# Showcase: Convection-Diffusion on a moving domain

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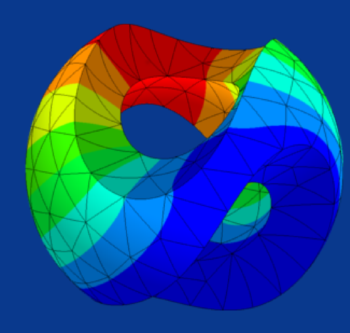


## Convection-Diffusion problem

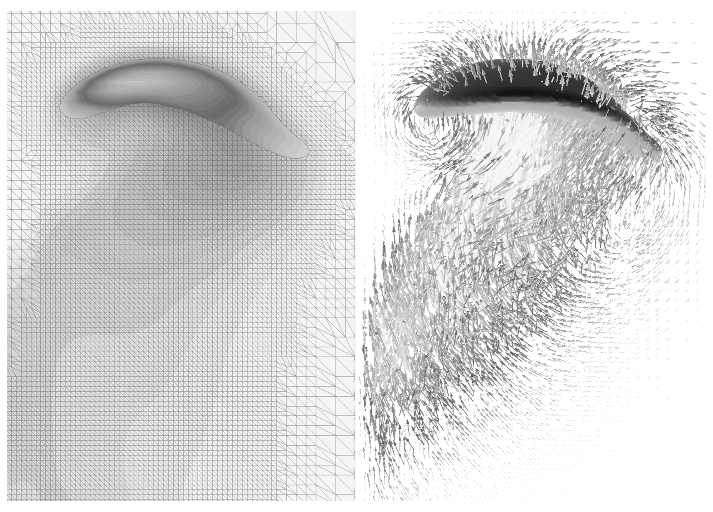
- Imagine a chemical species in a volume $\Omega$, such as a liquid.
- Concentration can be represented as $u = u(x, y, z, t)$.
- Physical process I: **Diffusion** will smear out close-by differences in concentration.
- Physical process II: **Convection** by a velocity field $\mathbf{w}$ represents transport.
- These are given by the following differential equation: Find $u(x, y, z, t)$ s.t.

$$\partial_t u + \mathbf{w} \cdot \nabla u - \alpha \Delta u = f \qquad \text{in } \Omega \text{ for all } t,$$
$$\nabla u \cdot \mathbf{n}_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \text{ for all } t.$$

- Example: Concentration of ink in water:

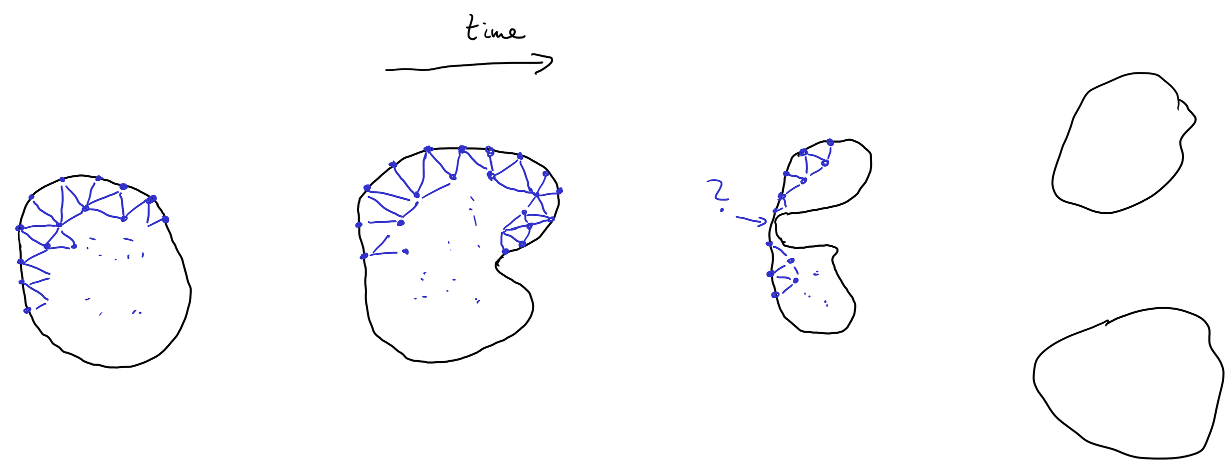
## Generalisation moving domain

- We consider a moving domain, i.e. $\Omega = \Omega(t)$.
- This is motivated e.g. by multi-phase flows.
- Example: droplet of oil in water.
- These are particulary challenging for computer simulations.



## Finite Element Method (Fitted)

- A mesh is generated in alignment with geometry.
- 2D: Triangles/ rectangles, 3D: Tetrahedrons/ boxes.
- Problem is posed on each element, yielding a linear algebra problem $A\mathbf{x} = \mathbf{b}$.
- Solving for $x$, one obtains a solution function, consisting of polynomials per element.
- Challenge: Mesh transfer can be complicated in flow cases with topology changes:

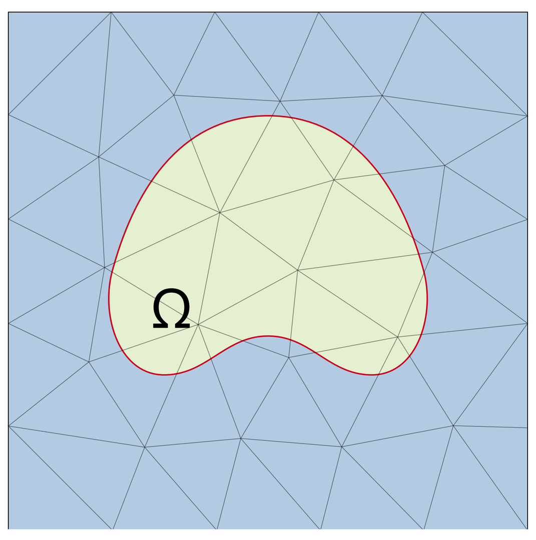



## Unfitted Finite Element Method

- To solve these problems, we decouple mesh generation and geometry.
- A levelset function $\phi(x, y, z)$ is used to define geometry:

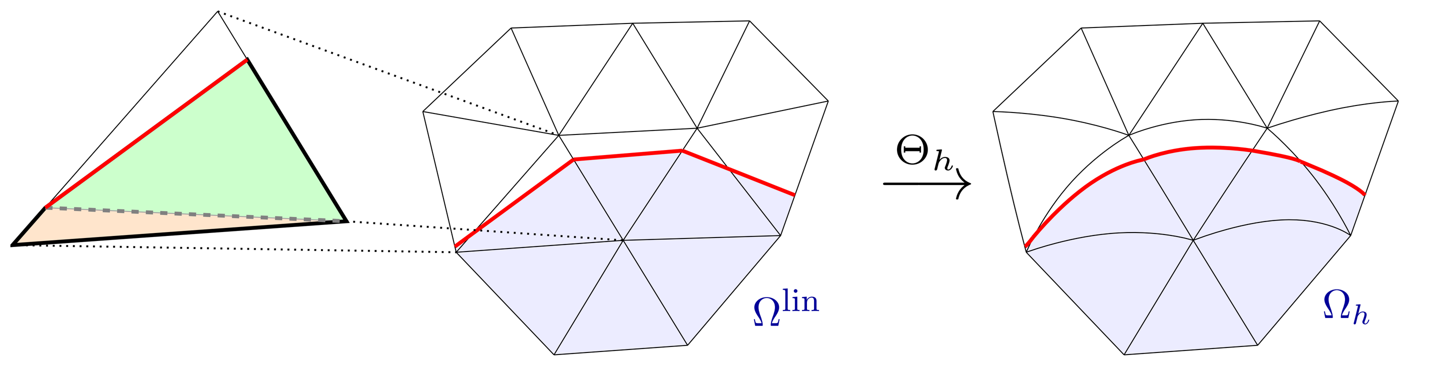
$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, \phi(x, y, z) \leq 0\}.$$

- In particular, elementwise linear $\phi^{\text{lin}}$ are computationally feasible. These correspond to polygonal geometry approximations.




## Higher order geometry approximation

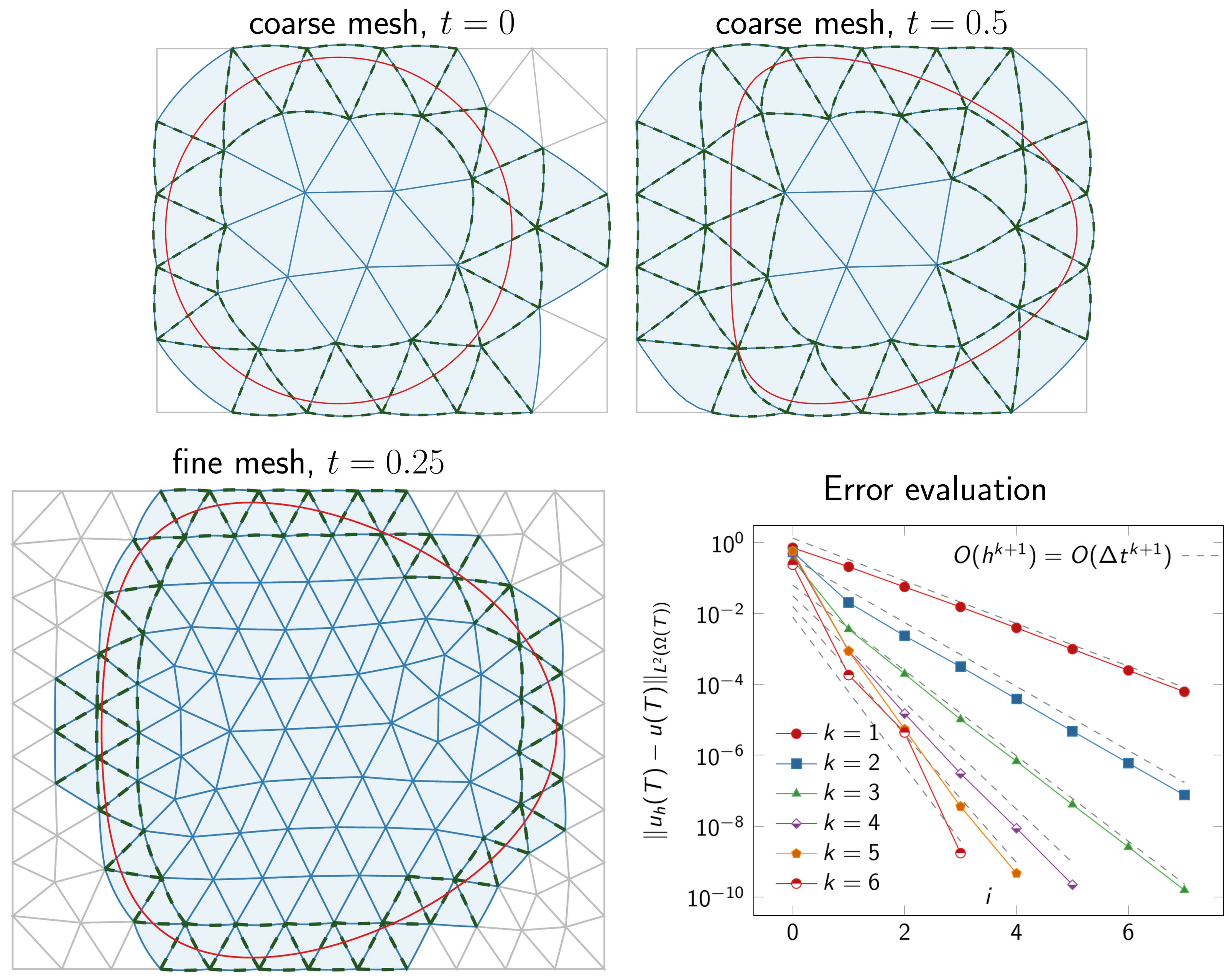
- Let $h$ denote the maximum size of an element.
- Then, the polygonal geometry approximation will lead to a error of order $h^2$.
- A mapping/ mesh deformation $\Theta$ is used to increase the accuracy of the approximation.




## Example 1: Kite Geometry

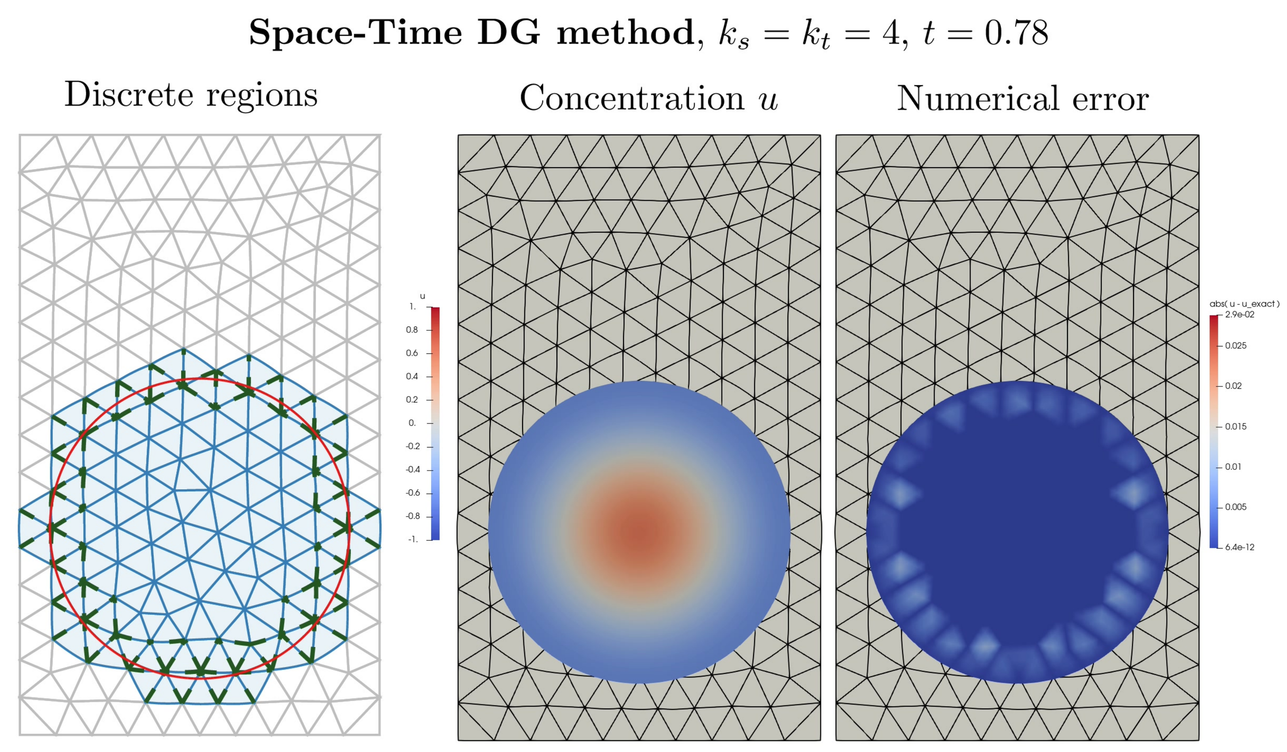
- Test geometry: Circle in 2D deforms into a kite shape.
- To assess numerical quality of the simulation, we prescribe a solution function $u$.
- The according right hand side $f$ is calculated by pen and paper.

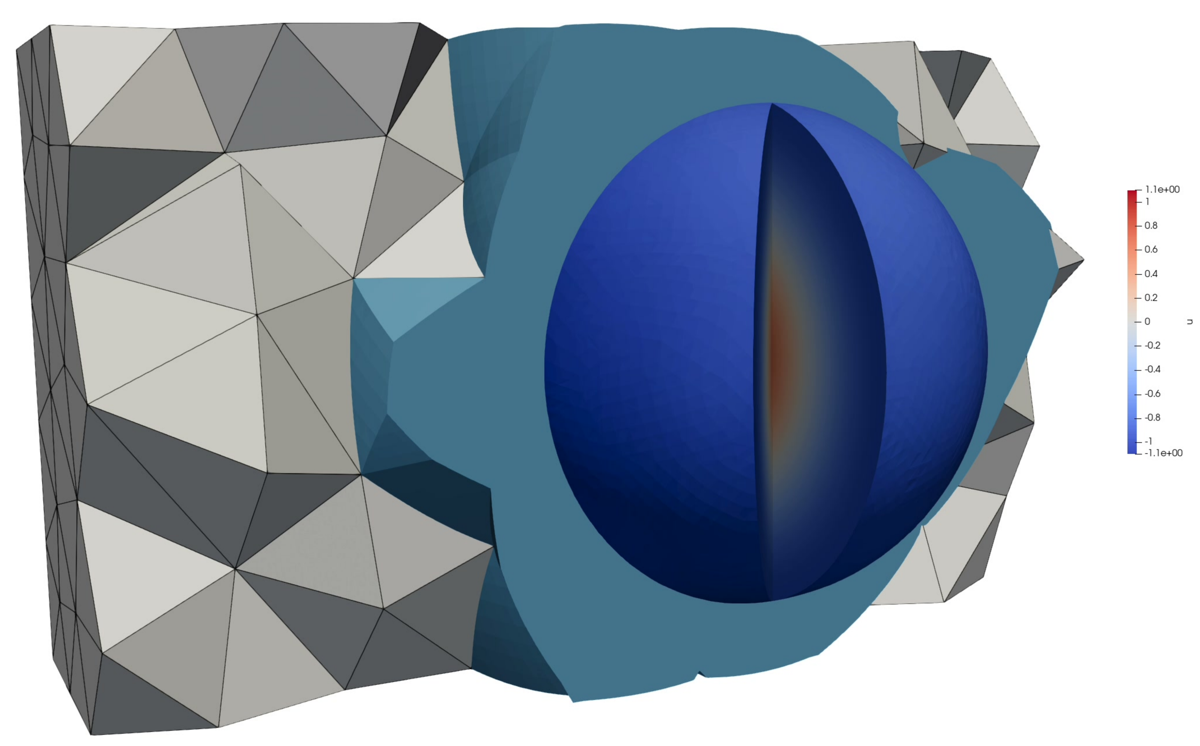
$\Rightarrow$ Measurement of the numerical error possible. We achieve errors of order $h^p$ for arbitrary $p$.




## Example 2: Moving $n$-sphere

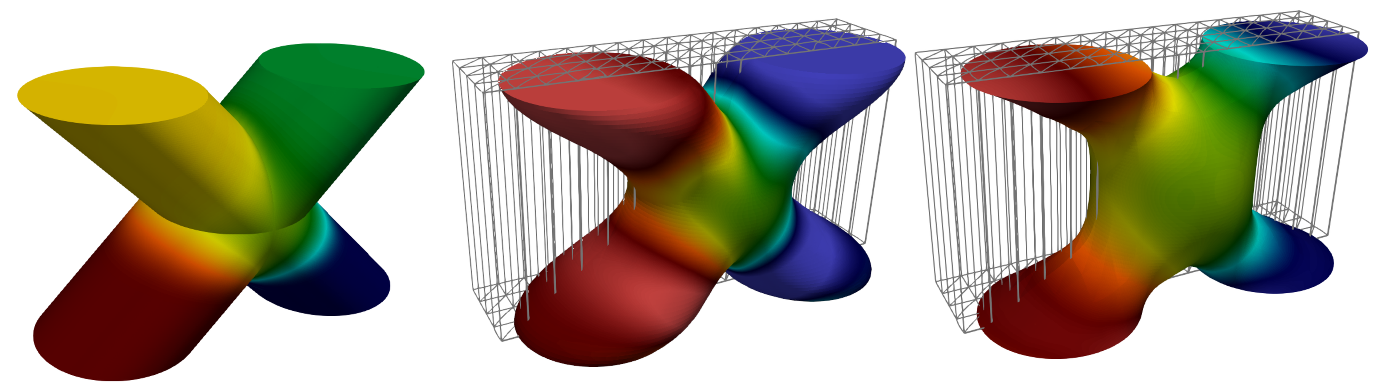
- Next, we consider a moving circle in 2D/ sphere in 3D. Method performs similarly.
- 1D convergence study in Binder available.



## Example 3: Colliding circles

- Two circles merge and seperate afterwards. $\Rightarrow$ Topologically challenging test case.
- Method captures geometry even in one time step (coarsest time resolution possible).



## Reference

F. Heimann, C. Lehrenfeld, J. Preuß (2022): *Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains*, arXiv preprint, `https://arxiv.org/abs/2202.02216`.