Wave Equation: Numerical Simulation and Experiment

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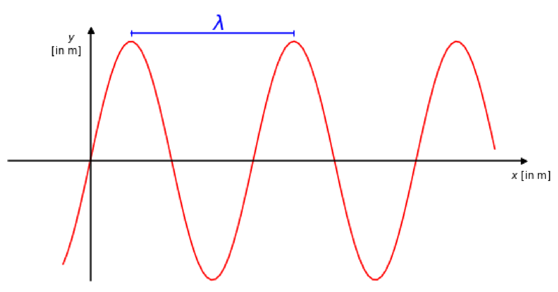
Numerical simulation (2D Model)

Physical motivation

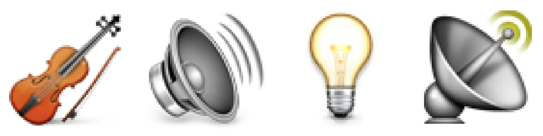
What are waves?

- periodically propagating dynamic disturbance
  - disturbance oscillates with **frequency** f (time)
  - **wavelength** λ: dist. between two waves (space)
  - $\lambda = \frac{c}{f}$, where c is the phase velocity



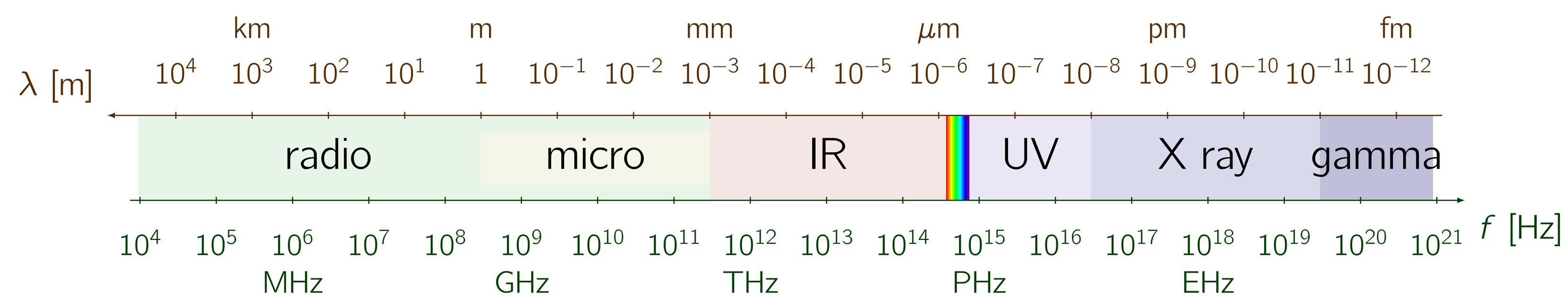
Examples:

- mechanical waves (deformation, e.g. string, water)
- acoustic waves (pressure)
- electromagnetic radiation (electric/magnetic field)



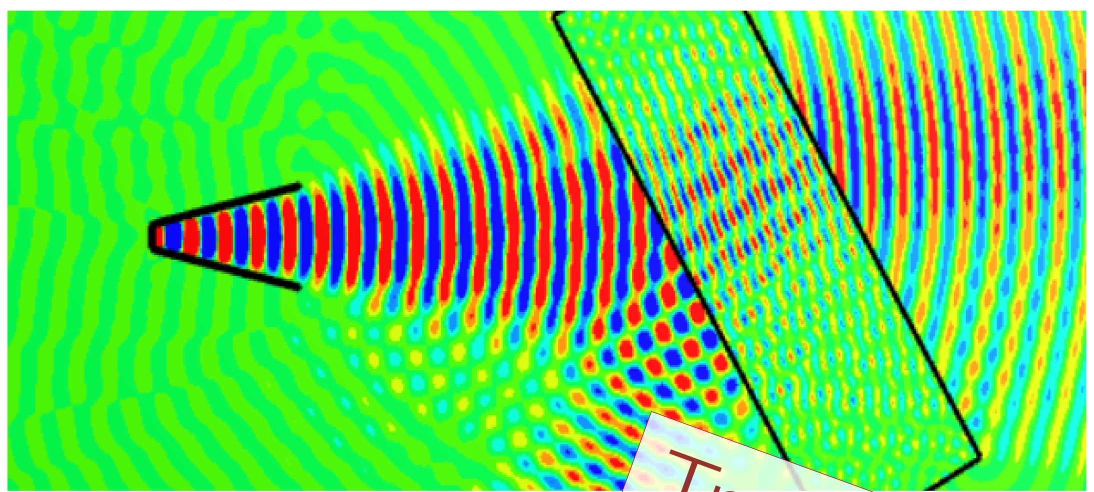
Electromagnetic radiation (EMR) as an example for waves

electromagnetic spectrum covers range of frequencies f of EMR:



spatially varying electric field is associated with a magnetic field that changes over time

Characteristic effects of waves

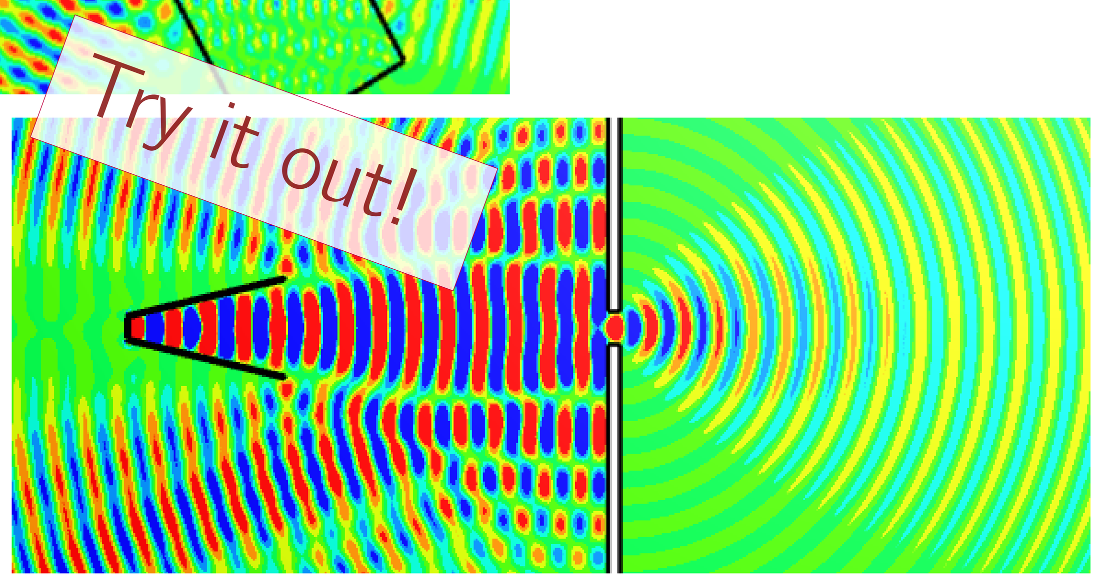


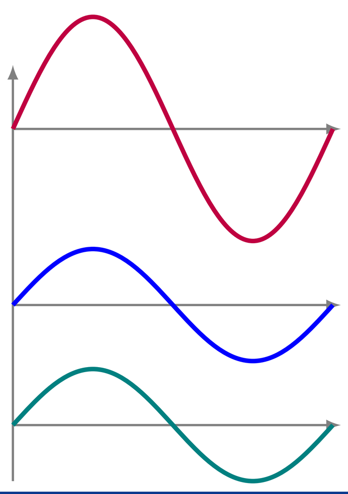
Refraction

Wave is **reflected** and **bent** at interface between two media with different refraction indices.

Diffraction

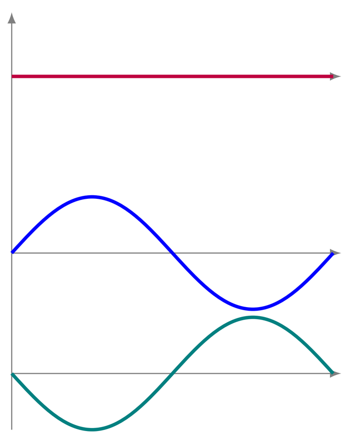
Wave **bends** around objects or form **new wavefronts** behind the slit (of size of the wave length).





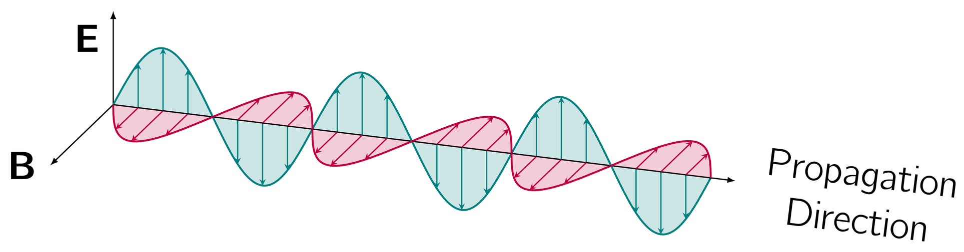
Interference

Superposition of waves [• + • = •]
Constructive [←]:
same sign ⇝ waves sum up.
Destructive [→]:
opposite sign ⇝ waves cancel out.



Two-dimensional model for electromagnetic wave propagation

Maxwell's equations: Relation between electric field E and magnetic field B.



Assumptions:

- z-component is fixed
- electric field only acts in x-y-plane: $E(x,y,z,t) = (v(x,y,t), w(x,y,t), 0)$
- absence of charge and electricity
- magnetic field only acts in z-direction: $B(x,y,z,t) = (0,0,u(x,y,t))$

From Maxwell's (partial differential) equations to the Helmholtz equations

Maxwell's equations with 2D ansatz becomes

$$-\Delta u(x,y,t) + \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} u(x,y,t) = 0$$

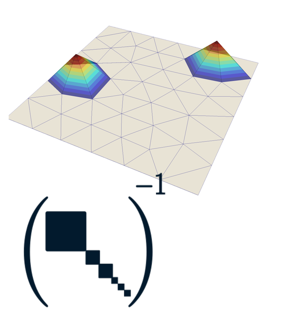
ansatz: u time-harmonic $u(x,y,t) = \hat{u}(x,y)e^{i2\pi f t}$
or superpos. $\sum_k \hat{u}_k(x,y)e^{i2\pi k t}$

$$-\Delta \hat{u}(x,y) - \nu \hat{u}(x,y) = 0$$

- $u(x,y,t)$: the unknown field
- μ_0, ε_0: electr./magn. constants
- $c = (\mu_0\varepsilon_0)^{-\frac{1}{2}}$: speed of light
- $\Delta u = \sum_{k=1}^{d} \frac{\partial^2}{\partial^2_{x_k}} u$
- $2\pi f = \omega$
- wave number $\nu = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

Challenges for the numerical simulation

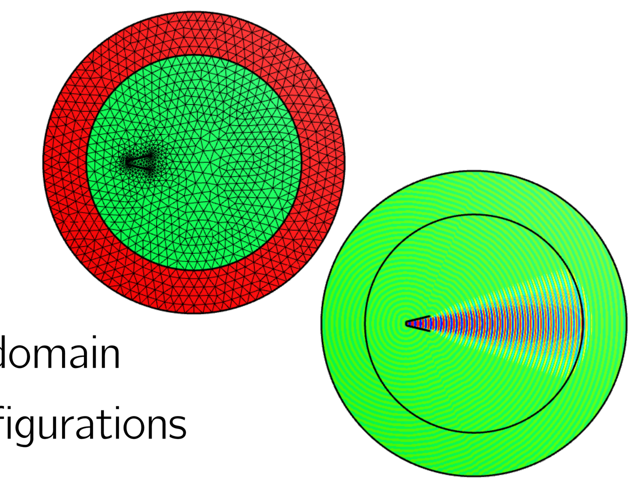
- approximation of unknown **functions** (fields) in **finite** dimensions
- setup of equation system for finite dimensional unknowns
- **efficient** solution of **huge** (linear, sparse) system of equations
- domain truncation (boundary conditions)
- **error estimation**, convergence to exact solution



Experiments

Basic setup of experiments

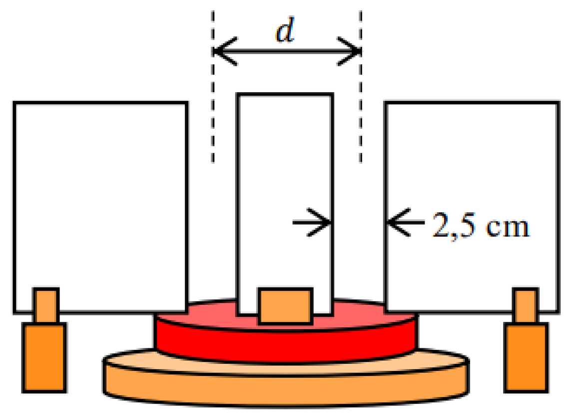
- EMR is emitted by a horn antenna
- wavelength $\lambda = 28$ mm ⇝ **microwaves**
- without any obstacle wave moves in a straight line
- receiver is another horn antenna
- simulation allows to display the **field** within the whole domain
- **simulation** vs. **experiment** for different **obstacle** configurations

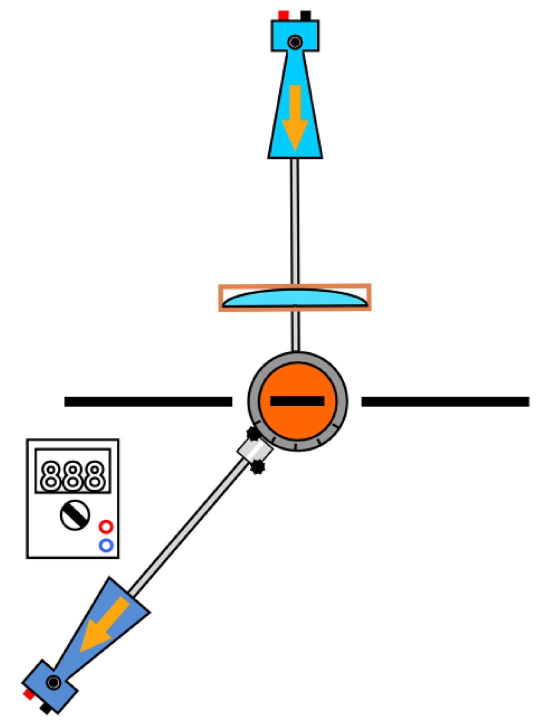


Setup 1: Double-slit experiment

Setup (to do)

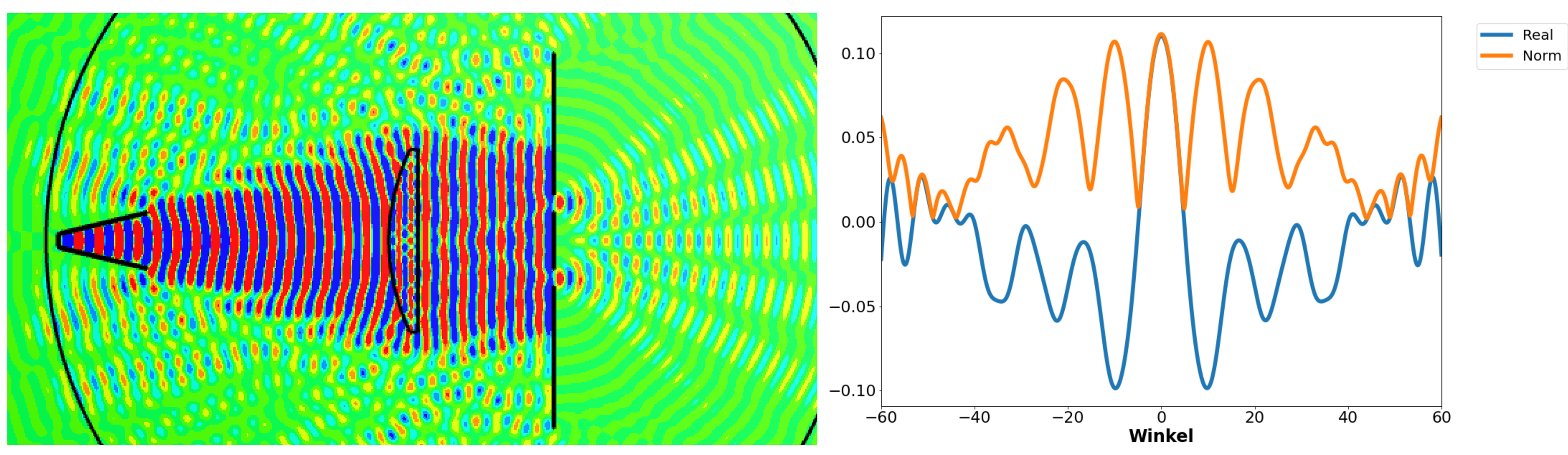
- three aluminum sheets are placed in a perpendicular to the transmitter direction (60 cm distance)
- distance between the sheets is 2.5 cm
- measure electric current behind the aluminum sheets for different angles behind the double-slit





Effects and possible observations

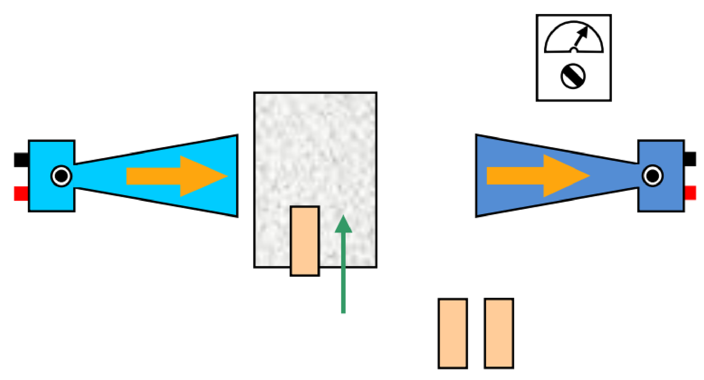
- behind the slits new **wave fronts** form
- constructive and destructive **interference** takes place
- distance of maxima allows to determine the wavelength $\lambda = \frac{d \cdot sin(\alpha_k)}{k}$



Setup 2: Wave propagation in acrylic glass

Setup (to do)

- blocks between the receiver/transmitter
- use different amounts of acrylic blocks
- observe electric current at the receiver



Effects and possible observations

- **refraction** since acrylic block has a refractive index of $n = 1.57$
- refractive index of air is $n \approx 1$
- at corner of acrylic block ⇝ **diffraction**

Hands-On!

- set up the experiments and **take measurements**
- **run the simulations**
- **compare experiments / simulations**
- **play around** and investigate your own (virtual) configuration

Special thanks go to

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Material for you!