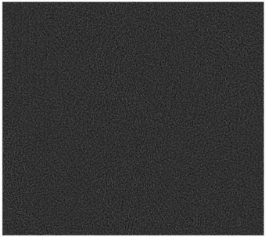
Denoising examples

(b) TV-L^2: u



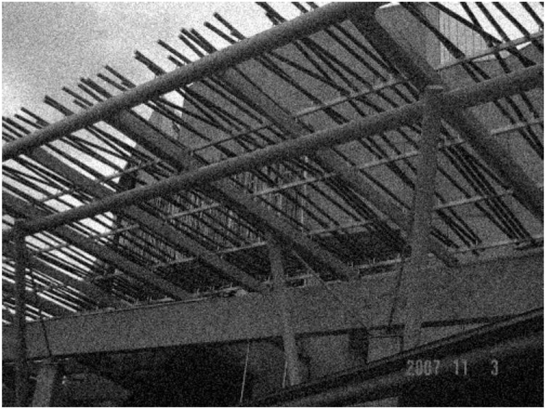
(c) TV-L^2: v

(d) TV-H^{-1}: u



(e) TV-H^{-1}: v

Denoising examples

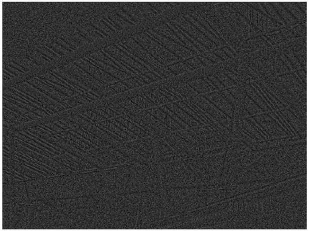


(a) $g = u + v$

Figure: Noisy image with $SNR = 29.4$

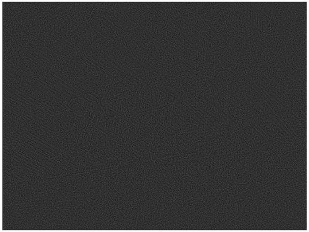
Denoising examples

(b) TV-L^2: u



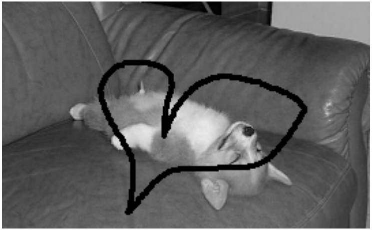
(c) TV-L^2: v

(d) TV-H^{-1}: u



(e) TV-H^{-1}: v

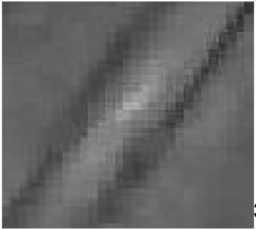
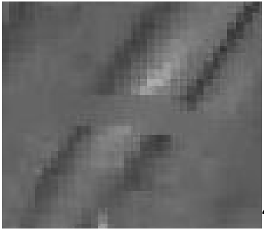
Inpainting examples




[2]

[2] u(1000) with $\lambda = 10^{-3}$.

4th order versus 2nd order method

[3] [4]

[3] TV-H^{-1} u(1000) with $\lambda = 10^{-3}$.
[4] TV-L^2 u(5000) with $\lambda = 10^{-3}$.

Outline

1. Unconditionally Stable Schemes
2. A Dual Approach for TV-H^{-1} Minimization
3. Domain Decomposition for TV-H^{-1} Inpainting

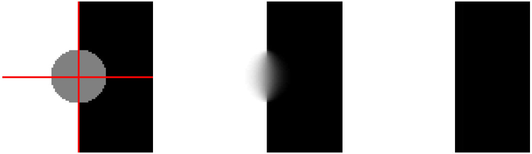
Domain Decomposition for TV-H^{-1} inpainting

Why domain decomposition?

Domain Decomposition for TV-H^{-1} inpainting

Why domain decomposition?

Speed up the numerical computation of minimizers! Parallel Computations are possible!



Domain Decomposition for TV-H^{-1} inpainting (cont.)

Domain Decomposition:

- Split the domain Ω into two arbitrary nonoverlapping domains $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$.
- Let $\mathcal{H} = L^2(\Omega)$ and $V_i = L^2(\Omega_i)$, where $\mathcal{H} = V_1 \oplus V_2$.

Domain Decomposition for TV-H^{-1} inpainting (cont.)

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Pick an initial $V_1 \oplus V_2 \ni u_1^0 + u_2^0 := u^0 \in BV(\Omega)$, for example $u^0 = 0$, and iterate

$$\begin{cases} u_1^{n+1} \approx \operatorname{argmin}_{u_1 \in V_1} \mathcal{J}(u_1 + u_2^n) \\ u_2^{n+1} \approx \operatorname{argmin}_{u_2 \in V_2} \mathcal{J}(u_1^{n+1} + u_2) \\ u^{n+1} := u_1^{n+1} + u_2^{n+1}. \end{cases}$$

Domain Decomposition for TV-H^{-1} inpainting (cont.)

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This is implemented by solving the subspace minimization problems via an **oblique thresholding iteration** (Fornasier, Schönlieb 08).

Domain Decomposition for TV-H^{-1} inpainting (cont.)

A minimizer u_1^{k+1} of the subproblem on Ω_1 can be iteratively computed (again by means of surrogate functionals) as

$$u_1^{k+1} = -\Delta \left(Id - \mathbb{P}^1_{\mu K}\right)\left(\Delta^{-1}(z + u_2) - \mu\eta\right) - u_2.$$

Domain Decomposition for TV-H^{-1} inpainting (cont.)

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$$u_1^{k+1} = -\Delta \left(Id - \mathbb{P}^1_{\mu K}\right) \left(\Delta^{-1}(z + u_2) - \mu\eta\right) - u_2.$$

where η fulfills

$$\eta = \frac{1}{\mu}\Pi_{V_2}\left[\mathbb{P}^1_{\mu K}\left(\mu\eta - \Delta^{-1}(u_2 + z)\right)\right].$$

which can be computed via the iteration

$$\eta^0 \in V_2, \quad \eta^{m+1} = \frac{1}{\mu}\Pi_{V_2}\left[\mathbb{P}^1_{\mu K}\left(\mu\eta^m - \Delta^{-1}(u_2 + z)\right)\right], \quad m \geq 0.$$

Domain Decomposition for TV-H^{-1} inpainting (cont.)

In sum we solve TV-H^{-1} inpainting by the alternating subspace minimizations: Pick an initial $V_1 \oplus V_2 \ni u_1^{0,L} + u_2^{0,M} := u^0 \in \mathcal{B}V(\Omega)$, for example $u^0 = 0$, and iterate

$$\begin{cases} \begin{cases} u_1^{n+1,0} = u_1^{n,L} \\ u_1^{n+1,\ell+1} = \operatorname{argmin}_{u_1 \in V_1} \mathcal{J}_1^s(u_1 + u_2^{n,M}, u_1^{n+1,\ell}) \quad \ell = 0, \ldots, L-1 \end{cases} \\ \begin{cases} u_2^{n+1,0} = u_2^{n,M} \\ u_2^{n+1,m+1} = \operatorname{argmin}_{u_2 \in V_2} \mathcal{J}_2^s(u_1^{n+1,L} + u_2, u_2^{n+1,m}) \quad m = 0, \ldots, M-1 \end{cases} \\ u^{n+1} := u_1^{n+1,L} + u_2^{n+1,M}, \end{cases}$$

Domain Decomposition for TV-H^{-1} inpainting (cont.)

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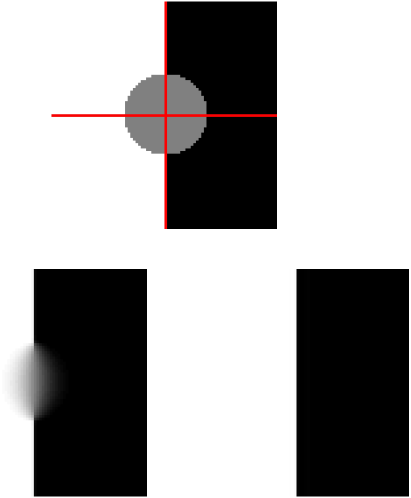
where each subminimization problem is computed by the oblique thresholding algorithm.

Domain decomposition results



Domain decomposition results

References

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- M. Fornasier, C.-B. Schönlieb, *Subspace correction methods for total variation and ℓ_1- minimization*, SIAM J. Numer. Anal., Vol.47, No.5, pp. 3397-3428 (2009).
- C.-B. Schönlieb, A. Bertozzi, *Unconditionally stable schemes for higher order inpainting*, UCLA-CAM report num. 09-78, 32 p.
- The Matlab Code for the domain decomposition method is available at: http://homepage.univie.ac.at/carola.schoenlieb/webpage_tvdode/tv_dode_numerics.htm

For more details see http://homepage.univie.ac.at/carola.schoenlieb

or write to: c.b.schonlieb@damtp.cam.ac.uk