HELICOIDS AND VORTICES

HAO CHEN AND DANIEL FREESE

Abstract. We point out an interesting connection between fluid dynamics and minimal surface theory: When gluing helicoids into a minimal surface, the limit positions of the helicoids correspond to a "vortex crystal", an equilibrium of point vortices in 2D fluid that move together as a rigid body. While vortex crystals have been studied for almost 150 years, the gluing construction of minimal surfaces is relatively new. As a consequence of the connection, we obtain many new minimal surfaces and some new vortex crystals by simply comparing notes.

In 2005, Traizet and Weber [TW05] glued helicoids into screw-motion invariant minimal surfaces with helicoidal ends. For the glue construction to succeed, the limit positions of the helicoids must satisfy a balancing condition and a nondegenerate condition. For simplicity, they assumed that the helicoids are aligned along a straight line, and noticed that the roots of Hermite polynomials provide examples of balanced and nondegenerate configurations. Recently, the second named author implemented a similar construction without this assumption [Fre21].

The main goal of this short note is to point out an interdisciplinary connection: The balanced configurations of helicoids correspond to binary vortex crystals. Here, a *vortex crystal* [ANS$^+$03], also known as *vortex equilibrium*, is a configuration of vortices in 2D fluids that moves as a rigid body, i.e. without any change of shape and size. A vortex crystal is *binary* if the circulations of vortices are ± 1. We will recall vortex dynamics in Section 1. For an example of this connection: the "definite" configurations in [TW05], given by roots of Hermite polynomials, correspond to vortex crystals that trace back to 19th century [Sti85].

Rotating vortex crystals correspond to screw-motion invariant minimal surfaces; this connection was readily established in [TW05, Fre21]. Our main results in Section 2 establish the other cases of the claimed connection. The construction will be given in Section 4, where we glue helicoids into translation-invariant minimal surfaces, corresponding to translating or periodic stationary vortex crystals. The construction will only be sketched because similar constructions have been repeated many times in the literature [Tra08a, Tra08b, CT21].

In view of the connection, we will compare notes between fluid dynamics and minimal surface theory, and obtain new examples in Section 3 for both vortex crystals and minimal surfaces. The minimal surface theory would benefit a lot because, in about 150 years, the fluid dynamics community has accumulated a large collection of examples of binary vortex crystals. In particular, stationary and translating vortex crystals have been analytically obtained with the help of Adler–Moser polynomials. When symmetries are imposed, nondegeneracy was recently verified for translating Adler–Moser examples [LW20]. They then lead to many

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examples. On the other hand, the idea to glue helicoids into minimal surfaces is only about 18 years old. Nevertheless the minimal surface community is also in possession of a few examples that would lead to new vortex crystals.

Remark 1. Minimal surfaces obtained by gluing helicoids find applications in natural sciences as models for topological defects (screw dislocations) in layered structures such as smectic liquid crystals [KS06,SK07,MKS12], biological membranes [TSK+13] and nuclear pasta [HBB+15,SBC+16].

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1. Vortex dynamics

We recommend [ANS+03] for general reference on vortex crystals.

Incompressible and inviscid flow in zero gravity is governed by the Euler equation

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) u = -\nabla p \tag{1}$$

under the incompressible condition $\nabla \cdot u = 0$, where u is the flow velocity field and p is the pressure. The pressure term can be eliminated by taking the curl of the Euler equation, which results in the reformulation

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) w = w \cdot \nabla u \tag{2}$$

in terms of the vorticity field $w := \nabla \times u$. For 2-dimensional flows, the right-hand side of (2) vanishes, giving the transport equation

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) w = 0.$$

We see that vorticity is transported in the velocity field as material elements.

A point vortex in the plane is given by the vorticity field

$$w(x) = \frac{\sigma}{2\pi}\delta(x),$$

where δ is the Dirac delta function, and $\sigma \in \mathbb{R}$ is the *circulation* of the vortex. In the complex coordinate, a point vortex gives rise to a velocity field

$$\overline{u(z)} = \frac{1}{2\pi \mathrm{i}}\frac{\sigma}{z}.$$

We use $(p_k, \sigma_k)_{1\le k\le n}$ to denote a configuration of n point vortices located at p_k with circulation σ_k, $k = 1, \cdots, n$. Each vortex is advected as a material particle by the velocity field produced by other vortices. So the dynamics of the vortex configuration is governed by the ordinary differential equation

$$\frac{d}{dt}\overline{p_j} = \frac{1}{2\pi \mathrm{i}}\sum_{k\neq j}\frac{\sigma_k}{p_j - p_k}, \quad \forall 1 \le j \le n. \tag{3}$$

We say that the configuration $(p_k, \sigma_k)_{1\le k\le n}$ is a *vortex crystal* if it moves as a rigid body. If this is the case, we have $dp_j/dt = v + \mathrm{i}\omega p_j$, where $v \in \mathbb{C}$ and $\omega \in \mathbb{R}$ are constant for all vortices. Hence vortex crystals are characterized by the algebraic equations

$$F_j := -\overline{v} + \mathrm{i}\omega\overline{p_j} + \frac{1}{2\pi \mathrm{i}}\sum_{k\neq j}\frac{\sigma_k}{p_j - p_k} = 0, \quad \forall 1 \le j \le n. \tag{4}$$

Multiply (4) by σ_j, and sum the conjugates over j, we obtain

$$v\sum_{j=1}^{n}\sigma_j + \mathrm{i}\omega\sum_{j=1}^{n}\sigma_j p_j = 0. \tag{5}$$

Multiply (4) by $\sigma_j p_j$, and take the sum over j, we obtain

$$\overline{v}\sum_{j=1}^{n}\sigma_j p_j - \mathrm{i}\omega\sum_{j=1}^{n}\sigma_j|p_j|^2 = \frac{1}{4\pi\mathrm{i}}\Big[\Big(\sum_{k=1}^{n}\sigma_k\Big)^2 - \sum_{k=1}^{n}\sigma_k^2\Big]. \tag{6}$$

In this paper, we only consider binary vortex crystals, so $\sigma_k = \pm 1$. Let $n_\pm$ be, respectively, the number of vortices with circulation ± 1, and write $m = n_+ - n_- = \sum \sigma_k$. We distinguish three cases,

(1) We say that the vortex crystal is *rotating* if $\omega \neq 0$. In this case, the governing equation (4) is invariant under Euclidean rotations. Moreover, we may assume that $v = 0$ up to a translation, and that $\omega = 1$ up to a Euclidean scaling. After these normalizations, (5) and (6) give

$$\sum_{k=1}^{n}\sigma_k p_k = 0 \quad \text{and} \quad \sum_{k=1}^{n}\sigma_k|p_k|^2 = \frac{m^2-n}{4\pi}.$$

(2) We say that the vortex crystal is *translating* if $\omega = 0$ but $v \neq 0$. Then (5) implies that $m = 0$ so $n_+ = n_- = n/2$. In this case, (4) is invariant under translations. We may assume that $v = 1$ up to a complex scaling (Euclidean scaling and rotation). After this normalization, (6) implies that

$$\sum_{k=1}^{n}\sigma_k p_k = -\frac{n}{4\pi\mathrm{i}}.$$

(3) We say that the vortex crystal is *stationary* if $\omega = 0$ and $v = 0$. Then (6) implies that $m^2 = n$, hence n_+ and n_- must be successive triangular numbers. In this case, (4) is invariant under Euclidean similarities (translations, rotations, and scalings).

A vortex crystal is said to be *stable* if any sufficiently small perturbation does not diverge. Linear and nonlinear analyses have been carried out on the stability. We say that a vortex crystal is *nondegenerate*[1] if the Jacobian matrix $DF = (\frac{\partial F_i}{\partial p_j})_{i,j}$ for the autonomous system (3) has the maximum possible rank. We have seen that, for a rotating (resp. translating, stationary) vortex crystal, the dynamics is invariant under Euclidean rotations (resp. translations, similarities), so the maximum possible real rank of its Jacobian is $2n-1$ (resp. $2n-2$, $2n-4$).

We may also consider singly or doubly periodic vortex crystals. In this case, it is convenient to consider a co-rotating reference frame in which the periods are fixed. Then the vortex crystal is either translating or stationary.

- Assume that the vortex crystal is *singly periodic*, i.e. invariant under a single translation $T \in \mathbb{C}$. Up to rotations and scalings, we may fix $T = 1$. Then the vortices can be seen as lying in the annulus $\mathbb{C}/\langle 1\rangle$. Up to translations, a vortex crystal in the annulus is governed by

$$F_j := -\overline{v} + \frac{1}{2\pi\mathrm{i}}\sum_{k\neq j}\pi\sigma_k\cot\pi(p_j - p_k) = 0, \quad \forall 1 \leq j \leq n$$

  in an appropriate reference frame.
- Assume that the vortex crystal is *doubly periodic*, i.e. invariant under two linearly independent translations T_1 and T_2. Up to rotations and scalings, we may fix $T_1 = 1$ and $T_2 = \tau \in \mathbb{C}$. Then the vortices can be seen as lying

[1] In fluid dynamics literature, degenerate vortex crystals were often said to be "neutrally stable".

in the flat torus $\mathbb{C}/\langle 1,\tau\rangle$. Up to translations, a vortex crystal in the torus is governed by

$$F_j := -\overline{v} + \frac{1}{2\pi \mathrm{i}} \sum_{k\neq j} \sigma_k \big(\zeta(p_j - p_k;\tau) - \xi(p_j - p_k;\tau)\big) = 0, \quad \forall 1 \le j \le n$$

in an appropriate reference frame, where $\zeta(z;\tau)$ is the Weierstrass zeta function on the torus $\mathbb{C}/\langle 1,\tau\rangle$ and $\xi(z;\tau) = 2x\zeta(1/2;\tau) + 2y\zeta(\tau/2;\tau)$ with $z = x + y\tau$, $x, y \in \mathbb{R}$.

In either case, the maximum possible real rank of the Jacobian is $2n-2$, which defines the degeneracy of these vortex crystals.

2. Main results

The connection between rotating vortex crystals and screw-motion invariant minimal surfaces was already established in the following theorem [Fre21].

Theorem 1 (Rotating vortex crystal)**.** *Let $(p_k^\circ, \sigma_k)_{1\le k\le n}$ be a normalized nondegenerate rotating vortex crystal with n vortices at $p_1^\circ, \dots, p_n^\circ \in \mathbb{C}$. Then there exists a one-parameter family $(M_\varepsilon)_{0<\varepsilon<\delta}$ of embedded minimal surfaces in $\mathbb{R}^3$ such that:*

(1) M_ε admits a screw symmetry S_ε composed of a vertical translation $2\pi(0,0,1)$ and a rotation around the vertical axis by an angle $2\pi\varepsilon^2$.
(2) The quotient $M_\varepsilon/S_\varepsilon$ is of genus $n-1$ and has two ends. The ends are helicoidal if $m = \sum \sigma_k \neq 0$, or planar if $m = 0$.
(3) As $\varepsilon \to 0$, up to a translation, M_ε converges to a helicoid of period $(0,0,2\pi)$ in the neighborhood of $(p_i^\circ/\varepsilon, 0)$ for each $1 \le i \le n$. The helicoid is right-handed (resp. left-handed) if $\sigma_i = 1$ (resp. -1).
(4) After rescaling the horizontal coordinates by ε, the resulting surface (no longer minimal) converges to the union of the multigraph of the multivalued function

$$f(z) = \sum_{i=1}^{n} \sigma_i \arg(z - p_i^\circ), \qquad z \in \mathbb{C} - \{p_1^\circ, \cdots, p_n^\circ\},$$

the multigraph of $f(z) + \pi$, and vertical lines over the points p_i°.

So we only need to construct translation invariant minimal surfaces, corresponding to translating or stationary vortex crystals. More specifically, we will prove the theorems below.

Theorem 2 (Finite translating vortex crystals)**.** *Let $(p_k^\circ, \sigma_k)_{1\le k\le n}$ be a nondegenerate translating vortex crystal normalized with velocity $v = 1$, with n vortices at $p_1^\circ, \dots, p_n^\circ \in \mathbb{C}$. Then there exists a one-parameter family $(M_\varepsilon)_{0<\varepsilon<\delta}$ of embedded minimal surfaces in $\mathbb{R}^3$ such that:*

(1) M_ε admits a translational symmetry $T_{0,\varepsilon} = 2\pi(-2\pi\varepsilon, 0, 1)$. So M_ε is a singly periodic minimal surface in $\mathbb{R}^3$.
(2) The quotient $M_\varepsilon/T_{0,\varepsilon}$ is of genus $n-1$ and has two helicoidal ends if the vortex crystal is translating;
(3) As $\varepsilon \to 0$, up to a translation, M_ε converges to a helicoid of period $(0,0,2\pi)$ in the neighborhood of $(p_i^\circ/\varepsilon, 0)$ for each $1 \le i \le n$. The helicoid is right-handed (resp. left-handed) if $\sigma_i = 1$ (resp. -1).
(4) After rescaling the horizontal coordinates by ε, the resulting surface (no longer minimal) converges to the union of the multigraph of the multivalued

function

$$f(z)=\sum_{i=1}^{n}\sigma_i\arg(z-p_i^\circ),\qquad z\in\mathbb{C}-\{p_1^\circ,\cdots,p_n^\circ\},$$

the multigraph of $f(z)+\pi$*, and vertical lines over the points* p_i°*.*

Unfortunately, we are aware of very few finite, nondegenerate, translating vortex crystals; see Examples 5. So the theorem above does not bring us many examples.

We will need a version of the theorem with imposed symmetry. For a vortex crystal, a *circulation-preserving* (resp. *-reversing*) symmetry is a Euclidean isometry that maps vortices to identical (resp. opposite) vortices. When a symmetry group is imposed, the vortex crystal is said to be *nondegenerate* if the only perturbations that preserve the balance as well as the symmetry are the trivial ones, namely Euclidean translations for translating vortex crystals.

Theorem 3 (Vortex crystals with imposed symmetry)**.** *Let* $(p_k^\circ,\sigma_k)_{1\le k\le n}$ *be a normalized translating vortex crystal with* n *vortices at* $p_1^\circ,\ \dots,\ p_n^\circ\in\mathbb{C}$*. Let* G *be a symmetry group of* (p_k°,σ_k)*. If the vortex crystal is nondegenerate with the symmetry group* G *imposed, then the conclusion of Theorem 2 holds. Moreover, the symmetry group* G *induces a symmetry group of the resulting minimal surfaces.*

In particular, a circulation-reversing reflection in the vortex crystal induces an order-2 rotational symmetry around a straight line in the minimal surface, and a circulation-preserving reflection induces a reflection symmetry for the minimal surface.

Remark 2. We did not manage to establish a similar connection between singly periodic minimal surfaces and finite stationary vortex crystals. See Remark 6 for detailed explanation.

Periodic vortex crystals will give rise to doubly or triply periodic minimal surfaces, as stated in the following theorems.

Theorem 4 (Singly periodic vortex crystals)**.** *Let* $(p_k^\circ,\sigma_k)_{1\le k\le n}$ *be a nondegenerate vortex crystal with* n *vortices at* $p_1^\circ,\ \dots,\ p_n^\circ\in\mathbb{C}/\langle 1\rangle$*. Then there exists a one-parameter family* $(M_\varepsilon)_{0<\varepsilon<\delta}$ *of embedded minimal surfaces in* $\mathbb{R}^3$ *such that:*

(1) M_ε *admits translational symmetries along the vectors*

$$T_{0,\varepsilon}=2\pi(\varepsilon\operatorname{Re}\nu,\varepsilon\operatorname{Im}\nu,1)\quad\text{and}\quad T_{1,\varepsilon}=(\varepsilon^{-1},0,m\pi),$$

where $m=\sum\sigma_k$*,* ν *is related to the velocity* v *of the translating vortex crystal by* $\nu=-2\pi v$*, and* $\nu=0$ *if the vortex crystal is stationary. So* M_ε *is a doubly periodic minimal surface.*

(2) The quotient $M_\varepsilon/\langle T_{0,\varepsilon},T_{1,\varepsilon}\rangle$ *is of genus* $n-1$ *and has four Scherk ends (asymptotic to half-planes).*

(3) The flux vector along any closed curve in $M_\varepsilon/\langle T_{0,\varepsilon},T_{1,\varepsilon}\rangle$ *has no vertical component.*

(4) As $\varepsilon\to 0$*, up to a translation,* $M_\varepsilon/\langle T_{1,\varepsilon}\rangle$ *converges to a helicoid of period* $(0,0,2\pi)$ *in the neighborhood of* $(p_i^\circ/\varepsilon,0)$ *for each* $1\le i\le n$*. The helicoid is right-handed (resp. left-handed) if* $\sigma_i=1$ *(resp.* -1*).*

Recall that the flux vector along a closed curve Γ is defined as the integral of the conormal vector along Γ [PR93]. It can be physically interpreted as the surface tension force along the curve.

Theorem 5 (Doubly periodic vortex crystals)**.** *Let* $(p_k^\circ,\sigma_k)_{1\le k\le n}$ *be a nondegenerate vortex crystal with* n *vortices at* $p_1^\circ,\ \dots,\ p_n^\circ\in\mathbb{T}=\mathbb{C}/\langle 1,\tau\rangle$*. Assume that* $m=\sum\sigma_k=0$*. Then there exists a one-parameter family* $(M_\varepsilon)_{0<\varepsilon<\delta}$ *of embedded minimal surfaces in* $\mathbb{R}^3$ *such that:*

(1) M_ε admits a translational symmetry along the vectors

$$\begin{aligned} T_{0,\varepsilon} &= 2\pi(\varepsilon \operatorname{Re}\nu, \varepsilon \operatorname{Im}\nu, 1), \\ T_{1,\varepsilon} &= (\varepsilon^{-1}, 0, \Psi_1(\varepsilon)), \\ T_{2,\varepsilon} &= (\varepsilon^{-1}\operatorname{Re}\tau, \varepsilon^{-1}\operatorname{Im}\tau, \Psi_2(\varepsilon)), \end{aligned}$$

where ν is related to the velocity v of the translating vortex crystal by $\nu = -2\pi v$, and $\nu = 0$ if the vortex crystal is stationary. So M_ε is a triply periodic minimal surface.

(2) The quotient $M_\varepsilon/\langle T_{0,\varepsilon}, T_{1,\varepsilon}, T_{2,\varepsilon}\rangle$ is of genus $n+1$.

(3) The flux vector along any closed curve in $M_\varepsilon/\langle T_{0,\varepsilon}, T_{1,\varepsilon}, T_{2,\varepsilon}\rangle$ has no vertical component.

(4) As $\varepsilon \to 0$, up to a translation, $M_\varepsilon/\langle T_{1,\varepsilon}, T_{2,\varepsilon}\rangle$ converges to a helicoid of period $(0,0,2\pi)$ in the neighborhood of $(p_i^\circ/\varepsilon, 0)$ for each $1 \le i \le n$. The helicoid is right-handed (resp. left-handed) if $\sigma_i = 1$ (resp. -1). Moreover, we have

$$\Psi_1(\varepsilon) \to -2\pi y \text{ and } \Psi_2(\varepsilon) \to 2\pi x \text{ as } \varepsilon \to 0,$$

where $(x, y) \in \mathbb{R}^2$ are defined by $\sum \sigma_k p_k = x + y\tau$.

Remark 3. In the Theorems, the surface is rotated into a position so that the flux vectors are horizontal. We find this choice best to reveal the connection to vortex crystals. The price is that Ψ_1 and Ψ_2 are left to vary with ε. One could also rotate the surface to fix $\Psi_1 = \Psi_2 \equiv 0$. Then the flux vectors are not horizontal, and the connection to vortex crystals is less direct.

3. Examples

The study on vortex crystals traces back to about 150 years ago, and has accumulated plenty of examples, many of which are binary hence would imply minimal surfaces. A nice survey of these examples is provided by Aref et al. [ANS+03]. Here, we examine the known vortex crystals and their corresponding minimal surfaces. Occasionally, we also have minimal surfaces that lead to new vortex crystals.

Example 1 (Linear configuration). Traizet and Weber [TW05] considered configurations of helicoids along a straight line. In particular, when p_i°, $1 \le i \le n$, are the roots of H_n, the Hermite polynomial of degree n, and $\sigma_i = -1$ for all i, then the configuration $(p_i^\circ, \sigma_i)_{1\le i\le n}$ is balanced and nondegenerate. In fluid dynamics, the corresponding rotating vortex crystal was first found by Stieltjes in 1885 [Sti85], and has been rediscovered and revisited many times [Sze59, Mar49, EFN51].

Traizet and Weber [TW05] also considered a configuration with $n = 2m + 1$ helicoids, $m + 1$ of which lie at the roots of H_{m+1} and have positive handedness, and the remaining m lie at the roots H_m and have negative handedness. This configuration is proved to be balanced and nondegenerate. We are not aware of any discussion of corresponding vortex crystals in the fluid dynamics community. So this is a new example of vortex crystal inspired by the minimal surfaces. □

Example 2 (Polygonal configuration). A Karcher–Scherk tower with $2n$ wings can be twisted into a configuration of n negatively handed helicoids lying at the vertices of a regular polygon; see Figure 1 and [Fre21, Prop. 8.11]. The Fischer–Koch surfaces can be twisted into a similar configuration, only with an extra helicoid, positively or negatively handed, at the center of the polygon; see [Fre21, Prop. 8.13].

The corresponding rotating vortex crystals were first investigated by Thomson in 1883 [Tho82]. In particular, he famously proved that identical vortices at the vertices of a regular n-gon is linearly stable if $n \le 6$, and linearly unstable if $n \ge 8$,

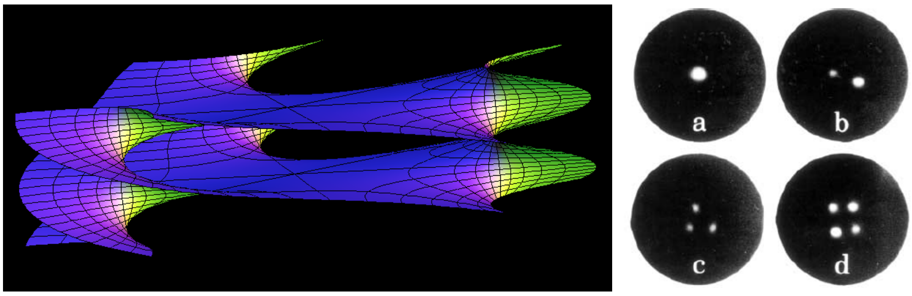


Figure 1. Left: A twisted Karcher–Scherk tower of six wings near the limit consisting of three helicoids at the vertices of an equiangular triangle (source: 3D-XplorMath Minimal Surface Gallery). Right: Thomson's vortex polygons seen in superfluid Helium [YGP79].

and "neutrally stable" if $n = 7$. The proof was improved and modified many times [Hav31, Dri85, Are95].

For our minimal surfaces, this stability analysis means that the Helicoid limit of twisted Scherk surface is nondegenerate for $n \neq 7$, for which Theorem 2 applies. The $n = 7$ case is degenerate, but becomes nondegenerate if we impose the dihedral symmetry of the heptagon [Fre21], so Theorem 3 applies. Even without imposed symmetry, it was proved that the nonlinear stability still holds for $n = 7$ [KY02], so a more elaborated version of the implicit function theorem might apply. □

Example 3 (Nested polygonal configurations). Fluid dynamists also investigated vortex crystals where vortices lie on the vertices of several concentric regular polygons [Hav31, CZ78, AvB05]. In some cases, e.g. when the vortices lie on two concentric polygons, with or without an extra vortex at the center, an algebraic approach is possible. However, we are not aware of any systematic investigation on such configurations.

This line of research overlaps with [Fre21], where the second named author considered configurations with dihedral symmetry, and helicoids all lie on the symmetry lines (including the center). Each dihedral configuration of helicoids can be seen as corresponding to a nested polygonal vortex crystal.

For instance, the Callahan-Hoffman-Meeks surface [CHM89] can be deformed to a helicoid limit where the segment between the two positively handed helicoids and the segment between the two negatively handed helicoids bisect each other perpendicularly. This corresponds to a vortex crystal where vortices lie on concentric 2-gons. □

Example 4 (Numerical examples). Campbell and Ziff [CZ78, CZ79] have obtained numerical examples of vortex crystals, and claimed to have found linearly stable configurations with up to 30 identical vortices. Their 1978 report is often referred to as the *Los Alamos Catalog*. Many of their examples were experimentally observed in superfluid Helium [YGP79]; see Figure 1, right. By our connection, they all correspond to screw-motion invariant minimal surfaces.

In many of their examples, the vortices seem to lie on concentric rings, but this impression is not precise. Rather, all their examples admit an axis of symmetry. Asymmetric examples were not found until [AV98]. □

Example 5 (Translating vortex crystal). A translating vortex crystal must consist of an even number n of vortices, with $n/2$ positive vortices and $n/2$ negative ones. The simplest case is a pair of opposite vortices, corresponding to Riemann minimal

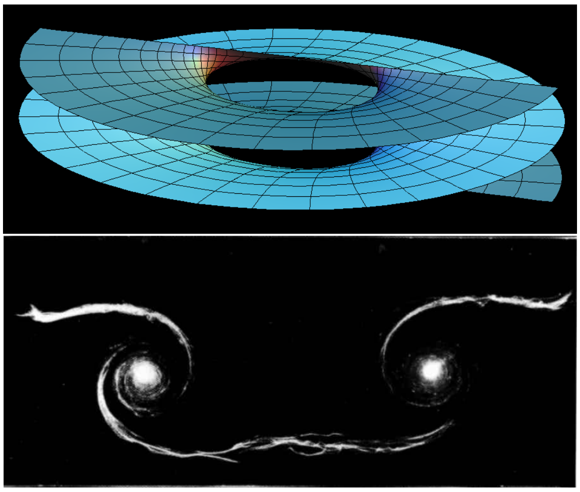


FIGURE 2. Top: A Riemann minimal example near the limit consisting of a pair of opposite helicoids (source: 3D-XplorMath Minimal Surface Gallery). Bottom: A pair of opposite vortices seen in a cross-section of the vortex sheet behind a wing. (source: H. Bipps [vD82, p. 50]).

examples; see Figure 2. Surprisingly, there is no solution for $n/2 = 2$. Some examples for $n/2 = 3$ and $n/2 = 6$ can be found in [KC87].

More generally, when $n/2 = j(j+1)/2$ is a triangular number, binary translating vortex crystals have been found [Bar84, CK87, Cla09] with positive vortices at the roots of a j-th Adler–Moser polynomial Θ_j, and negative vortices at the roots of the corresponding modified Adler–Moser polynomial $\tilde{\Theta}_j$.

The Adler–Moser polynomial $\Theta_j(z)$ actually depends on m complex parameters $\kappa_1 = z, \kappa_2, \dots, \kappa_j$. Changing these parameters preserves the balance. In fact, these perturbations are linearly independent and span the kernel of the Jacobian [LW20]. As a consequence, the Adler–Moser translating configurations are degenerate except for the trivial case $n = 2$.

However, under the assumption that Θ_j has only simple roots, these configurations are proved to be nondegenerate [LW20] if we impose (up to Euclidean motions) a reflection symmetry in the real axis that preserves circulations and a reflection symmetry in the imaginary axis that reverses circulations. In fact, this symmetry is realized by a unique choice of parameters $\kappa_2, \dots, \kappa_j$. With this choice, the assumption that Θ_j has only simple roots is verified for $j \leq 34$ [LW20].

By Theorem 3, these symmetric translating Adler–Moser configurations give rise to singly periodic minimal surfaces. The reflection in the real axis becomes a rotational symmetry in the x-axis, and the reflection in the imaginary axis becomes a reflection in the yz-plane. See Figure 3 for an example with $n/2 = 6$. □

Example 6 (Singly periodic vortex crystals). The doubly periodic Scherk surface can be deformed to a periodic helicoid limit. It corresponds to a singly periodic vortex crystal with a single vortex in the period.

The famous vortex street of von Kármán [vK12], and more general cases of Dolaptschiew and Maue [Mau40], are the only singly periodic vortex crystals with two (opposite) vortices in the period. They correspond to the helicoid limits of Karcher–Meeks–Rosenberg surfaces [Kar88, MR88]. See Figure 4. □

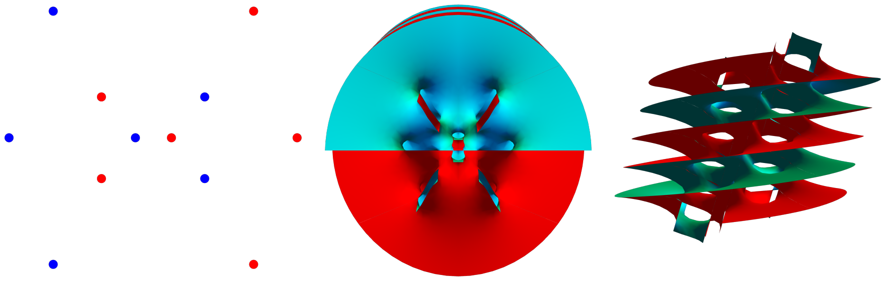


FIGURE 3. Left: An Adler–Moser translating vortex crystal with $n/2 = 6$. Middle: Top view of a singly periodic minimal surface arising from the corresponding helicoid configuration. Right: Side view of the same minimal surface.

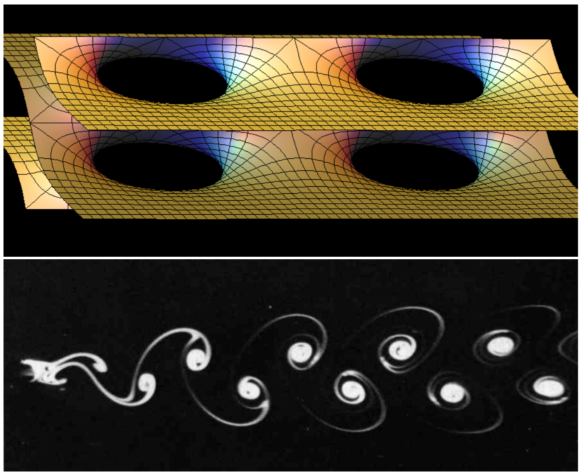
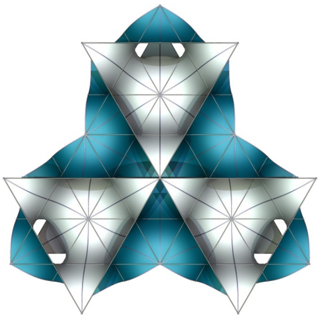


FIGURE 4. Top: A doubly periodic Karcher–Meeks–Rosenberg surface near the limit consisting of a row of alternating helicoids (source: 3D-XplorMath Minimal Surface Gallery). Bottom: A von-Kármán vortex street. (source: Sadatoshi Taneda [vD82, p. 57]).

Example 7 (Doubly periodic vortex crystals). If a doubly periodic vortex crystal has two vortices in the period, they must be of opposite circulations. Such a configuration is generically nondegenerate. They give rise to triply periodic minimal surfaces of genus 3 (TPMSg3); see Figure 5 for an example. In fact, our construction will assume an orientation-reversing translation, hence the produced examples must all belong to the 5-parameter family of TPMSg3s constructed by Meeks [Mee90]. □

4. Sketched construction

In [TW05, Fre21], the main technical issue was the multivaluedness of the Weierstrass data. For translating invariant minimal surfaces, the Weierstrass data is single-valued, so the construction is much easier. The construction below is adapted from [Tra08a]. This approach has been repeated many times in the literature [Tra08a, Tra08b, CT21] and is only simpler in our context; hence we will only present a sketch.



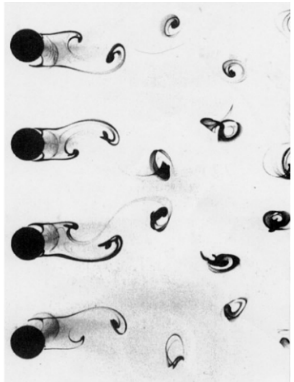


FIGURE 5. Left: A triply periodic rPD surface near the limit consisting of helicoids arranged in a hexagonal lattice (source: Matthias Weber). Right: A doubly periodic vortex crystal seen in the wake behind a row of cylinders. (source: Toshio Kobayashi [JSME84, p. 43]).

4.1. **Weierstrass data.** Let Σ_+ be

- the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ if the vortex crystal is finite;
- the annuli $\mathbb{C}/\langle 1 \rangle$ if the vortex crystal is singly periodic;
- the torus $\mathbb{T}_+ = \mathbb{C}/\langle 1, \tau_+ \rangle$ if the vortex crystal is doubly periodic.

Moreover, let Σ_- be

- Another copy of Σ_+ if the vortex crystal is finite or singly periodic;
- the torus $\mathbb{T}_- = \mathbb{C}/\langle 1, \tau_- \rangle$ if the vortex crystal is doubly periodic.

Consider n points $p = (p_k)_{1 \le k \le n}$ in Σ_+ and n points $q = (q_k)_{1 \le k \le n}$ in Σ_-.

The node-opening is parameterized by n complex numbers $t = (t_k)_{1 \le k \le n}$. If $t = 0$, we identify $p_k \in \mathbb{C}_+$ and $q_k \in \mathbb{C}_-$ to form a node. The resulting singular Riemann surface with nodes is denoted Σ_0. If $t \ne 0$, we open the nodes as follows. Let $z_\pm$ be the standard coordinates of $\mathbb{C}_\pm$. Consider local coordinates $w_k^+ = z_+ - p_k$ in the neighborhood of p_k and $w_k^- = z_- - q_k$ in the neighborhood of q_k. Fix a small $\delta > 0$ such that the disks $|w_k^\pm| < \delta$ are all disjoint. Then for every $1 \le k \le n$, we remove the disk $|w_{k,\pm}| < |t_k|/\delta$, and identify the annuli

$$|t_k|/\delta \le |w_k^+| \le \delta \qquad \text{and} \qquad |t_k|/\delta \le |w_k^-| \le \delta$$

by

$$w_k^+ w_k^- = t_k.$$

This produces a Riemann surface that we denote Σ_t.

We construct the minimal surface using the Weierstrass parameterization

$$\Sigma_t \ni z \mapsto \operatorname{Re} \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3)$$

where ϕ_1, ϕ_2, and ϕ_3 are meromorphic forms on Σ_t satisfying the conformality condition

$$Q = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \tag{7}$$

Then the flux vector along a closed curve Γ is given by [PR93]

$$\operatorname{Im} \int_\Gamma (\phi_1, \phi_2, \phi_3) \, .$$

4.2. **Equations.** We define

$$\Omega_\pm = \{z \in \Sigma_\pm \mid |w_k^\pm| > \delta \,\text{for all}\, 1 \le k \le n\}.$$

Let A_k be an anticlockwise circle in Ω_+ around p_k and A'_k be an anticlockwise circle in Ω_- around q_k. Note that A_k is homologous in Σ_t to $-A'_k$. Let B_k be a cycle in Σ_ε which goes "half-way up" from Σ_+ to Σ_- through the helicoind near p_1 then "half-way down" through the helicoid near p_k, as in [Fre21]. We need to solve the period problems

$$\begin{aligned} &\operatorname{Re}\int_{A_k} (\phi_1, \phi_2, \phi_3) = 2\pi\sigma_k(\varepsilon \operatorname{Re}\nu, \varepsilon \operatorname{Im}\nu, 1), \\ &\operatorname{Re}\int_{B_k} (\phi_1, \phi_2, \phi_3) = (0, 0, 0). \end{aligned} \tag{8}$$

We close the A-period by defining ϕ's as the unique meromorphic forms satisfying

$$\int_{A_k} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = 2\pi\mathrm{i}(\alpha_k - \mathrm{i}\sigma_k\varepsilon^2 \operatorname{Re}\nu, \beta_k - \mathrm{i}\sigma_k\varepsilon^2 \operatorname{Im}\nu, \gamma_k - \mathrm{i}\sigma_k\varepsilon),$$

where $\tilde\phi_i := \varepsilon\phi_i$ are the rescaled Weierstrass data. Depending on the type of the vortex crystal, we also require the following

- if the vortex crystal is finite and translating, we want $\tilde\phi_1$ and $\tilde\phi_2$ to have double poles at $\infty_\pm$. Up to rotations and scalings, we assume that

$$\tilde\phi_1 \sim dz_\pm + \mathcal{O}(z_\pm^{-2})dz_\pm \quad \text{and} \quad \tilde\phi_2 \sim \mp\mathrm{i}dz_\pm + \mathcal{O}(z_\pm^{-2})dz_\pm \quad \text{at } \infty_\pm. \tag{9}$$

  On the other hand, since the minimal surfaces have planar ends, $\tilde\phi_3$ must be holomorphic at $\infty_\pm$. Consequently, we must have $m = \sum \sigma_k = 0$.
- if the vortex crystal is singly periodic, let $A_\pm \subset \Sigma_\pm$ be the segments $\{t \mp K\mathrm{i} \mid 0 \le t \le 1\}$, where $K > |\operatorname{Im} p_k|$ and $K > |\operatorname{Im} q_k|$ for all k. We want that

$$\operatorname{Re}\int_{A_\pm} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = (1, 0, \varepsilon m\pi).$$

  So we require that

$$\int_{A_\pm} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = (1 + \mathrm{i}\alpha_\pm, \mathrm{i}\beta_\pm, \varepsilon m\pi + \mathrm{i}\gamma_\pm).$$

  The flux vectors $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ (or rather their inverse) can be physically interpreted as the surface tension forces along the Scherk ends. Up to a rotation around the x-axis (corresponding to the real axis), we may assume that $\gamma_+ \equiv 0$.
- if the vortex crystal is doubly periodic, let $A_\pm \subset \Omega_\pm$ be curves homologous in $\Sigma_\pm$ to the segment from 0 to 1, and $B_\pm \subset \Omega_\pm$ be curves homologous to the segment from 0 to $\tau_\pm$. We want that

$$\begin{aligned} &\operatorname{Re}\int_{A_\pm} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = (1, 0, \varepsilon\Psi_1), \\ &\operatorname{Re}\int_{B_\pm} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = (\operatorname{Re}\tau, \operatorname{Im}\tau, \varepsilon\Psi_2). \end{aligned} \tag{10}$$

  So we require that

$$\int_{A_\pm} (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3) = (1 + \mathrm{i}\alpha_\pm, \mathrm{i}\beta_\pm, \varepsilon\Psi_1 + \mathrm{i}\gamma_\pm).$$

  Again, the flux vectors can be physically interpreted as the surface tension forces. Up to a Euclidean rotation, we may assume that the periods of $\tilde\phi_3$ over A_+ and

B_+ are real. That is

$$\gamma_+ = \operatorname{Im} \int_{A_+} \tilde{\phi}_3 = 0 \quad \text{and} \quad \operatorname{Im} \int_{B_+} \tilde{\phi}_3 = 0.$$

In any of these cases, the ϕ's are uniquely determined by the requirements above.

Write $\tilde{Q} := \varepsilon^2 Q$. The conformality condition (7) is equivalent to

$$\mathcal{E}_k := \int_{A_k} \frac{\tilde{Q} w_k^+}{dz_+} = 0, \qquad 1 \le k \le n, \tag{11}$$

$$\mathcal{F}_k := \int_{A_k} \frac{\tilde{Q}}{dz_+} = 0, \qquad 1 \le k \le n, \tag{12}$$

$$\mathcal{F}'_k := \int_{A'_k} \frac{\tilde{Q}}{dz_-} = 0, \qquad 1 \le k \le n, \tag{13}$$

and, if the vortex crystal is periodic,

$$\int_{A_\pm} \frac{\tilde{Q}}{dz_\pm} = 0. \tag{14}$$

Note that (12) are not independent. One dependence comes from the residue theorem, namely that

$$\sum_{k=1}^{n} \mathcal{F}_k = 0. \tag{15}$$

4.3. **Solutions.** To facilitate the solution, we will construct minimal surfaces with an orientation-reversing translational symmetry R_ε such that $R_\varepsilon^2 = T_{0,\varepsilon}$. We want R_ε to correspond to the symmetry

$$\iota : \Sigma_+ \ni z \mapsto \overline{z} \in \Sigma_-.$$

More specifically, we want that

$$\iota^*(\phi_1, \phi_2, \phi_3) = (\overline{\phi_1}, \overline{\phi_2}, \overline{\phi_3}).$$

This can be achieved by assuming that

$$q_k \equiv \overline{p}_k, \quad t_k \in \mathbb{R} \quad \text{and} \quad (\alpha_k, \beta_k, \gamma_k) \equiv 0.$$

In the doubly periodic case, we also assume that $\tau_- \equiv \overline{\tau_+}$. As a consequence, the B-period problem (8) are automatically solved. Indeed, we can choose B-curves such that $B_k + \iota(B_k)$ is homologous to $\sigma_1 A_1 - \sigma_k A_k$, for which all ϕ periods are pure imaginary; see [Fre21]. In the periodic cases, since $\iota(A_+) = A_-$, we also have $\gamma_- = \gamma_+ \equiv 0$. In the doubly periodic case, since $\iota(B_+) = B_-$, the period of $\tilde{\phi}_3$ over B_- must also be real. Note that the A-curves and B-curves form a homology basis. Now that the vertical components of the flux vectors vanish along all these curves, they must vanish along any closed curve.

Moreover, since $\iota^*(Q) = \overline{Q}$ and $\iota(A_k) = A_k = -A'_k$, we have

$$\mathcal{E}_k = \int_{A_k} \frac{\tilde{Q} w_k^+}{dz_+} = \int_{\iota(A_k)} \iota^* \Big(\frac{\tilde{Q} w_k^+}{dz_+} \Big) = - \int_{A'_k} \frac{\overline{\tilde{Q}} w_k^-}{dz_-} = -\overline{\mathcal{E}_k},$$

and

$$\mathcal{F}_k = \int_{A_k} \frac{\tilde{Q}}{dz_+} = \int_{\iota(A_k)} \iota^* \Big(\frac{\tilde{Q}}{dz_+} \Big) = - \int_{A'_k} \frac{\overline{\tilde{Q}}}{dz_-} = -\overline{\mathcal{F}'_k}.$$

This means that $\operatorname{Re} \mathcal{E}_k = 0$ and (13) is automatically solved if (12) is solved.

Remark 4. Without the symmetry ι, we can use the Implicit Function Theorem to prove the existence of q_k, $\operatorname{Im} t_k$, α_k, β_k, γ_k, and τ_- that solve the B-period problem and the conformality equation (13).

Similar arguments as in [Tra08a], using the Implicit Function Theorem, show that

Proposition 6. *For ε in a neighborhood of* 0 *and p_k in a neighborhood of its central values p_k°, there exist unique values for parameters t_k, $\alpha_\pm$, $\beta_\pm$, and τ_+, depending smoothly on ε and p_k, such that the imaginary part of* (11)*, as well as* (14) *in the periodic cases, and* (10) *in the doubly periodic case, are solved. At $\varepsilon = 0$, we have $t_k = 0$ in all cases, $\alpha_\pm = 0$, $\beta_\pm = \mp 1$ in the periodic cases, and $\tau_+ = \tau$ in the doubly periodic case, where τ is the torus parameter for the given doubly periodic vortex crystal.*

A sketched proof for the proposition is delayed to Appendix A.

At $\varepsilon = 0$, we have

$$\tilde\phi_1 = dz_\pm \quad \text{and} \quad \tilde\phi_2 = \mp \mathrm{i} dz_\pm \quad \text{on} \quad \Sigma_\pm.$$

On Σ_+, $\phi_3 = \tilde\phi_3/\varepsilon$ extends smoothly to $\varepsilon = 0$ with the explicit form

$$\sum -\mathrm{i}\sigma_k \Upsilon(z - p_k) dz \tag{16}$$

where

$$\Upsilon(z) := \begin{cases} 1/z & \text{if } (p_k, \sigma_k) \text{ is finite;} \\ \pi\cot(\pi z) & \text{if } (p_k, \sigma_k) \text{ is singly periodic;} \\ \big(\zeta(z;\tau) - \xi(z;\tau)\big) & \text{if } (p_k, \sigma_k) \text{ is doubly periodic.} \end{cases}$$

One then verifies that, at $\varepsilon = 0$, we have indeed

$$\int_{A_\pm} \phi_3 = m\pi \in \mathbb{R}$$

if (p_k, σ_k) is singly periodic and (cf. [Tra08a, § 4.3.1] and [CT21, § 5])

$$\int_{A_\pm} \phi_3 = -2\pi y \in \mathbb{R} \quad \text{and} \quad \int_{B_\pm} \phi_3 = 2\pi x \in \mathbb{R}$$

if (p_k, σ_k) is doubly periodic, as we have assumed. Here $(x, y) \in \mathbb{R}^2$ are defined by $\sum \sigma_k p_k = x + y\tau$.

Then, as $\varepsilon \to 0$, $\mathcal{F}_{j,+}/\varepsilon^2$ converges smoothly to the value

$$\begin{aligned} \frac{\partial \mathcal{F}_{j,+}}{\partial \varepsilon^2}\Big|_{\varepsilon=0} &= \int_{A_j} \Big(\frac{2\tilde\phi_1}{dz_+} \frac{\partial \tilde\phi_1}{\partial \varepsilon^2} + \frac{2\tilde\phi_2}{dz_+} \frac{\partial \tilde\phi_2}{\partial \varepsilon^2} \Big) + 2\pi i \operatorname{Res}_{p_j} \frac{\phi_3^2}{dz_+} \\ &= \frac{\partial}{\partial \varepsilon^2} \int_{A_j} (2\tilde\phi_1 - 2i\tilde\phi_2) - 2\pi i \operatorname{Res}_{p_j} \Big(\sum_k \sigma_k \Upsilon(z - p_k) \Big)^2 \\ &= 4\pi\sigma_j \operatorname{Re}\nu - 4\pi \mathrm{i}\sigma_j \operatorname{Im}\nu - 4\pi\mathrm{i}\sigma_j \sum_{k\neq j} \sigma_k \Upsilon(p_j - p_k) \\ &= 4\pi\sigma_j \Big[\overline{\nu} - \mathrm{i} \sum_{k\neq j} \sigma_k \Upsilon(p_j - p_k) \Big] = 8\pi^2 F_j, \end{aligned}$$

which vanishes if and only if (p_k) and (σ_k) are the positions and circulations of a binary vortex crystal with velocity $v = -\nu/2\pi$. Recall that $\mathcal{F}_k$ are related by (15). Therefore, if the vortex crystal is not a finite stationary one, and is nondegenerate (possibly with imposed symmetry), we may apply the Implicit Function Theorem to prove the following proposition that concludes the construction of a family of immersed surfaces.

Proposition 7. *If* (p_k°) *and* (σ_k) *are the positions and circulations of a binary vortex crystal that is nondegenerate (possibly with imposed symmetry). Then for* ε *in a neighborhood of* 0*, there exist unique values for* p_k*, depending smoothly on* ε*, such that* $p_k(0) = p_k^\circ$ *and the conformality condition* (12) *is solved.*

Finally, a similar argument as in [TW05, Fre21] proves that the constructed surfaces are all embedded for ε sufficiently small.

Remark 5. The same construction can be carried out without the symmetry ι (see Remark 4) and the conclusion is the same. By uniqueness of the implicit functions, this implies that all minimal surfaces sufficiently close to a balanced and nondegenerate configuration of helicoids admit an orientation-reversing symmetry.

Remark 6. For finite stationary vortex crystals, we may fix two vortices to quotient out Euclidean similarities, leaving $n-2$ free complex parameters. But in the last step of the construction, we have n complex equations $F_k = 0$ to solve. The relation (15) eliminates one complex equation, Riemann Bilinear relation can eliminate a real equation (see [Tra02a, Tra08b]), but we still have one real equation too many. Hence the construction does not work. This is compatible with Traizet's observation that catenoids cannot be glued into a single periodic minimal surface with vertical periods [Tra02b].

Remark 7. In [Tra15], a correspondence was established between hollow vortices and minimal surfaces bounded by horizontal symmetry curves. More specifically, the height differential of the minimal surface corresponds to the velocity field of the fluid. A similar correspondence can be seen in our construction by noticing from (16) that, at $\varepsilon = 0$, $\phi_3 = 2\pi\overline{u}dz$, where u is the flow vector field generated by the vortex crystal (away from the vortices).

Appendix A. Proof of proposition 6

Similar arguments as in [Tra08a] apply here.

When $\varepsilon = 0$, all the A_k-periods of $\tilde\phi$'s vanish, and thus $\tilde\phi_3$ converges to 0. If the vortex crystal if finite, $\tilde\phi_1$ and $\tilde\phi_2$ converge to holomorphic forms in $\mathbb{C}$, and their limits are determined by their behavior at ∞. In view of the assumptions (9), we have

$$\tilde\phi_1 \to dz_\pm \quad \text{and} \quad \tilde\phi_2 \to \mp \mathrm{i} dz_\pm$$

in $\Sigma_\pm$ as $\varepsilon \to 0$, and one easily verifies that $\tilde Q$ converges to 0.

If the vortex crystal is periodic, $\tilde\phi_1$ and $\tilde\phi_2$ converge to holomorphic forms in $\Sigma_\pm$ and are determined by their $A_\pm$ periods. More specifically, at $\varepsilon = 0$, we have

$$\tilde\phi_1 \to (1 + i\alpha_\pm) dz_\pm \quad \text{and} \quad \tilde\phi_2 \to \mathrm{i}\beta_\pm dz_\pm$$

in $\Sigma_\pm$ as $\varepsilon \to 0$. Whence we have,

$$\tilde Q = (\tilde\phi_1)^2 + (\tilde\phi_2)^2 \to (1 + 2i\alpha_\pm - \alpha_\pm^2 - \beta_\pm^2) dz_\pm^2.$$

In order for this to vanish, we need $\alpha_\pm = 0$ and $\beta_\pm = \mp 1$. The sign of $\beta_\pm$ is chosen so the surface has the desired orientation. Hence, again, we have the limit

$$\tilde\phi_1 \to dz_\pm \quad \text{and} \quad \tilde\phi_2 \to \mp \mathrm{i} dz_\pm$$

in $\Sigma_\pm$ as $\varepsilon \to 0$.

The remaining proof focuses on Σ_+. To ease the notations, we write dz in the place of dz_+. For periodic vortex crystals, we compute the partial derivatives

$$\frac{\partial}{\partial \alpha_+} \int_{A+} \frac{\tilde Q}{dz} = 2\mathrm{i}, \qquad \frac{\partial}{\partial \beta_+} \int_{A_+} \frac{\tilde Q}{dz} = 2,$$

and all other partial derivatives of $\int_{A+} \tilde{Q}/dz$ vanish. By [Tra08a, Lemma 3], we have the partial derivatives

$$\frac{\partial \mathcal{E}_k}{\partial t_k} = \frac{\partial}{\partial t_k} \int_{A_k} \frac{(z-p_k)\tilde{Q}}{dz} = \int_{A_k} (z-p_k) \left(2\frac{\partial \tilde{\phi}_1}{\partial t_k} - 2\mathrm{i}\frac{\partial \tilde{\phi}_2}{\partial t_k} \right) = -8\pi\mathrm{i},$$

and all other partial derivatives of $\mathcal{E}_k$ vanish.

Finally, we compute, at $\varepsilon = 0$,

$$(\operatorname{Re}\tau, \operatorname{Im}\tau) = \operatorname{Re}\int_{B_+} (\tilde{\phi}_1, \tilde{\phi}_2) = \operatorname{Re}\int_0^{\tau_+} (dz, -\mathrm{i}dz) = (\operatorname{Re}\tau_+, \operatorname{Im}\tau_+),$$

which determines $\tau_+ = \tau$.

The proposition then follows from the Implicit Function Theorem.

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(Chen) Georg-August-Universität Göttingen, Institut für Numerische und Angewandte Mathematik
Email address: hao.chen.math@gmail.com

(Freese) Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
Email address: djfreese@iu.edu