Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes

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Abstract

The present paper is a survey aiming at characterizing all solutions of the discrete phase retrieval problem. Restricting ourselves to complex signals with finite support, we will give a full classification of all trivial and nontrivial ambiguities of the phase retrieval problem. In our classification, trivial ambiguities are caused either by signal shifts in space, by multiplication with a rotation factor $\mathrm{e}^{\mathrm{i}\alpha}$, $\alpha \in [0, 2\pi)$, or by conjugation and reflection of the signal. Furthermore, we show that all nontrivial ambiguities of the finite discrete phase retrieval problem can be characterized by signal convolutions.

In the second part of the paper, we examine the usually employed a priori conditions regarding their ability to reduce the number of ambiguities of the phase retrieval problem or even to ensure uniqueness indeed. For the corresponding proofs we can employ our findings on the ambiguity classification. The considerations on the structure of ambiguities also show clearly the ill-posedness of the phase retrieval problem even in cases where uniqueness is theoretically shown.

Key words. discrete one-dimensional phase retrieval for complex signals, autocorrelation polynomial, signal convolution, compact support, interference measurements

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1 Introduction

1.1 Ambiguities in one-dimensional discrete phase retrieval

In many fields of physics and engineering, one is faced with the problem to determine a signal from the modulus of its Fourier transform, or equivalently, from its autocorrelation function. This phase retrieval problem occurs in different applications, e.g., in crystallography [28, 23], astronomy [11] and laser optics [37].

The solution of the phase retrieval problem is generally challenging due to the fact that it is not uniquely solvable. Therefore, it is of essential importance to employ suitable additional conditions on the desired solution signal in order to ensure its uniqueness.

Let us shortly survey the rich literature on the problem of ambiguities in phase retrieval with main emphasis to the one-dimensional case.

In the one-dimensional continuous setting, the phase retrieval problem can be stated as follows. Find a function $f: \mathbb{R} \to \mathbb{C}$ from $\left|\widehat{f}\right|$ where $\widehat{f}$ denotes the Fourier transform of f. To solve this problem, we need to pose additional conditions on f, e.g., f is real, has compact support and finite energy. But still, even with these side conditions, the continuous one-dimensional phase retrieval problem can have infinitely many solutions. Using the Laplace transform of the autocorrelation function, all ambiguities have been characterized by using the Hadamard's Factorization Theorem for entire functions in [39, 19] and by logarithmic Hilbert transform in [5], respectively. Assuming that the unknown function is symmetric or monotone on the support one can enforce uniqueness of the phase retrieval problem [25]. Also additional constraints being given by the specific experiment can reduce the set of ambiguities. For example, under the assumption that a lens has a finite aperture, the corresponding phase retrieval problem can be solved uniquely [32, 39].

Another approach to avoid ambiguities in the continuous case is to extend the set of given data. Further intensity measurements [40, 20] or a specially constructed reference signal [5] can be used to determine the unknown function without ambiguities. In [38], it has been shown that a band-limited real function f can be uniquely recovered from the modulus of its function values being sampled at twice its sampling rate. Generalizing this idea to the complex case, [33] applies a combination of oversampling and modulations with complex exponentials to recover complex signals with compact support from intensity measurements in Fourier domain.

For numerical purposes one needs to restrict to a discrete space model for the signal. Therefore, we will consider only finitely supported signals $(x[n])_{n\in\mathbb{Z}}$ with $x[n] = 0$ for $n < 0$ and $n \geq N$ for some $N \in \mathbb{N}$. In this case, the (nontrivial) ambiguities of the phase retrieval problem can be described by zeros or poles of the z-transform of the autocorrelation signal, see [7, 31].

In the past, there have been several approaches to reduce the set of nontrivial ambiguities to a unique solution. For example, restricting the solution sets of zeros and poles of the z-transform of the autocorrelation signal suitably, uniqueness can be ensured [17], see also Subsection 4.2. Unfortunately, this additional condition can only be applied if all zeros of the z-transform are known, and we are not aware of a special physical meaning of this zero restriction.

In [16], the phase retrieval problem with signed phase information has been studied. Fixing the finite support of a real signal $\mathbf{x}$ and knowing whether the phase of the Fourier transform $\widehat{x}(\omega)$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or in $\left[-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, uniqueness of the solution can be shown.

Other approaches use some additional knowledge about signal values, especially the endpoint of the finite length signal [41, 36, 42, 43], see also Subsection 4.3. In [37, 26, 27] the case of complex finite signals has been considered with the additional condition that also the magnitudes of the signal in space domain are available. This approach is investigated in Subsection 4.4.

A further idea is to replace the Fourier transform by the so-called short-time Fourier transform, where the unknown signal is overlapped with a small analysis window at different positions. Under some additional assumptions it is possible to recover the unknown signal only from the magnitudes of the short-time Fourier transform [29, 30].

Being interested in additional measurement conditions for the phase retrieval problem that are physically feasible, the idea of measuring intensities of interferences has been extensively considered. We study this approach in Section 5. For example, the interference with a known [21, 22] or unknown [24] reference signal can be used. In [34, 35] this idea is generalized to the reconstruction of complex signals. A special case of interference signals is considered in [8], where the two reference signals are shifts of the desired signal $\mathbf{x}$ in Fourier space.

The Fourier transform of a signal vector $\mathbf{x}$ can be interpreted as scalar products of a vector with the rows $\mathbf{e}_k$ of the Fourier matrix, and the phase retrieval problem can be stated as the problem to reconstruct $\mathbf{x} \in \mathbb{C}$ from the magnitudes $|\langle \mathbf{x}, \mathbf{e}_k \rangle|$, $k = 0, \dots, N-1$. Generalizing this orthonormal basis $\{\mathbf{e}_k : k = 0, \dots, N-1\}$ to a frame, one may ask the question, how the frame vectors have to be constructed in order to uniquely recover $\mathbf{x}$ from the magnitudes of its frame coefficients, and how many frame vectors are needed to ensure uniqueness. This problem has been extensively studied within the last years, see, e.g., [3, 2, 1, 4, 6] and references therein. Unfortunately, one cannot construct a suitable frame just by adding further vectors of the form $(\mathrm{e}^{-\mathrm{i}\omega n})_{n=0}^{N-1}$ with some $\omega \in [0, 2\pi)$ to the Fourier basis since the autocorrelation function of the finite vector $\mathbf{x}$ is already completely determined by $|\langle \mathbf{x}, \mathbf{e}_k \rangle|$, $k = 0, \dots, N-1$.

Finally we want to mention that the higher-dimensional phase retrieval problem, however, has a completely different behavior. The reason is that the multidimensional polynomials usually cannot be factorized in linear factors corresponding to the zeros and poles. Instead we can only obtain a factorization of the z-transform of the signal into a product of irreducible polynomials with normalized support [15]. Since the reducible polynomials form a set of measure zero [18] in the space of all polynomials (up to a certain degree), almost all multidimensional signals can be recovered uniquely. Nevertheless, in some applications, such as in the crystallography [28], the factorization into reducible polynomials of small degree is the usual case.

To ensure global irreducibility of the z-transform of the autocorrelation polynomial it is enough to place a single reference point outside the unknown object [13, 10]. Other approaches work with random illuminations [12] or random masks [14]. Here the reconstruction is unique with high probability.

1.2 Our contribution and outline

In Section 2, we shall give a complete mathematical classification of solutions of the discrete phase retrieval problem for compactly supported complex signals. Using the zero set of the autocorrelation function, we show that each nontrivial solution of the discrete phase retrieval problem can be constructed by convolution. For that purpose, we have to determine all complex trigonometric polynomials $B(\omega)$ being a root of the nonnegative autocorrelation polynomial $A(\omega)$ of the solution signal $\mathbf{x}$, i.e., satisfying

$|B(\omega)|^2 = A(\omega)$. The properties of the autocorrelation polynomial, particularly its factorization into linear factors and the consequences for the solutions of the discrete phase retrieval problem are investigated in Section 3. We will observe that the number of nontrivial ambiguities depends on the zero set of an algebraic polynomial that is closely related to $A(\omega)$. In contrast to statements in earlier literature, we show that the number of nontrivial ambiguities of the discrete phase retrieval problems is bounded by 2^{N-2} (where N denotes the signal length), but can be also considerably smaller depending on the data. We will illustrate our findings with suitable examples.

Using the obtained classification of ambiguities, we reconsider additional a priori conditions that have been proposed in earlier literature (often with the restriction to real signals) to ensure uniqueness of the phase retrieval solutions in Sections 4 and 5. In particular, the a priori assumption that the desired signal is real and positive, being frequently applied in phase retrieval algorithms, does usually not lead to uniqueness in the one-dimensional case. If beside the Fourier intensities one signal value is known [41], or alternatively one or more magnitudes of the signal $\mathbf{x}$ in time domain are known [37, 26, 27], then we show that the signal can be uniquely reconstructed with high probability (up to multiplication with an unimodular constant). However, we can also construct "counterexamples", where the discrete phase retrieval problem is not uniquely determined by these a priori conditions.

Finally, in Section 5, we especially consider the case when beside $|\hat{x}(\omega)|^2$ also the intensities of interference signals can be measured. Here we distinguish the cases, where the reference signal itself is known or unknown. In the latter case, we give a new proof for uniqueness based on our representations of solution ambiguities derived in Section 2, see Theorem 5.4.

2 Trivial and nontrivial ambiguities

We consider the discrete one-dimensional phase retrieval problem where we want to reconstruct the complex signal $\mathbf{x} = (x\,[n])_{n\in\mathbb{Z}} \in \ell^2$ from its Fourier intensities. In the following we assume that the unknown signal $\mathbf{x}$ has finite support with support length $N \in \mathbb{N}$, i.e., there exists an $n_0 \in \mathbb{Z}$ such that $x(k) = 0$ for $k < n_0$ and $k \geq n_0 + N$. Furthermore, we suppose that the squared magnitude of the discrete Fourier transform

$$\widehat{x}(\omega) := \sum_{n\in\mathbb{Z}} x\,[n]\ \mathrm{e}^{-\mathrm{i}\omega n}, \qquad \omega \in [-\pi, \pi)$$

is measured at $2N - 1$ data points $\frac{2\pi k}{N}$, $k = -N + 1, \dots, N - 1$, i.e., the vector

$$\left|\widehat{\mathbf{x}}\right|^2 := \left(\left|\widehat{x}\left(\tfrac{2\pi k}{N}\right)\right|^2\right)_{k=-N+1}^{N-1} \tag{1}$$

is given.

We recall that knowing the Fourier intensity vector $\left|\widehat{\mathbf{x}}\right|^2 \in \mathbb{R}_+^{2N+1}$ of $(x\,[n])_{n\in\mathbb{Z}}$ is equivalent to knowing the autocorrelation signal $\mathbf{a} := (a[n])_{n\in\mathbb{Z}}$ with

$$a[n] := \sum_{\ell\in\mathbb{Z}} \overline{x[\ell]}\, x[\ell+n] \qquad (n \in \mathbb{Z}). \tag{2}$$

Indeed, for complex signals $\mathbf{x}$ the autocorrelation signal $\mathbf{a}$ is conjugate symmetric, i.e., $a[n] = \overline{a[-n]}$, and we simply observe that

$$A(\omega) := \left|\widehat{x}(\omega)\right|^2 = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[n]\ \overline{x[k]}\, \mathrm{e}^{-\mathrm{i}\omega(n-k)} = \sum_{\ell\in\mathbb{Z}} a[\ell]\ \mathrm{e}^{-\mathrm{i}\omega\ell} = \widehat{a}(\omega).$$

If the signal $\mathbf{x}$ has support length N, then the nonnegative autocorrelation function $A(\omega)$ is a trigonometric polynomial of degree $N-1$,

$$A(\omega) = \sum_{\ell=-N+1}^{N-1} a[\ell]\ \mathrm{e}^{-\mathrm{i}\omega\ell}$$

that is already determined uniquely by the given $2N-1$ data points

$$A\left(\tfrac{2\pi k}{N}\right) = \left|\widehat{x}\left(\tfrac{2\pi k}{N}\right)\right|^2 \qquad (k = -N+1, \ldots, N-1).$$

If $\mathbf{x} \in \ell^1$ is a real signal with finite support, we have $a[n] = a[-n]$, i.e., the autocorrelation function $A(\omega)$ is a nonnegative even trigonometric polynomial with

$$A(\omega) = a[0] + 2\sum_{n\in\mathbb{Z}} a[n]\cos(\omega n).$$

The problem of phase retrieval for finitely supported signals $\mathbf{x}$ can now be stated as follows. For a given intensity vector $|\widehat{\mathbf{x}}|^2$ in (1) or equivalently, for its given autocorrelation function $A(\omega)$, we want to reconstruct the signal $\mathbf{x}$ with support length less than or equal to N.

It is well-known that this phase retrieval problem is not uniquely solvable. There exist ambiguities being caused by translation, reflection and conjugation of the vector $\mathbf{x}$, or multiplication of $\mathbf{x}$ with an unimodular constant. These ambiguities are trivial and cannot be avoided.

We will be especially interested in the complete characterization of the nontrivial ambiguities of this phase reconstruction problem. However, let us first summarize all trivial ambiguities.

Proposition 2.1. *Let $A(\omega)$ be the autocorrelation function given by the squared magnitude $\left|\widehat{x}(\omega)\right|^2$ of the finite complex signal $\mathbf{x}$. Then we have:*

(i) *For each $n_0 \in \mathbb{Z}$ the shifted signal $(y[n]) := (x[n-n_0])$ has the same autocorrelation $A(\omega)$, i.e.,*

$$\left|\widehat{x}(\omega)\right|^2 = \left|\widehat{y}(\omega)\right|^2$$

for all $\omega \in [-\pi, \pi)$.

(ii) *The reflected conjugated signal* $(y[k]) := \left(\overline{x[-k]}\right)$ *has the same autocorrelation function* $A(\omega)$.

(iii) *The rotated signal* $(y[n]) := \left(\mathrm{e}^{\mathrm{i}\alpha} x[n]\right)$, *where* $\alpha \in [-\pi, \pi)$, *has the same autocorrelation* $A(\omega)$.

Proof. (i) Obviously, we have

$$\widehat{y}(\omega) = \sum_{n\in\mathbb{Z}} x[n-n_0]\ \mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n\in\mathbb{Z}} x[n]\ \mathrm{e}^{-\mathrm{i}\omega(n+n_0)} = \mathrm{e}^{-\mathrm{i}\omega n_0}\,\widehat{x}(\omega),$$

i.e., $\widehat{y}(\omega)$ and $\widehat{x}(\omega)$ only differ by a factor of modulus 1. Hence the autocorrelation determines the length of the support of the signal $\mathbf{x}$, but it is invariant regarding support shifts.

(ii) From $y[k] = \overline{x[-k]}$ it follows

$$\widehat{y}(\omega) = \sum_{k\in\mathbb{Z}} \overline{x[-k]}\,\mathrm{e}^{-\mathrm{i}\omega k} = \overline{\sum_{k\in\mathbb{Z}} x[k]\ \mathrm{e}^{-\mathrm{i}\omega k}} = \overline{\widehat{x}(\omega)},$$

and $\left|\widehat{y}(\omega)\right|^2 = \left|\widehat{x}(\omega)\right|^2$ for all $\omega \in [-\pi, \pi)$.

(iii) Obviously, we have $\widehat{y}(\omega) = \mathrm{e}^{\mathrm{i}\alpha}\,\widehat{x}(\omega)$. □

Remark 2.2. The trivial ambiguity caused by shifts of $\mathbf{x}$ in Proposition 2.1 (i) can be avoided by normalizing the unknown finite support of $\mathbf{x}$ to $\{0, \ldots, N-1\}$. Note that for $\mathbf{x}$ with support $\{0, \ldots, N-1\}$ the reflected conjugated signal in Proposition 2.1 (ii) has the support $\{-N+1, \ldots, 0\}$. Therefore, after support normalization the ambiguity caused by reflection and conjugation is of the form $(y[k]) := (\overline{x[N-k]})$. ○

Beside these trivial ambiguities, there are also nontrivial ambiguities that we want to classify in detail in this paper. In particular, the following main theorem shows that each nontrivial ambiguity of the considered phase retrieval problem can be characterized by a convolution.

Theorem 2.3. *Let* $A(\omega)$ *be the autocorrelation function given from* $|\widehat{\mathbf{x}}|^2$ *of the finite complex signal* $\mathbf{x}$. *Further, let* $\mathbf{x}_1 := (x_1[n])_{n\in\mathbb{Z}}$, $\mathbf{x}_1 := (x_2[n])_{n\in\mathbb{Z}}$ *be two finite signals with*

$$\mathbf{x} = \mathbf{x}_1 * \mathbf{x}_2,$$

i.e.,

$$x[n] := \sum_{k\in\mathbb{Z}} x_1[k]\,x_2[n-k].$$

Then $\mathbf{y} := \mathrm{e}^{\mathrm{i}\alpha}\left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$ *with* $\alpha \in [-\pi, \pi)$, $n_0 \in \mathbb{Z}$ *has the same autocorrelation function* $A(\omega)$, *i.e.,*

$$\left|\widehat{x}(\omega)\right|^2 = \left|\widehat{y}(\omega)\right|^2$$

for all $\omega \in [-\pi, \pi)$.

Moreover, for a signal $\mathbf{y}$ *being a solution of the phase retrieval problem, i.e., possessing the autocorrelation function* $A(\omega)$ *of* $\mathbf{x}$*, there exist finite signals* $\mathbf{x}_1$*,* $\mathbf{x}_2$ *with* $\mathbf{x} = \mathbf{x}_1 * \mathbf{x}_2$ *and* $\mathbf{y} := \mathrm{e}^{\mathrm{i}\alpha}\left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$.

Proof. Let $\mathbf{y} := \mathrm{e}^{\mathrm{i}\alpha}\left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$, then

$$\widehat{y}(\omega) = \mathrm{e}^{-\mathrm{i}\omega n_0 + \mathrm{i}\alpha}\, \overline{\widehat{x_1}(\omega)}\, \widehat{x_2}(\omega)$$

and hence

$$\left|\widehat{y}(\omega)\right|^2 = \left|\widehat{x_1}(\omega)\right|^2 \left|\widehat{x_2}(\omega)\right|^2 = \left|\widehat{x}(\omega)\right|^2 .$$

The proof that each solution of the phase retrieval problem can be represented in this way is postponed to Section 3. □

3 Investigation of the autocorrelation function

In Fourier space, the considered phase retrieval problem for finitely supported complex signals $\mathbf{x}$ can be reformulated as follows: For a given real nonnegative autocorrelation polynomial

$$A(\omega) = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n},$$

of degree $N-1$ with $a[N-1] \neq 0$ find all finite trigonometric polynomials

$$\widehat{x}(\omega) = \sum_{n\in\mathbb{Z}} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$$

with $x[n] \in \mathbb{C}$ such that

$$\left|\widehat{x}(\omega)\right|^2 = A(\omega).$$

Avoiding the trivial shift ambiguity, we assume that $\operatorname{supp} \mathbf{x} \subset \{0, \ldots, N-1\}$, i.e.,

$$\widehat{x}(\omega) = \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$$

with $x[0] \neq 0$ and $x[N-1] \neq 0$. We obtain the following theorem.

Theorem 3.1. *Let* $A(\omega)$ *be a nonnegative trigonometric polynomial*

$$A(\omega) = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \tag{3}$$

with $a[n] \in \mathbb{C}$ *and* $a[-n] = \overline{a[n]}$ *for* $n = 0, \ldots, N-1$*, and* $a[N-1] \neq 0$*. Then each*

$$B(\omega) = \sum_{n=0}^{N-1} b[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$$

with $b[n] \in \mathbb{C}$ *satisfying* $|B(\omega)|^2 = A(\omega)$ *can be written in the form*

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha}\left[|a[N-1]| \cdot \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1}\right]^{\frac{1}{2}} \cdot \prod_{j=1}^{N-1}\left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right) \tag{4}$$

where $\alpha \in [-\pi, \pi)$*. Assuming that* $A(\omega) = \mathrm{e}^{\mathrm{i}\omega(N-1)} P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)$ *where* $P_A(z)$ *is an algebraic polynomial of degree* $2N-2$ *with complex coefficients and the factorization*

$$P_A(z) := a[N-1] \prod_{j=1}^{N-1}\left(z - \gamma_j\right)\left(z - \overline{\gamma}_j^{-1}\right),$$

we have $\gamma_j \neq 0$*, and the parameters* β_j *in* (4) *can be chosen as*

$$\beta_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}.$$

Proof. We consider the complex algebraic polynomial

$$P_A(z) := a[0]\, z^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]}\, z^{N-1-n} + \sum_{n=1}^{N-1} a[n]\, z^{N-1+n}$$

of degree $2N-2$ with the complex coefficients $a[n]$ being given by the trigonometric polynomial $A(\omega)$. By construction, P_A is related to the trigonometric polynomial $A(\omega)$ in (3) by

$$P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right) = \mathrm{e}^{-\mathrm{i}\omega(N-1)} A(\omega).$$

Let γ_j be a root of the polynomial P_A, i.e.,

$$P_A\left(\gamma_j\right) = a[0]\, \gamma_j^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]}\, \gamma_j^{N-1-n} + \sum_{n=1}^{N-1} a[n]\, \gamma_j^{N-1+n} = 0.$$

As a consequence of $\overline{a[N-1]} \neq 0$, we have $\gamma_j \neq 0$. For $|\gamma_j| \neq 1$ it follows

$$\begin{aligned}
\overline{\gamma}_j^{2N-2} P_A\left(\overline{\gamma}_j^{-1}\right) &= \overline{\gamma}_j^{2N-2}\left[a[0]\, \overline{\gamma}_j^{-N+1} + \sum_{n=1}^{N-1} \overline{a[n]}\, \overline{\gamma}_j^{-N+1+n} + \sum_{n=1}^{N-1} a[n]\, \overline{\gamma}_j^{-N+1-n}\right] \\
&= \overline{a[0]}\, \overline{\gamma}_j^{N-1} + \sum_{n=1}^{N-1} \overline{a[n]}\, \overline{\gamma}_j^{N-1+n} + \sum_{n=1}^{N-1} a[n]\, \overline{\gamma}_j^{N-1-n} \\
&= \overline{P_A\left(\gamma_j\right)} = 0.
\end{aligned}$$

Therefore, all roots of P_A lying not on the circle occur in pairs $\left(\gamma_j, \overline{\gamma}_j^{-1}\right)$. For $|\gamma_j| = 1$ we can write $\gamma_j = \mathrm{e}^{\mathrm{i}\phi_j}$, where ϕ_j is a physical real zero of $A(\omega)$. In order not to cause any contradiction with $A(\omega) \geq 0$ for all ω, this zero must have even multiplicity.

Hence, we find a factorization of P_A in the form

$$P_A(z) = a[N-1] \prod_{j=1}^{N-1} \left(z - \gamma_j\right)\left(z - \overline{\gamma}_j^{-1}\right). \tag{5}$$

Observing that

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right)\right| = \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|\left|\overline{\gamma}_j^{-1}\right|\left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right| = \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2$$

and remembering that $A(\omega)$ is nonnegative, we have

$$\begin{aligned} A(\omega) = |A(\omega)| &= \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| \\ &= |a[N-1]| \prod_{j=1}^{N-1} \left|\gamma_j\right|^{-1} \left|\prod_{j=1}^{N-1} (\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j)\right|^2 = |B(\omega)|^2 \end{aligned}$$

and the representation of $B(\omega)$ in (4) follows. □

Remark 3.2. For the case of a nonnegative polynomial $A(\omega)$ with real coefficients, it has been shown in [9, Lemma 6.1.3.] that there always exists a real trigonometric root polynomial $B(\omega)$ with $A(\omega) = |B(\omega)|^2$. Restricting Theorem 3.1 to this real case, where the coefficients of $A(\omega)$ satisfy $a[n] \in \mathbb{R}$ and $a[n] = a[-n]$, the algebraic polynomial $P_A(z)$ has only real coefficients. According to the proof of Theorem 3.1, its real roots appear in pairs

$$\left\{\gamma_j, \gamma_j^{-1}\right\},$$

and its complex roots appear in quads

$$\left\{\gamma_j, \overline{\gamma}_j, \gamma_j^{-1}, \overline{\gamma}_j^{-1}\right\}.$$

○

We now consider the question, how many nontrivial ambiguities can occur depending on the zero set of $P_A(z)$.

Corollary 3.3. *Let A be a nonnegative autocorrelation polynomial of degree $N-1$, and let a solution B of $|B(\omega)|^2 = A(\omega)$ be defined by* (4) *with $\alpha = 0$ and the zero set*

$$\left\{\beta_j \in \mathbb{C} \colon\ j = 1, \ldots, N-1\right\}.$$

Then considering all 2^{N-1} solutions of $|B(\omega)|^2 = A(\omega)$, which can be constructed by choosing for each $j = 1, \ldots, N-1$ one root of the root pair

$$\beta_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\},$$

of P_A in Theorem 3.1, we obtain up to 2^{N-2} nontrivial ambiguities.

Proof. Since the support of the coefficient sequence of $B(\omega)$ is already fixed to be $\{0, \ldots, N-1\}$, and since the rotation factor $\mathrm{e}^{\mathrm{i}\alpha}$ in (4) has been fixed as $\mathrm{e}^{\mathrm{i}\cdot 0} = 1$, there can only occur trivial ambiguities caused by reflection and conjugation as considered in Proposition 2.1 (ii). We consider the polynomials $B(\omega)$ corresponding to the zero set Λ,

$$B(\omega) = B_\Lambda(\omega) = |a[N-1]|^{\frac{1}{2}} \prod_{\beta_j \in \Lambda} |\beta_j|^{-\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right),$$

where Λ contains either $\beta_j = \gamma_j$ or $\beta_j = \overline{\gamma}_j^{-1}$ as the j-th entry.

First, we observe that the 2^{N-1} trigonometric polynomials B_Λ, which are produced by taking all choices of the set Λ, are pairwise different if all zeros $\{\gamma_j, \overline{\gamma}_j^{-1},\ j = 1, \ldots, N-1\}$ of the polynomial P_A in (5) are pairwise different. Further, if we fix a set $\Lambda = \{\beta_1, \ldots, \beta_{N-1}\}$ and consider the "reflected" set $\widetilde{\Lambda} = \left\{\overline{\beta}_1^{-1}, \ldots, \overline{\beta}_{N-1}^{-1}\right\}$, then

$$\begin{aligned}
B_{\widetilde{\Lambda}}(\omega) &= |a[N-1]|^{\frac{1}{2}} \prod_{j=1}^{N-1} \left|\overline{\beta}_j\right|^{\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\beta}_j^{-1}\right) \\
&= |a[N-1]|^{\frac{1}{2}} \prod_{j=1}^{N-1} \mathrm{e}^{-\mathrm{i}\arg\overline{\beta}_j} \left|\overline{\beta}_j\right|^{-\frac{1}{2}} \left(\overline{\beta}_j \mathrm{e}^{-\mathrm{i}\omega} - 1\right) \\
&= (-1)^{N-1}\, \mathrm{e}^{-\mathrm{i}(N-1)\omega}\, |a[N-1]|^{\frac{1}{2}} \prod_{j=1}^{N-1} \mathrm{e}^{-\mathrm{i}\arg\overline{\beta}_j} \left|\overline{\beta}_j\right|^{-\frac{1}{2}} \left(\mathrm{e}^{\mathrm{i}\omega} - \overline{\beta}_j\right) \\
&= (-1)^{N-1}\, \mathrm{e}^{-\mathrm{i}(N-1)\omega}\, \overline{B_\Lambda(\omega)} \prod_{j=1}^{N-1} \mathrm{e}^{-\mathrm{i}\arg\overline{\beta}_j},
\end{aligned}$$

i.e., the signal that corresponds to $B_{\widetilde{\Lambda}}$ is a shift of the reflected, conjugated and perhaps rotated signal corresponding to B_Λ. Thus $B_{\widetilde{\Lambda}}$ is a trivial ambiguity of B_Λ.

Assuming that the products $\prod_{j=1}^{N-1} \left|\beta_j\right|$ for all choices of Λ are pairwise different, this is the only trivial ambiguity that occurs because in this case the modulus of the leading coefficients of all the trigonometric polynomials B_Λ are pairwise different. For example, this is the case if the absolute values of all roots β_j are pairwise different primes. Thus we can obtain up to 2^{N-2} nontrivial solutions of the considered phase retrieval problem. □

Remark 3.4. Similarly as in the Proposition above we can also generate up to 2^{N-2} nontrivial solutions $B(\omega)$ of $|B(\omega)|^2 = A(\omega)$ when $A(\omega)$ is the autocorrelation polynomial of a real signal, i.e., $x[n] \in \mathbb{R}$. For each pair

$$\left\{\gamma_j, \gamma_j^{-1}\right\}$$

of real roots of P_A, we can choose $\beta_j = \gamma_j$ or $\beta_j = \gamma_j^{-1}$. For each quad

$$\left\{\gamma_j, \overline{\gamma}_j, \gamma_j^{-1}, \overline{\gamma}_j^{-1}\right\}$$

of complex roots of P_A in (5), we can only choose either $\gamma_j, \overline{\gamma}_j$ or $\gamma_j^{-1}, \overline{\gamma}_j^{-1}$ to construct

$B(\omega)$. Therefore, the upper bound of 2^{N-2} nontrivial solutions can only be attained if all zeros of P_A are real. If all zeros of P_A are complex and appear in quads, we can only have at most $2^{\frac{N-1}{2}-1}$ nontrivial solutions $B(\omega)$. ○

Corollary 3.5. *For the phase retrieval problem with a given real nonnegative autocorrelation polynomial $A(\omega)$ the number of nontrivial solutions $B(\omega)$ of $|B(\omega)|^2 = A(\omega)$ may vary from 1 up to 2^{N-2} depending on the zero set. In particular, the phase problem is uniquely solvable up to trivial ambiguities if either $P_A(z)$ has only zeros on the unit circle or if all zeros up to one pair $\left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}$ lie on the unit circle.*

Proof. 1. As shown in Theorem 3.1, the phase retrieval problem $|B(\omega)|^2 = A(\omega)$ has at least one solution $B(\omega)$. Further solutions being nontrivially different from $B(\omega)$ may occur by switching between the zeros β_j and $\overline{\beta}_j^{-1}$.

A unique solution of the phase retrieval problem up to trivial ambiguities occurs if all roots of the polynomial P_A lie on the unit circle. In this case, the zero pair $\left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}$ reduces to one two-fold zero of the form $\mathrm{e}^{\mathrm{i}\alpha_j}$ for some $\alpha_j \in [-\pi, \pi)$. Therefore, the zeros β_j for the construction of $B(\omega)$ are uniquely determined and all ambiguities coincide. We also obtain a unique solution up to reflection and conjugation when all roots lie on the unit circle up to one pair $\left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}$ with $|\gamma_j| \neq 1$. As shown in the proof of Corollary 3.3, we only obtain two solutions by switching from γ_j to $\overline{\gamma}_j^{-1}$ for this one zero pair. But then one solution can be obtained by reflection and conjugation of the other as shown in the proof of Corollary 3.3.

2. The number of nontrivial solutions depends on the number of different zero pairs of P_A lying not on the unit circle and their multiplicities. The largest number 2^{N-2} of nontrivial ambiguities has been constructed already in Corollary 3.3. □

Example 3.6. We want to give examples where exactly two, exactly three, or exactly 2^{N-2} nontrivial solutions of the phase retrieval problem occur.

(i) Two nontrivial solutions of $A(\omega) = |B(\omega)|^2$ occur, for example, for a given autocorrelation function of the form

$$A(\omega) = \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| = |a[N-1]| \left|\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right|^{2N-6} \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \tfrac{1}{\gamma_1}\right)\right| \cdot \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_2\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \tfrac{1}{\gamma_2}\right)\right|$$

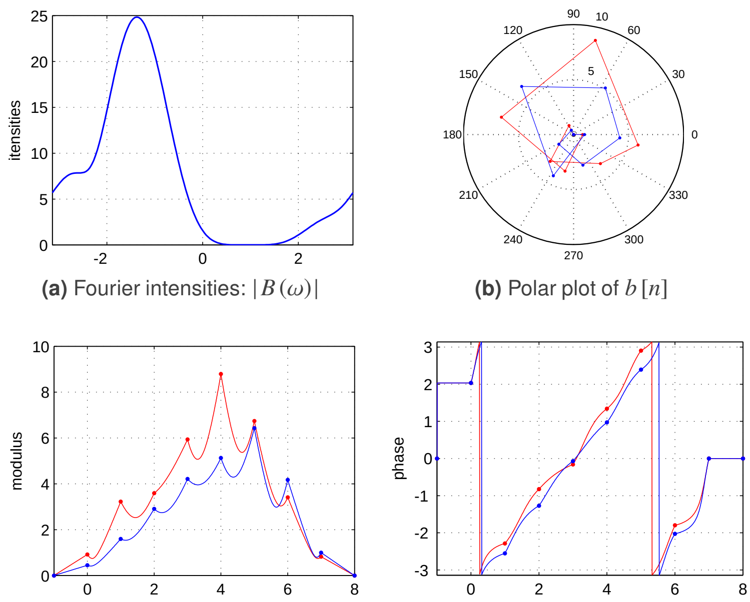
with $\gamma_1, \gamma_2 \in \mathbb{C}$; $\gamma_1 \neq \gamma_2$; $|\gamma_1|, |\gamma_2| \neq 1$; and $\alpha \in [-\pi, \pi)$, namely

$$B_1(\omega) = |a[N-1]|^{\frac{1}{2}} |\gamma_1\gamma_2|^{-\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right)^{N-3} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_2\right)$$

and

$$B_2(\omega) = |a[N-1]|^{\frac{1}{2}} \left|\frac{\gamma_1}{\gamma_2}\right|^{-\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right)^{N-3} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \frac{1}{\gamma_2}\right).$$

In Figure 1, a specific example is shown for the chosen zeros $\gamma_1 := -0.6 + 0.2\mathrm{i}$, $\gamma_2 := -0.5 - 0.5\mathrm{i}$, $\alpha := 0.3\pi$, and the support length $N := 8$. Figure 1 (b) shows



(a) Fourier intensities: $|B(\omega)|$

(b) Polar plot of $b[n]$

(c) Modulus: $|b[n]|$

(d) Phase: $\arg(b[n])$

Figure 1: Two nontrivial solutions of $A(\omega) = |B(\omega)|^2$ as given in Example 3.6 (i) for $N = 8$.

the signals $b_1[n]$ and $b_2[n]$, which are determined by the different solutions

$$B_1(\omega) = \sum_{n=0}^{N-1} b_1[n]\, \mathrm{e}^{-\mathrm{i}\omega n} \qquad \text{and} \qquad B_2(\omega) = \sum_{n=0}^{N-1} b_2[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$$

respectively, being represented as polygonal chains in the complex plane. In Figure 1 (c) and (d), the modulus and the phase of these chains are plotted. Since modulus and phase are nonlinear, the graphs are not piecewise linear.

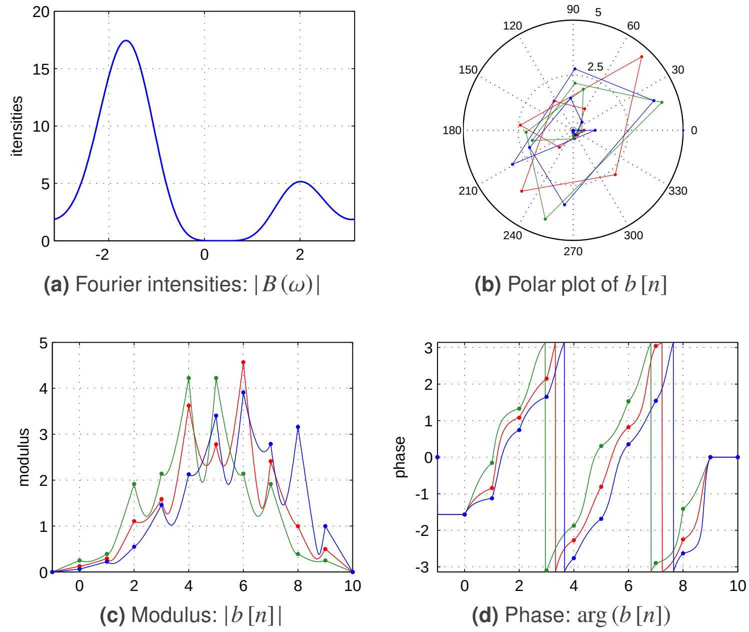
(ii) It is also possible to have an odd number of nontrivial solutions for a given autocorrelation function. For example, we consider an autocorrelation function of the form

$$A(\omega) = \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| = |a[N-1]| \left|\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right|^{2N-10} \cdot \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \tfrac{1}{\overline{\gamma_1}}\right)\right|^4$$

where $\gamma_1 \in \mathbb{C}$, $|\gamma_1| \neq 1$, and $\alpha \in [-\pi, \pi)$ with three nontrivial solutions, namely

$$B_1(\omega) = |a[N-1]|^{\frac{1}{2}} |\gamma_1|^{-2} \left(\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right)^{N-5} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)^4,$$

$$B_2(\omega) = |a[N-1]|^{\frac{1}{2}} |\gamma_1|^{-1} \left(\mathrm{e}^{-\mathrm{i}\omega} + \mathrm{e}^{\mathrm{i}\alpha}\right)^{N-5} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_1\right)^3 \left(\mathrm{e}^{-\mathrm{i}\omega} - \tfrac{1}{\overline{\gamma_1}}\right),$$



(a) Fourier intensities: $|B(\omega)|$ **(b)** Polar plot of $b[n]$

(c) Modulus: $|b[n]|$ **(d)** Phase: $\arg(b[n])$

Figure 2: Three nontrivial solutions of $A(\omega) = |B(\omega)|^2$ as in Example 3.6 (ii) for $N = 10$.

and

$$B_3(\omega) = |a[N-1]|^{\frac{1}{2}} \left(e^{-i\omega} + e^{i\alpha}\right)^{N-5} \left(e^{-i\omega} - \gamma_1\right)^2 \left(e^{-i\omega} - \tfrac{1}{\gamma_1}\right)^2 .$$

A specific example is given in Figure 2. Here, we have chosen $\gamma_1 := -0.5$, $\alpha := 0.1\pi$, and $N := 10$.

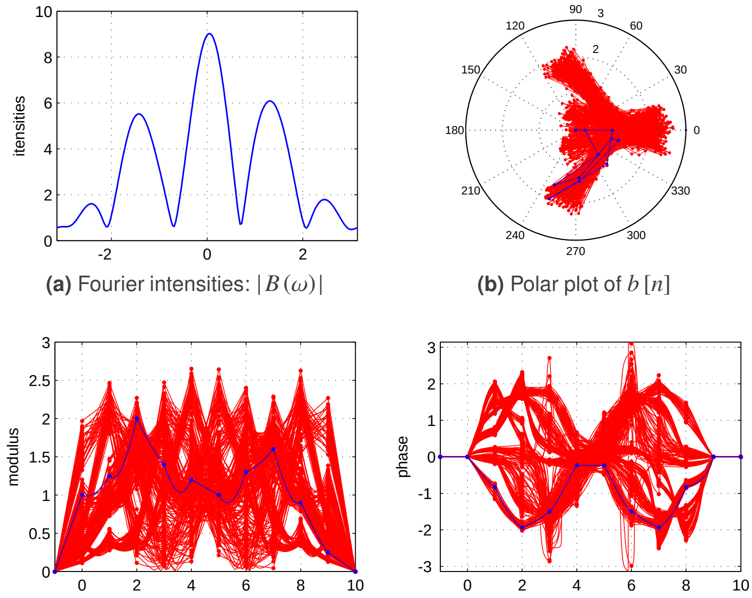
(iii) In Figure 3, we consider an autocorrelation polynomial $A(\omega)$ of degree $N-1 = 9$ where for all possible zero sets $\Lambda = \left\{\beta_j : j = 1, \ldots, N-1\right\}$ determining $B(\omega)$ in Theorem 3.1 the values $|b[0]| = |\prod_{\beta_j \in \Lambda} \beta_j|$ are pairwise different. Particularly, we consider the signal $\mathbf{x}$ (marked in Figure 3) whose modulus and phase are given by

$$(|x[n]|)_{n=0}^{N-1} = (1, 1.25, 2, 1.4, 1.2, 1, 1.3, 1.6, 0.9, 0.25)^{\mathrm{T}}$$

and

$$\arg x[n] = \left(\cos \tfrac{4\pi n}{9}\right) - 1$$

respectively, and let $A(\omega) := |\widehat{x}(\omega)|^2$. Here, the minimal difference of the coefficients $|b[0]|$ for two different solutions $B(\omega)$ is $1.3233 \cdot 10^{-4}$. Hence, the complete solution set of the phase retrieval problem $A(\omega) = |B(\omega)|^2$ contains $2^8 = 256$ different nontrivial solutions. All these solutions are presented in Figure 3. The example illustrates that the different solutions can possess very different shapes. ○



(a) Fourier intensities: $|B(\omega)|$ **(b)** Polar plot of $b[n]$

(c) Modulus: $|b[n]|$ **(d)** Phase: $\arg(b[n])$

Figure 3: 2^{N-2} nontrivial solutions of $A(\omega) = |B(\omega)|^2$ as in Example 3.3 (iii) for $N = 10$.

We are now ready to prove our main result stated in Theorem 2.3 that all nontrivial ambiguities for the phase retrieval problem can be represented by convolutions.

Proof (Theorem 2.3). Let $\mathbf{x}_1, \mathbf{x}_2$ be two signals with support length greater than one such that $\mathbf{x} = \mathbf{x}_1 * \mathbf{x}_2$ solves the phase retrieval problem, i.e., $|\widehat{x}(\omega)|^2 = |B(\omega)|^2 = A(\omega)$, where $A(\omega)$ denotes the autocorrelation polynomial of $\mathbf{x}$. Assuming that $\mathbf{x}$ has support length N, $A(\omega)$ is a nonnegative trigonometric polynomial of order $N-1$. We consider the symbols

$$\widehat{x}(\omega) := \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{\mathrm{i}\omega n},$$

$$\widehat{x_1}(\omega) := \sum_{n=0}^{N_1-1} x_1[n]\, \mathrm{e}^{\mathrm{i}\omega n}, \qquad \widehat{x_2}(\omega) := \sum_{n=0}^{N_2-1} x_2[n]\, \mathrm{e}^{\mathrm{i}\omega n},$$

where we have normalized the signals without loss of generality by Proposition 2.1 such that the support of each signal starts at zero. The Fourier transform of $\mathbf{x}$ is $\widehat{x}(\omega) = \widehat{x_1}(\omega)\,\widehat{x_2}(\omega)$ and is a trigonometric polynomial of the form (4) as given in Theorem 3.1. Since $\widehat{x_1}(\omega)$, $\widehat{x_2}(\omega)$ are polynomials of degree greater than one, they can be composed by products of factors $(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j)$ of $\widehat{x}(\omega)$ in (4).

We show that all nontrivial ambiguities with normalized support of the phase retrieval problem can be given as a product of the form $\mathrm{e}^{-\mathrm{i}\omega(N_1-1)}\,\overline{\widehat{x_1}(\omega)}\,\widehat{x_2}(\omega)$. Let $\widehat{x}(\omega)$ and $\widehat{\widehat{x}}(\omega)$ be two nontrivially different solutions of the phase retrieval problem, i.e.,

$$\left|\widehat{x}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega)\right|^2 = A(\omega),$$

where $\widehat{x}(\omega)$ corresponds to the zero set

$$\left\{\beta_j : \; j = 1 \ldots N-1\right\}$$

and $\widehat{\widetilde{x}}(\omega)$ to

$$\left\{\overline{\beta}_j^{-1} : \; j = 1, \ldots, J\right\} \cup \left\{\beta_j : \; j = J+1, \ldots, N-1\right\}$$

for some $J \in \{1, \ldots, N-2\}$. Then choose

$$\widehat{x_1}(\omega) := \prod_{j=1}^{J} \left|\beta_j\right|^{-\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right)$$

such that $\widehat{x_1}(\omega)$ is composed by the zero set $\left\{\beta_j : \; j = 1, \ldots, J\right\}$. It follows that

$$\mathrm{e}^{-\mathrm{i}\omega J}\, \overline{\widehat{x_1}(\omega)} = \mathrm{e}^{-\mathrm{i}\omega J} \prod_{j=1}^{J} |\beta_j|^{-\frac{1}{2}} \left(\mathrm{e}^{\mathrm{i}\omega} - \overline{\beta}_j\right) = \prod_{j=1}^{J} |\beta_j|^{-\frac{1}{2}}\, \overline{\beta}_j \left(\overline{\beta}_j^{-1} - \mathrm{e}^{-\mathrm{i}\omega}\right),$$

i.e., $\mathrm{e}^{-\mathrm{i}\omega J}\, \overline{\widehat{x_1}(\omega)}$ corresponds to the zero set $\left\{\overline{\beta}_j^{-1} : \; j = 1, \ldots, J\right\}$. Further, let

$$\widehat{x_2}(\omega) := \frac{\widehat{x}(\omega)}{\widehat{x_1}(\omega)}.$$

Hence, the second solution is up to an unimodular constant

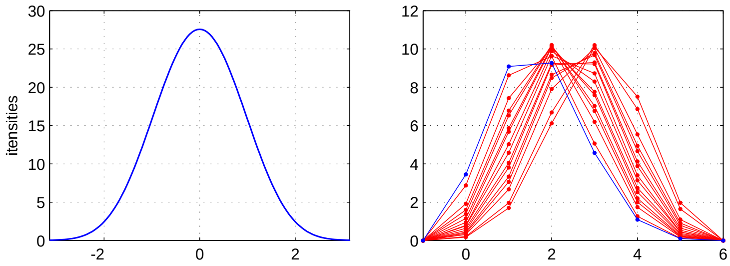
$$\widehat{\widetilde{x}}(\omega) = \mathrm{e}^{-\mathrm{i}\omega J}\, \overline{\widehat{x_1}(\omega)}\, \widehat{x_2}(\omega),$$

i.e., we have $\widetilde{\mathbf{x}} = \mathrm{e}^{-\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - J])$. □

In particular, the observations of Theorem 2.3 and Theorem 3.1 provide us with the opportunity to construct all nontrivial solutions $\widehat{x}(\omega)$ of $\left|\widehat{x}(\omega)\right|^2 = A(\omega)$ from one known solution $\widehat{x}(\omega)$ of the form (4).

4 Enforcing uniqueness of the one-dimensional phase retrieval problem

In order to evaluate a meaningful solution of the phase retrieval problem numerically, one needs to pose appropriate a priori conditions that ensure unique solvability. In the literature on one-dimensional phase retrieval, there have been many attempts to incorporate further conditions on the signal in order to achieve this goal. However, often there are no theoretical considerations, whether certain additional conditions indeed ensure uniqueness. Using our new insights on the representation of trivial and nontrivial ambiguities of the phase retrieval problem, we want to find out, to what extent the usually applied a priori conditions are indeed sufficient to ensure a unique



(a) Fourier intensities: $\left|\widehat{x}(\omega)\right|$ **(b)** Real nontrivial solutions: $x[n]$

Figure 4: Full nonnegative solution set for $A(\omega) = \left|\widehat{x}(\omega)\right|^2$

solution or to reduce the number of nontrivial ambiguities.

These considerations are also of essential importance in order to judge beforehand whether a numerical procedure can be able to provide a meaningful solution in a stable manner.

Observe that we have restricted ourselves already to the case of complex signals with compact support $\{0, \ldots, N-1\}$, thereby we avoid the trivial ambiguities caused by signal shifts.

4.1 Nonnegativity of the real signal

Often, phase retrieval is considered with the a priori assumption that the signal $\mathbf{x}$ to be recovered is compactly supported, real, and nonnegative. However, as already shown in [7], this condition does not necessarily lead to a smaller number of ambiguities. With $\widehat{x}(\omega)$ of the form

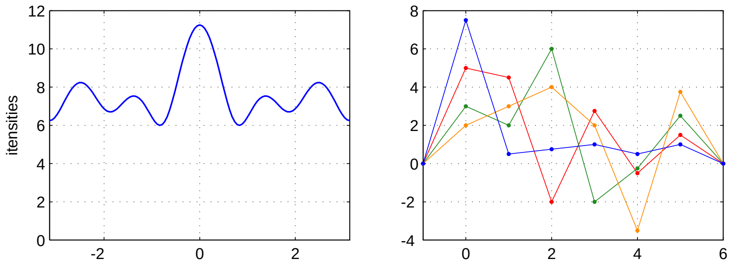
$$\widehat{x}(\omega) = |a[N-1]|^{\frac{1}{2}} \prod_{j=1}^{N-1} \left|\beta_j\right|^{-\frac{1}{2}} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right) \tag{6}$$

where $\beta_j \in \mathbb{R}$ and $\beta_j < 0$ even the highest possible number of 2^{N-2} nontrivial ambiguities is still attained. Therefore, nonnegativity of a real $\mathbf{x}$ is generally not sufficient to ensure unique solvability of the phase retrieval problem. On the other hand, in rare cases, the restriction to nonnegative signals can lead to an inconsistence with the given autocorrelation polynomial $A(\omega)$, such that no solution $\widehat{x}(\omega)$ with nonnegative coefficients exists.

Example 4.1. Figures 4–6 show some different cases which can occur under the restriction of nonnegativity. In Figure 4, every nontrivial ambiguity that can be constructed from $|\widehat{x}(\omega)|^2$, where $\mathbf{x}$ is the marked signal of length 6 being determined by the zero set

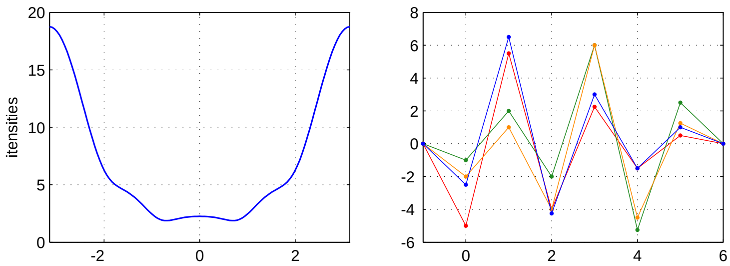
$$\left\{\beta_j\right\} := \left\{-3.65, -2.5, -1.8, -1.75, -1.2\right\}$$

via (6), is real and nonnegative, i.e., is a solution of the corresponding discrete phase retrieval problem. Again, we have plotted the solution set without reflected, conjugated signals. Here, we have $2^4 = 16$ different solutions, which by Corollary 3.3 is the maximal number of nontrivial ambiguities.



(a) Fourier intensities: $\left|\widehat{x}(\omega)\right|$ **(b)** Real nontrivial solutions: $x[n]$

Figure 5: Unique nonnegative solution of $A(\omega) = \left|\widehat{x}(\omega)\right|^2$



(a) Fourier intensities: $\left|\widehat{x}(\omega)\right|$ **(b)** Real nontrivial solutions: $x[n]$

Figure 6: Empty nonnegative solution set for $A(\omega) = \left|\widehat{x}(\omega)\right|^2$

In a different example, see Figure 5, the condition of nonnegativity is strong enough to ensure uniqueness of the problem. Here, the marked nonnegative solution $\mathbf{x}$ corresponds to the zero set

$$\{\beta_j\} := \{-1.5, -0.5 + 1.5\mathrm{i}, -0.5 - 1.5\mathrm{i}, 1 + 1\mathrm{i}, 1 - 1\mathrm{i}\}.$$

Note that the problem has only four nontrivial ambiguities in this example because in Theorem 3.1 the complex zeros must be chosen as complex conjugated pairs.

In the last example, Figure 6, the restriction of positivity is too strong. Here, every solution of the phase retrieval problem possesses some negative coefficients, i.e., the given phase retrieval problem cannot be solved by a real nonnegative signal. The blue signal, in Figure 6 (b), corresponds to the zero set

$$\{\beta_j\} := \{0.5, -0.5 + 1.5\mathrm{i}, -0.5 - 1.5\mathrm{i}, 1 + 1\mathrm{i}, 1 - 1\mathrm{i}\}. \qquad \circ$$

4.2 Restriction of the zero set

Based on the characterization of nontrivial solutions of the discrete phase retrieval problem using the zero sets of $P_A(z)$ in Theorem 3.1, we may consider only the solution $\widehat{x}(\omega)$ of $\left|\widehat{x}(\omega)\right|^2 = A(\omega)$ where the zero set $\{\beta_j : j = 1, \ldots, N-1\}$ is fixed in a way such that $|\beta_j| \leq 1$ for all $j = 1, \ldots, N-1$. Then, the zero set of $\widehat{x}(\omega)$ and hence the solution $\mathbf{x}$ is uniquely determined up to trivial ambiguities by Theorem 3.1. Alternatively, one may take the zero set, where $|\beta_j| \geq 1$ for all j.

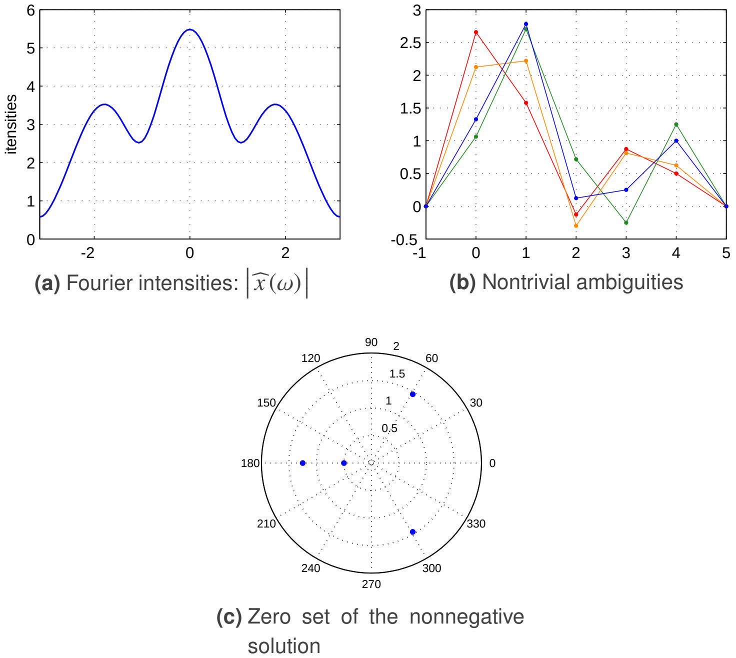


Figure 7: Unique solution of $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ for real and nonnegative signals with zeros inside and outside the unit circle

For the real case, this approach has been proposed already in [17, Theorem 8]. Unfortunately, we are not aware of any special physical feature of the phase retrieval solution that is obtained choosing the zero sets in the one or the other way.

Remark 4.2. Note that fixing the zeros inside or outside the unit circle is not compatible with the restriction to nonnegativity for real-valued signals. For example, in Figure 7, we consider a phase retrieval problem which has a unique solution under the restriction of nonnegativity. However, this solution corresponds to the zero set

$$\left\{\beta_j\right\} := \left\{-1.25, -0.5, 0.75 + 1.25\mathrm{i}, 0.75 - 1.25\mathrm{i}\right\},$$

where some zeros are inside and some zeros outside the unit circle as shown in Figure 7 (c). ○

4.3 Using additional endpoints of the signal

A different idea to enforce uniqueness of the phase retrieval problem is to use additionally known values of the signal $\mathbf{x}$ with fixed support $\{0, \ldots, N-1\}$. In [41], it had been assumed that for the real phase retrieval problem besides the autocorrelation function also the last signal value $x[N-1]$ is given. We want to examine in more detail, how far this additional condition leads towards a unique phase retrieval solution, thereby extending the considerations to the complex case.

Theorem 4.3. *Let a nonnegative trigonometric autocorrelation polynomial $A(\omega)$ of degree $N-1$ and a constant $C \in \mathbb{C}$ be given. Further, let $P_A(z)$ be the corresponding algebraic polynomial with the zero set*

$$\left\{\gamma_j,\, \overline{\gamma}_j^{-1} : j = 1, \ldots, N-1\right\}$$

as in (5).

The phase retrieval problem

$$A(\omega) = \left|\widehat{x}(\omega)\right|^2 \qquad \text{with endpoint} \qquad x[N-1] = C$$

has a unique solution $\mathbf{x} = (x[n])_{n=0}^{N-1} \in \mathbb{C}^N$ if and only if there exists a zero set

$$\left\{\beta_j : j = 1, \ldots, N-1\right\}$$

where $\beta_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}$ for $j = 1, \ldots, N-1$ such that the consistency condition

$$|C|^2 = |a[N-1]| \cdot \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \tag{7}$$

and

$$\prod_{\beta_j \in \Lambda} \left|\beta_j\right|^2 \neq 1 \tag{8}$$

for each nonempty proper subset $\Lambda \subset \{|\beta_j| \neq 1 : j \in \{1, \ldots, N-1\}\}$ is fulfilled.

Proof. 1. Using Theorem 3.1 with

$$B(\omega) = \widehat{x}(\omega) = \sum_{n=0}^{N-1} x[n] e^{-\mathrm{i}\omega n},$$

the endpoint of the signal $\mathbf{x}$ is of the form

$$x\,[N-1] = \mathrm{e}^{\mathrm{i}\alpha} \left[|a\,[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \right]^{\frac{1}{2}}, \tag{9}$$

where we assume that $\mathbf{x}$ is constructed by the zero set $\{\beta_j : j = 1, \ldots, N-1\}$ as given above. Hence, the endpoint condition $x[N-1] = C$ can only be satisfied if the consistency condition (7) is fulfilled. In this case, there always exists at least one solution of the phase retrieval problem $|\widehat{x}(\omega)|^2 = A(\omega)$.

2. We consider uniqueness of the solution vector $\mathbf{x}$. Suppose that we have a further solution $\widetilde{\mathbf{x}}$ of the phase retrieval problem. Then the endpoint $\widetilde{x}\,[N-1]$ again satisfies

$$C = \widetilde{x}\,[N-1] = \mathrm{e}^{\mathrm{i}\widetilde{\alpha}} \left[|a\,[N-1]| \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-1} \right]^{\frac{1}{2}},$$

where $\widetilde{\beta}_j \in \left\{\beta_j, \overline{\beta}_j^{-1}\right\}$ by Theorem 3.1. For simplifying the notation of the following, let

$$\widetilde{\beta}_j = \overline{\beta}_j^{-1} \quad (j = 1, \ldots, L) \qquad \text{and} \qquad \widetilde{\beta}_j = \beta_j \quad (j = L+1, \ldots, N-1)$$

for some $L \in \{1, \ldots, N-1\}$.

Thus, comparison of the endpoints $x[N-1] = \widetilde{x}[N-1]$ yields the identity

$$\mathrm{e}^{\mathrm{i}\alpha} \prod_{j=1}^{L} \left|\beta_j\right|^{-\frac{1}{2}} = \mathrm{e}^{\mathrm{i}\widetilde{\alpha}} \prod_{j=1}^{L} \left|\overline{\beta}_j\right|^{\frac{1}{2}},$$

i.e.,

$$\mathrm{e}^{\mathrm{i}(\alpha-\widetilde{\alpha})} = \prod_{j=1}^{L} \left|\beta_j\right|.$$

Since the right hand side is real and positive, it follows that $\alpha = \widetilde{\alpha}$ and hence

$$\prod_{j=1}^{L} \left|\beta_j\right| = 1. \tag{10}$$

In this product, we can omit all zeros β with $|\beta_j| = 1$. The remaining equality contradicts condition (8). Therefore, there exists no further solution of the phase retrieval problem.

Observe that also trivial ambiguities do not occur. Shift ambiguities are avoided by fixing the support of $\mathbf{x}$ to $\{0, \ldots, N-1\}$, the rotation angle α is determined by $\mathrm{e}^{\mathrm{i}\alpha} = \frac{C}{|C|}$ using $x[N-1] = C$ in (9), and finally the ambiguity caused by conjugation and reflection is already covered by the consideration above for $L = N-1$, where all zeros β_j switch to $\overline{\beta}_j^{-1}$. □

Assuming that $A(\omega)$ is the autocorrelation polynomial of the complex-valued signal $\mathbf{x} = (x[n])_{n=0}^{N-1}$, it is very likely that the phase retrieval problem is uniquely solvable if $x[n-1]$ is already known since the submanifold of all $(\beta_j)_{j=1}^{N-1} \in \mathbb{C}^{N-1}$ satisfying (**??**) has intrinsic (real) dimension smaller than $2N-2$, where $\mathbb{C}^{N-1}$ is considered to be embedded into $\mathbb{R}^{2N-2}$. Therefore, we call all autocorrelation polynomials satisfying (**??**) *generic*.

Corollary 4.4. *A signal $\mathbf{x} = (x[n])_{n=0}^{N-1}$ with a generic autocorrelation polynomial $A(\omega)$ can be uniquely reconstructed from $|\widehat{x}(\omega)|^2 = A(\omega)$ and $x[N-1]$.*

Remark 4.5. A similar result can be proved for real signals, that has been also considered in [41, Theorem 1]. In this case the zeros β_j are real or occur in complex conjugated pairs. All steps of the proof of Theorem 4.3 can then be similarly obtained, and the signal $\mathbf{x}$ can be recovered from its autocorrelation function and $x[N-1]$ almost surely.

One may now ask the question, how many signal points are needed to know beforehand in order to solve the phase retrieval problem always uniquely? In [30, Theorem 2] it has been shown for real signals $\mathbf{x} = (x[n])_{n=0}^{N-1}$ that the solution is unique if the $\left\lfloor \frac{N}{2} \right\rfloor$ signal values $x[n]$ for $n = \left\lceil \frac{N}{2} \right\rceil, \ldots, N-2, N-1$ are already given. Here, $\left\lfloor \frac{N}{2} \right\rfloor$ denotes the largest integer less than or equal $\frac{N}{2}$ and $\left\lceil \frac{N}{2} \right\rceil$ the smallest integer greater than or equal to $\frac{N}{2}$. Following the lines of that proof we can easily generalize the result to complex signals.

For the given values $x\left[\left\lfloor \frac{N}{2} \right\rfloor\right], \ldots, x[N-1]$, the remaining coefficients are directly encoded in the autocorrelation signal $\mathbf{a}$ in (2). They can be reconstructed by solving the linear equation system

$$\begin{pmatrix} \overline{x[N-1]} & & & \\ \overline{x[N-2]} & \overline{x[N-1]} & & \\ \vdots & \vdots & \ddots & \\ \overline{x\left[\left\lceil \frac{N}{2} \right\rceil\right]} & \overline{x\left[\left\lceil \frac{N}{2} \right\rceil + 1\right]} & \cdots & \overline{x[N-1]} \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x\left[\left\lfloor \frac{N}{2} \right\rfloor - 1\right] \end{pmatrix} = \begin{pmatrix} a[N-1] \\ a[N-2] \\ \vdots \\ a\left[\left\lceil \frac{N}{2} \right\rceil\right] \end{pmatrix}.$$

Since the first matrix is a lower left triangle matrix and $x[N-1] \neq 0$, this equation system has a unique solution.

Remark 4.6. Similarly as shown in the next Subsection 4.4, it is sufficient to know an arbitrary signal value $x[n]$, $n \in \{0, \ldots, N-1\} \setminus \{\frac{N-1}{2}\}$ instead of $x[N-1]$ in order to ensure a unique solution of the discrete phase retrieval problem with high probability.

4.4 Using moduli of the unknown signal

Instead of considering a given endpoint of the unknown signal $\mathbf{x} = (x[n])_{n=0}^{N-1}$, we now assume that besides the moduli of the Fourier transform $\left|\widehat{x}(\omega)\right|$ either all or at least some of the magnitudes $|x[n]|$, $n = 0, \ldots, N-1$ are given. Phase retrieval problems of this kind have been considered for example in [37, 26, 27], where a multilevel Gauss-Newton method has been proposed as a numerical approach. Again, we want to investigate, whether these conditions on $\mathbf{x}$ lead to a reduction of ambiguities of the phase retrieval problem.

Suppose that $\mathbf{x}$ and $\widetilde{\mathbf{x}}$ are two nontrivial solutions of a phase retrieval problem satisfying $|\widehat{x}(\omega)|^2 = |\widehat{\widetilde{x}}(\omega)|^2$ for all $\omega \in [0, 2\pi)$ and $|x[n]| = |\widetilde{x}[n]|$ for one or more indices $n \in \{0, \ldots, N-1\}$. We assume again that $P_A(z)$ determined by the autocorrelation polynomial $A(\omega) = |\widehat{x}(\omega)|^2$ has the zero set

$$\left\{\gamma_j, \overline{\gamma_j}^{-1} : j = 1, \ldots, N-1\right\}.$$

By Theorem 3.1, the solutions of the phase retrieval problem have a Fourier transform of the form

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha} \left[|a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \right]^{\frac{1}{2}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right) = \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\mathrm{i}\omega n}$$

and

$$\widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\widetilde{\alpha}} \left[|a\,[N-1]| \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-1} \right]^{\frac{1}{2}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \widetilde{\beta}_j \right) = \sum_{n=0}^{N-1} \widetilde{x}\,[n]\, \mathrm{e}^{-\mathrm{i}\omega n},$$

where $\beta_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\}, \widetilde{\beta}_j \in \left\{\gamma_j, \overline{\gamma}_j^{-1}\right\} = \left\{\beta_j, \overline{\beta}_j^{-1}\right\}$, and $\alpha, \widetilde{\alpha} \in [-\pi, \pi)$.

Now the additional conditions

$$|x\,[n]| = |\widetilde{x}\,[n]|$$

for $n \in \{0, \ldots, N-2\}$ imply by Vieta's formulas

$$\begin{aligned} &\prod_{j=1}^{N-1} \left|\beta_j\right|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \beta_{k_1} \cdots \beta_{k_n} \right| \\ &\qquad = \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_n} \right|, \end{aligned} \tag{11}$$

and particularly for the leading coefficient $|x\,[N-1]| = |\widetilde{x}\,[N-1]|$ it follows that

$$\prod_{j=1}^{N-1} \left|\beta_j\right|^{-\frac{1}{2}} = \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-\frac{1}{2}}. \tag{12}$$

Again, let us assume that $\widetilde{\beta}_j = \overline{\beta}_j^{-1}$ for $j = 1, \ldots, L$ and $\widetilde{\beta}_j = \beta_j$ else. Then (12) already leads to the condition $\prod_{j=1}^{L} \left|\beta_j\right| = 1$ similarly as the endpoint condition in the last subsection. Therefore, we get the following Corollary.

Corollary 4.7. *Let a nonnegative trigonometric autocorrelation polynomial $A(\omega)$ of degree $N-1$ and a nonnegative constant $C \in \mathbb{R}$ be given. Let $P_A(z)$ be the corresponding algebraic polynomial with the zero set*

$$\left\{\gamma_j, \overline{\gamma}_j^{-1} : j = 1, \ldots, N-1\right\}$$

as in (5).

The phase retrieval problem $|\widehat{x}(\omega)|^2 = A(\omega)$ with the additional condition $|x[N-1]| = C$ has a solution $\mathbf{x} = (x[n])_{n=0}^{N-1} \in \mathbb{C}^N$ if and only if there exists a zero set

$$\left\{\beta_j : j = 1, \ldots, N-1\right\},$$

where $\beta_j \in \{\gamma_j, \overline{\gamma}_j^{-1}\}$ for $j = 1, \ldots, N-1$ such that consistency condition

$$|C|^2 = |a[N-1]| \cdot \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \tag{13}$$

is satisfied. Moreover, this signal $\mathbf{x}$ *is uniquely determined up to trivial ambiguities caused by multiplication with an unimodular constant if*

$$\prod_{\beta_j \in \Lambda} |\beta_j| \neq 1 \tag{14}$$

for each nonempty proper subset $\Lambda \subset \{|\beta_j| \neq 1 : j \in \{1, \ldots N-1\}$ *is fulfilled.*

The conditions (11) are equivalent to

$$\left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \beta_{k_1} \cdots \beta_{k_n} \right| = \prod_{j=1}^{L} |\beta_j| \cdot \left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_n} \right|. \tag{15}$$

Each of these equations can be regarded as polynomial equation in the real and imaginary parts of the zeros β_j.

Hence, for any additionally fixed modulus value $|x[n]|$ being given for some $n \in \{0, \ldots, N-1\}$ and being consistent with the zero set corresponding to the autocorrelation polynomial $A(\omega) = |\widehat{x}(\omega)|^2$, we only obtain multiple nontrivial solutions if the zero set satisfies the polynomial equation (15) for this n. Thus we can equivalently replace the condition (14) in Corollary 4.7 by the condition

$$\left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \beta_{k_1} \cdots \beta_{k_n} \right| \neq \prod_{\beta_j \in \Lambda} |\beta_j| \cdot \left| \sum_{1 \le k_1 < \cdots < k_n \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_n} \right|$$

for some $n \in \{1, \ldots, N-1\}$ and with Λ as before.

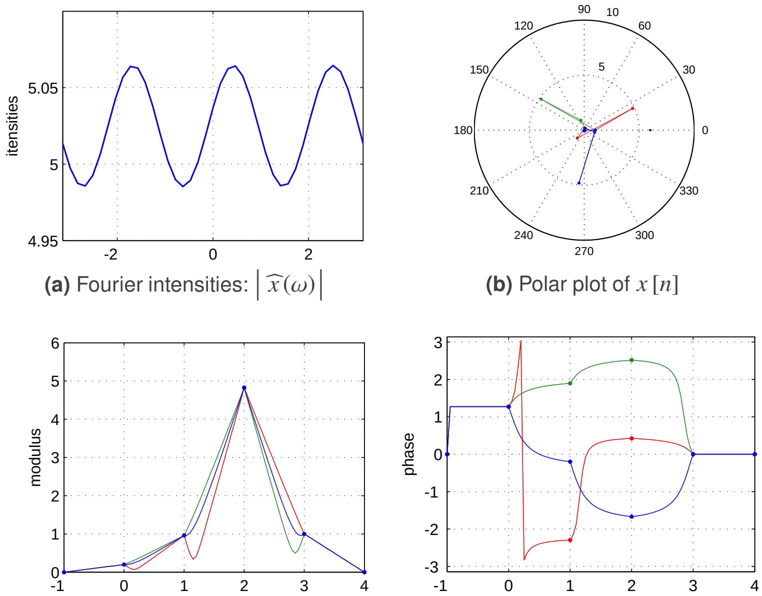
It can be shown that the polynomial equation (15) is nontrivial, except for the case $L = N-1$ and $n = \frac{N-1}{2}$ that leads to a trivial ambiguity caused by reflection and conjugation. Therefore, similarly as in Corollary 4.4, it is almost sure that the phase retrieval problem is uniquely solvable up to the trivial rotation ambiguity if besides the autocorrelation polynomial also the modulus of one signal value $x[n]$ is given. In the case $n = \frac{N-1}{2}$, the reconstruction is only unique up to rotated or reflected, conjugated signals.

One may ask the question, whether it is possible to determine the phase retrieval solution always uniquely (up to rotation), if more then one value $|x[n]|$ or even all values $|x[n]|$ for $n = 0, \ldots, N-1$ are given. As the next example shows, this is not the case.

Example 4.8. Figure 8 shows a phase retrieval problem for a given $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ and additionally given moduli $|x[n]|$ for $n = 1, \ldots, N-1$. The zero set corresponding to the marked signal $\mathbf{x}$ of length 4 is given by

$$\beta_1 := \tfrac{1}{2} + 5\mathrm{i}, \qquad \beta_2 := \frac{\mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi}}{\frac{1}{2} - 5\mathrm{i}}, \qquad \text{and} \qquad \beta_3 := \frac{\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}}{\frac{1}{2} - 5\mathrm{i}}.$$

In the specific example, the phase retrieval problem of dimension $N = 4$ has three different nontrivial ambiguities. ○



(a) Fourier intensities: $\left|\widehat{x}(\omega)\right|$

(b) Polar plot of $x[n]$

(c) Modulus: $|x[n]|$

(d) Phase: $\arg(x[n])$

Figure 8: Three nontrivial solutions of $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ with given moduli $|x[n]|$ for $n = 0, \ldots, 3$ as in Example 4.8.

5 Enforcing uniqueness by interference measurements

In this section, we investigate signal reconstruction where besides Fourier intensity measurements of the wanted signal also some intensity pattern resulting from interference with a known or unknown reference signal is available, as in Fourier holography. There have been different attempts to use interference with a known or unknown reference signal, and to exploit intensity measurements in order to achieve uniqueness of the phase retrieval problem. Let us shortly examine some different cases.

5.1 Interference with a known Dirac signal

First, as in [22], we consider the interference with a known Dirac signal and generalize this idea to finite length complex signals.

Theorem 5.1. *Let the autocorrelation polynomial* $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ *of the finitely supported signal* $\mathbf{x} = (x[n])_{n=0}^{N-1}$ *and the autocorrelation polynomial* $\widetilde{A}(\omega) = \left|\widehat{y}(\omega)\right|^2$ *be given, where*

$$y[n] = x[n] + C\delta[n - n_0]$$

and $C \in \mathbb{C} \setminus \{0\}$, $n_0 \in \mathbb{Z}$ *are known, i.e.,* $y[n_0] = x[n_0] + C$ *and* $y[n] = x[n]$ *for* $n \in \mathbb{Z} \setminus \{n_0\}$. *Then* $\mathbf{x}$ *can be uniquely reconstructed from the autocorrelation polynomials* $A(\omega)$ *and* $\widetilde{A}(\omega)$, *and the constants* C *and* n_0 *up to one trivial ambiguity.*

Proof. We can follow the lines of the proof given in [22]. The Fourier transform yields
$$\widehat{y}(\omega) = \widehat{x}(\omega) + C\mathrm{e}^{-\mathrm{i}\omega n_0}$$
and
$$\left|\widehat{y}(\omega)\right|^2 = \left|\widehat{x}(\omega)\right|^2 + 2\,|C|\left|\widehat{x}(\omega)\right|\,\cos\left(\phi\left(\omega\right) + n_0\omega - \alpha\right) + |C|^2,$$
where $\phi(\omega) \in [0, 2\pi)$ denotes the phase of $\widehat{x}(\omega)$, i.e., $\widehat{x}(\omega) = \left|\widehat{x}(\omega)\right|\,\mathrm{e}^{\mathrm{i}\phi(\omega)}$, and $\alpha \in [0, 2\pi)$ the phase of C, i.e., $C = |C|\,\mathrm{e}^{\mathrm{i}\alpha}$. Here, the unknown phase $\phi(\omega)$ is directly encoded, and we find
$$\cos\left(\phi\left(\omega\right) + n_0\omega - \alpha\right) = \frac{\left|\widehat{y}(\omega)\right|^2 - \left|\widehat{x}(\omega)\right|^2 - |C|^2}{2\,|C|\left|\widehat{x}(\omega)\right|} \qquad \text{for} \qquad \left|\widehat{x}(\omega)\right| \neq 0.$$

Since $\widehat{x}(\omega)$ is a nonvanishing trigonometric polynomial, we have $\widehat{x}(\omega) \neq 0$ almost everywhere, and $\cos(\phi(\omega) + n_0\omega - \alpha)$ is also for $\widetilde{\omega}$ with $\widehat{x}(\widetilde{\omega}) = 0$ well determined by taking the limit value $\omega \to \widetilde{\omega}$ on the right hand side of the equation above. Hence, we obtain for all $\omega \in [-\pi, \pi)$ the values
$$\phi\left(\omega\right) + n_0\omega - \alpha = \pm\arccos\left(\frac{\left|\widehat{y}(\omega)\right|^2 - \left|\widehat{x}(\omega)\right|^2 - |C|^2}{2\,|C|\left|\widehat{x}(\omega)\right|}\right) + 2\pi k, \quad k \in \mathbb{Z}. \tag{16}$$

Let now $\widetilde{\mathbf{x}}$ be a second solution of the phase retrieval problem, i.e.,
$$\left|\widehat{x}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega)\right|^2, \qquad \left|\widehat{x}(\omega) + C\,\mathrm{e}^{-\mathrm{i}n_0\omega}\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + C\,\mathrm{e}^{-\mathrm{i}n_0\omega}\right|^2$$
With $\widehat{\widetilde{x}}(\omega) = \left|\widehat{\widetilde{x}}(\omega)\right|\mathrm{e}^{\mathrm{i}\widetilde{\phi}}$, it hence follows from (16) that
$$\phi(\omega) = \pm\left[\widetilde{\phi}(\omega) + n_0\omega - \alpha\right] - n_0\omega + \alpha + 2\pi k$$
Since differences of phases by a multiple of 2π do not give different solutions and because $\widehat{x}$ resp. $\widehat{\widetilde{x}}$ are trigonometric polynomials and hence continuous, we need to consider only two cases, namely either that $\phi(\omega) = \widetilde{\phi}(\omega)$ or $\phi(\omega) = (-\widetilde{\phi}(\omega) - 2n_0\omega + 2\alpha)$. Thus, there is only one possible second solution of the form
$$\widehat{\widetilde{x}}(\omega) = \left|\widehat{x}(\omega)\right|\,\mathrm{e}^{-\mathrm{i}\phi(\omega) - 2\mathrm{i}n_0\omega + 2\mathrm{i}\alpha}$$
resulting after support shift to $\{0, \ldots, N-1\}$ in the trivial ambiguity
$$(\widetilde{x}[n]) = \left(\mathrm{e}^{2\mathrm{i}\alpha}\,\overline{x[N-1-n]}\right).$$
This proves the assertion. □

Remark 5.2. (i) Assuming that $n_0 \in \{0, \ldots, N-1\}$, we need in the complex case $2N-1$ measurements to determine the autocorrelation polynomial $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ and further $2N-1$ measurements to determine the autocorrelation polynomial $\widetilde{A}(\omega) = \left|\widehat{y}(\omega)\right|^2$. In the real case, we need only N measurements for each of the autocorrelation polynomials.

(ii) One may also take $n_0 \in \mathbb{Z} \setminus \{0, \dots N-1\}$ thereby enlarging the support of $\mathbf{y}$ (compared to the support of $\mathbf{x}$). In this case we need more than $2N-1$ measurements to recover $\left|\widehat{y}(\omega)\right|^2$ depending on the support length of the signal $\mathbf{y}$. One exception remarked in [22] is the case $n_0 = 2N-1$, where $4N-1$, in the real case $2N$, measurements are needed to recover $\left|\widehat{y}(\omega)\right|^2$ but where $\left|\widehat{x}(\omega)\right|^2$ does not need to be measured. The autocorrelation polynomial $\left|\widehat{x}(\omega)\right|^2$ can be directly recovered from $\left|\widehat{y}(\omega)\right|^2$ observing that

$$\widehat{y}(\omega) = \widehat{x}(\omega) + C\,\mathrm{e}^{\mathrm{i}\omega(2N-1)}$$

and

$$\left|\widehat{y}(\omega)\right|^2 = \left|\widehat{x}(\omega)\right|^2 + |C|^2 + \widehat{x}(\omega)\,\overline{C}\,\mathrm{e}^{-\mathrm{i}\omega(2N-1)} + \overline{\widehat{x}(\omega)}\,C\,\mathrm{e}^{\mathrm{i}\omega(2N-1)},$$

where the coefficients of the three polynomials $\left|\widehat{x}(\omega)\right|^2$, $\widehat{x}(\omega)\,\overline{C}\,\mathrm{e}^{-\mathrm{i}\omega(2N-1)}$ and $\overline{\widehat{x}(\omega)}\,C\,\mathrm{e}^{\mathrm{i}\omega(2N-1)}$ do not superpose. ○

5.2 Interference with a known reference signal

Let us now consider the interference with a known reference signal $\mathbf{h} := (h[n])_{n\in\mathbb{Z}}$ with finite support thereby generalizing and simplifying the results in [22].

Theorem 5.3. *Let the autocorrelation polynomial $A(\omega) = \left|\widehat{x}(\omega)\right|^2$ of the finitely supported signal $\mathbf{x} = (x[n])_{n=0}^{N-1}$ and the autocorrelation polynomial $\widetilde{A}(\omega) = \left|\widehat{y}(\omega)\right|^2$ be given, where*

$$y[n] = x[n] + h[n]$$

for some known finitely supported reference signal $\mathbf{h} = (h[n])_{n\in\mathbb{Z}}$. If the signal $\mathbf{h}$ possesses a linear phase, then $\mathbf{x}$ can be uniquely reconstructed up to one trivial ambiguity. If $\mathbf{h}$ does not have a linear phase, then we generally obtain two nontrivial solutions.

Proof. Note that also $\mathbf{y} := (y[n])_{n\in\mathbb{Z}}$ is a finite length signal. As before, we observe that

$$\begin{aligned}\left|\widehat{y}(\omega)\right|^2 &= \left(\widehat{x}(\omega) + \widehat{h}(\omega)\right)\left(\overline{\widehat{x}(\omega)} + \overline{\widehat{h}(\omega)}\right)\\ &= \left|\widehat{x}(\omega)\right|^2 + \left|\widehat{h}(\omega)\right| + 2\,\mathrm{Re}\left(\widehat{x}(\omega)\,\overline{\widehat{h}(\omega)}\right).\end{aligned}$$

With $\widehat{x}(\omega) = \left|\widehat{x}(\omega)\right|\,\mathrm{e}^{\mathrm{i}\phi(\omega)}$ and $\widehat{h}(\omega) = \left|\widehat{h}(\omega)\right|\,\mathrm{e}^{\mathrm{i}\psi(\omega)}$, where $\phi(\omega)$ and $\psi(\omega)$ denote the phase functions of $\widehat{x}$ and $\widehat{h}$, it follows that

$$\left|\widehat{y}(\omega)\right|^2 = \left|\widehat{x}(\omega)\right|^2 + \left|\widehat{h}(\omega)\right|^2 + 2\left|\widehat{x}(\omega)\right|\left|\widehat{h}(\omega)\right| \cos\left(\phi(\omega) - \psi(\omega)\right)$$

such that

$$\phi(\omega) - \psi(\omega) = \pm \arccos\left(\frac{\left|\widehat{y}(\omega)\right|^2 - \left|\widehat{x}(\omega)\right|^2 - \left|\widehat{h}(\omega)\right|^2}{2\left|\widehat{x}(\omega)\right|\left|\widehat{h}(\omega)\right|}\right) + 2\pi k$$

for ω with $\widehat{x}(\omega)\widehat{h}(\omega) \neq 0$ and $k \in \mathbb{Z}$.

Similarly as in the proof of Theorem 5.1, we can restrict our considerations to $k = 0$. Again, we only find two different solutions since $\widehat{x}$ and $\widehat{\overline{x}}$ are continuous. Both solutions are related by

$$\phi_1(\omega) - \psi(\omega) = -\phi_2(\omega) + \psi(\omega),$$

i.e.,

$$\phi_2(\omega) = -\phi_1(\omega) + 2\psi(\omega).$$

Considering the corresponding signals

$$\widehat{x_1}(\omega) = \left|\widehat{x}(\omega)\right| \mathrm{e}^{\mathrm{i}\phi_1(\omega)} \qquad \text{and} \qquad \widehat{x_2}(\omega) = \left|\widehat{x}(\omega)\right| \mathrm{e}^{-\mathrm{i}\phi_1(\omega)+2\mathrm{i}\psi(\omega)},$$

we want to examine if this second solution is a trivial ambiguity.

According to Proposition 2.1, the ambiguity is trivial if and only if $\psi(\omega)$ is of the form

$$\psi(\omega) = -n_0\omega + \alpha \qquad \text{or} \qquad \psi(\omega) = \phi_1(\omega) - n_0\omega + \alpha$$

for some $n_0 \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. In the second case, $\mathbf{x}_2$ is obtained from $\mathbf{x}_1$ by a $2n_0$-shift and rotation, and in the first case, beside shift and rotation also conjugation and reflection of $\mathbf{x}_1$ are involved. For $\psi(\omega) = -n_0\omega + \alpha$ the reference signal $\mathbf{h}$ possesses a linear phase, i.e.,

$$\widehat{h}(\omega) = \left|\widehat{h}(\omega)\right| \mathrm{e}^{-\mathrm{i}(n_0\omega-\alpha)}.$$

This is equivalent to

$$\mathrm{e}^{-\mathrm{i}\alpha}\, \mathrm{e}^{\mathrm{i}n_0\omega}\, \widehat{h}(\omega) = \left|\widehat{h}(\omega)\right| = \overline{\left|\widehat{h}(\omega)\right|} = \mathrm{e}^{\mathrm{i}\alpha}\, \mathrm{e}^{-\mathrm{i}n_0\omega}\, \overline{\widehat{h}(\omega)},$$

i.e.,

$$\mathrm{e}^{-\mathrm{i}\alpha} h[n + n_0] = \mathrm{e}^{\mathrm{i}\alpha} h[n_0 - n].$$

Thus, when the known reference signal $\mathbf{h}$ has linear phase, the phase retrieval problem is uniquely solvable up to one trivial ambiguity. If $\mathbf{h}$ does not have a linear phase, we obtain up to two nontrivial solutions. Observe here that it is impossible to fulfill $\psi(\omega) = \phi(\omega) - n_0\omega + \alpha$ since the phase $\psi(\omega)$ is unknown and needs to be reconstructed. □

5.3 Interference with an unknown reference signal with known intensity

Let us finally consider the problem of phase reconstruction for two unknown finite length signals $\mathbf{x}$, $\mathbf{h}$ from the intensities

$$\left|\widehat{x}(\omega)\right|^2, \qquad \left|\widehat{h}(\omega)\right|^2, \qquad \text{and} \qquad \left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2.$$

In case of real signals, the idea goes back to [24]. For complex signals, we refer to [34], where beside the three intensities given above, a forth intensity

$$\left|\widehat{x}(\omega) + \mathrm{i}\,\widehat{h}(\omega)\right|^2$$

is used for reconstruction. Here, we state the result for complex signals $\mathbf{x}$, $\mathbf{h}$ and give a new complete proof based on our findings in Theorem 2.3.

Theorem 5.4. *Let $(x[n])$ and $(h[n])$ be two complex finite support sequences, and assume that the factorizations of their discrete Fourier transforms*

$$\widehat{x}(\omega) = x[N_1 - 1] \prod_{j=1}^{N_1-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right),$$

$$\widehat{h}(\omega) = \mathrm{e}^{-\mathrm{i}\omega k_1} h[N_2 - 1] \prod_{j=1}^{N_2-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)$$

with an integer shift $k_1 \in \mathbb{Z}$ have no common roots, i.e., $\eta_j \neq \gamma_k$ for all $j = 1, \dots N_1 - 1$, $k = 1, \dots N_2 - 1$. Then

$$x[n] \qquad \text{and} \qquad h[n]$$

can be uniquely recovered from $\left|\widehat{x}(\omega)\right|^2$, $\left|\widehat{h}(\omega)\right|^2$ and $\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2$ up to trivial ambiguities.

Proof. 1. Assume that the phase retrieval problem has two nontrivial solution pairs $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$ such that $|\widehat{x}(\omega)|^2 = |\widehat{\widetilde{x}}(\omega)|^2$, $|\widehat{h}(\omega)|^2 = |\widehat{\widetilde{h}}(\omega)|^2$ and $|\widehat{y}(\omega)|^2 = |\widehat{\widetilde{y}}(\omega)|^2$, where

$$\widehat{y}(\omega) = \widehat{x}(\omega) + \widehat{h}(\omega), \qquad \widehat{\widetilde{y}}(\omega) = \widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega).$$

According to Theorem 2.3, there exist convolution representations, shifts, and rotations such that

$$\widehat{x}(\omega) = \widehat{x_1}(\omega)\,\widehat{x_2}(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\,\widehat{x_1}(\omega)\,\overline{\widehat{x_2}(\omega)}, \tag{17}$$

and

$$\widehat{h}(\omega) = \mathrm{e}^{-\mathrm{i}k_1\omega}\widehat{h_1}(\omega)\,\widehat{h_2}(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\,\mathrm{e}^{-\mathrm{i}k_2\omega}\widehat{h_1}(\omega)\,\overline{\widehat{h_2}(\omega)} \tag{18}$$

for some $\alpha_1, \alpha_2 \in [-\pi, \pi)$ and shifts $k_1, k_2 \in \mathbb{Z}$, where w.l.o.g. we consider shifts only in $\widehat{h}$ and $\widehat{\widetilde{h}}$. Here, we assume that

$$\operatorname{supp} x_1 = \{0, \ldots, m_1\}, \quad \operatorname{supp} x_2 = \{0, \ldots, N_1 - m_1 - 1\},$$
$$\operatorname{supp} h_1 = \{0, \ldots, m_2\}, \quad \operatorname{supp} h_2 = \{0, \ldots, N_2 - m_2 - 1\},$$

i.e., $\widehat{x_1}$ corresponds to m_1 linear factors and $\widehat{h_1}$ to m_2 factors. Now,

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2$$

together with $|\widehat{x}(\omega)|^2 = |\widehat{\widetilde{x}}(\omega)|^2$ and $|\widehat{h}(\omega)|^2 = |\widehat{\widetilde{h}}(\omega)|^2$ implies

$$\widehat{x}(\omega)\,\overline{\widehat{h}(\omega)} + \overline{\widehat{x}(\omega)}\,\widehat{h}(\omega) = \widehat{\widetilde{x}}(\omega)\,\overline{\widehat{\widetilde{h}}(\omega)} + \overline{\widehat{\widetilde{x}}(\omega)}\,\widehat{\widetilde{h}}(\omega)\,.$$

Incorporating (17) and (18) yields

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}k_1\omega}\,\widehat{x_1}(\omega)\,\widehat{x_2}(\omega)\,\overline{\widehat{h_1}(\omega)}\,\overline{\widehat{h_2}(\omega)} + \mathrm{e}^{-\mathrm{i}k_1\omega}\,\overline{\widehat{x_1}(\omega)}\,\overline{\widehat{x_2}(\omega)}\,\widehat{h_1}(\omega)\,\widehat{h_2}(\omega)\\
&\quad = \mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{\mathrm{i}k_2\omega}\,\widehat{x_1}(\omega)\,\overline{\widehat{x_2}(\omega)}\,\overline{\widehat{h_1}(\omega)}\,\widehat{h_2}(\omega)\\
&\qquad + \mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}k_2\omega}\,\overline{\widehat{x_1}(\omega)}\,\widehat{x_2}(\omega)\,\widehat{h_1}(\omega)\,\overline{\widehat{h_2}(\omega)},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left[\mathrm{e}^{\mathrm{i}k_1\omega}\,\widehat{x_1}(\omega)\,\overline{\widehat{h_1}(\omega)} - \mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}k_2\omega}\,\overline{\widehat{x_1}(\omega)}\,\widehat{h_1}(\omega)\right]\\
&\qquad\cdot\left[\widehat{x_2}(\omega)\,\overline{\widehat{h_2}(\omega)} - \mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}\omega(k_1-k_2)}\,\overline{\widehat{x_2}(\omega)}\,\widehat{h_2}(\omega)\right] = 0.
\end{aligned}$$

Hence, either

$$\widehat{x_1}(\omega)\,\overline{\widehat{h_1}(\omega)} = \mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}\omega(k_1+k_2)}\,\overline{\widehat{x_1}(\omega)}\,\widehat{h_1}(\omega)\,, \tag{19}$$

or

$$\widehat{x_2}(\omega)\,\overline{\widehat{h_2}(\omega)} = \mathrm{e}^{\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}\omega(k_1-k_2)}\,\overline{\widehat{x_2}(\omega)}\,\widehat{h_2}(\omega) \tag{20}$$

has to be fulfilled.

2. Suppose that (19) is true, i.e., $\widehat{x_1}(\omega)\,\overline{\widehat{h_1}(\omega)}$ has linear phase. Considering the factorizations

$$\widehat{x_1}(\omega) = x_1[m_1]\prod_{j=1}^{m_1}\left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right),$$

$$\widehat{h_1}(\omega) = h_1[m_2]\prod_{\ell=1}^{m_2}\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_\ell\right)$$

(19) leads to

$$x_1[m_1]\,\overline{h_1[m_2]}\prod_{j=1}^{m_1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\eta_j\right)\prod_{\ell=1}^{m_2}\left(\mathrm{e}^{\mathrm{i}\omega}-\overline{\gamma}_\ell\right)$$
$$=\mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{-\mathrm{i}\omega(k_1+k_2)}\,\overline{x_1[m_1]}\,h_1[m_2]\prod_{j=1}^{m_1}\left(\mathrm{e}^{\mathrm{i}\omega}-\overline{\eta}_j\right)\prod_{\ell=1}^{m_2}\left(\mathrm{e}^{-\mathrm{i}\omega}-\gamma_\ell\right).$$

Hence, with $C := x_1[m_1]\,\overline{h_1[m_2]}$,

$$C\,\mathrm{e}^{\mathrm{i}\omega m_2}\prod_{\ell=1}^{m_2}(-\overline{\gamma}_\ell)\prod_{j=1}^{m_1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\eta_j\right)\prod_{\ell=1}^{m_2}\left(\mathrm{e}^{-\mathrm{i}\omega}-\frac{1}{\overline{\gamma}_\ell}\right)$$
$$=\overline{C}\,\mathrm{e}^{-\mathrm{i}(\alpha_1-\alpha_2)}\,\mathrm{e}^{\mathrm{i}\omega(-k_1-k_2+m_1)}\prod_{j=1}^{m_1}\left(-\overline{\eta}_j\right)\prod_{j=1}^{m_1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\frac{1}{\overline{\eta}_j}\right)\prod_{\ell=1}^{m_2}\left(\mathrm{e}^{-\mathrm{i}\omega}-\gamma_\ell\right). \tag{21}$$

As we have on both sides trigonometric polynomials of a fixed degree, it follows that

$$m_2+k_2+k_1=m_1. \tag{22}$$

If we interpret the left and right hand side of the above equation (21) as polynomials in $z := \mathrm{e}^{-\mathrm{i}\omega}$, then we have on both sides the same zero set, i.e.,

$$\left\{\eta_j : j=1,\dots,m_1\right\}\cup\left\{\overline{\gamma}_\ell^{-1} : \ell=1,\dots,m_2\right\}$$
$$=\left\{\overline{\eta}_j^{-1} : j=1,\dots,m_1\right\}\cup\left\{\gamma_\ell : \ell=1,\dots,m_2\right\}.$$

Using the assumption of the theorem that $\eta_j \neq \gamma_\ell$ for all j and ℓ, we can conclude that the roots on both sides of (21) lying not on the unit circle can only occur in pairs $\left(\eta_j,\overline{\eta}_j^{-1}\right)$ and $\left(\gamma_\ell,\overline{\gamma}_\ell^{-1}\right)$. Therefore we can assume that $\widehat{x_1}(\omega)$ possesses the zeros η_j, $\overline{\eta_j}^{-1}$ for $j=1,\dots,L_1$ as well as possible zeros on the unit circle $\eta_{j+2L_1}=\mathrm{e}^{\mathrm{i}\nu_j}$, $j=1,\dots m_1-2L_1$, and similarly $\widehat{h_1}(\omega)$ possesses the zeros γ_j, $\overline{\gamma_j}^{-1}$ for $j=1,\dots,L_2$ as well as possible zeros on the unit circle $\gamma_{j+2L_2}=\mathrm{e}^{\mathrm{i}\mu_j}$, $j=1,\dots m_2-2L_2$. Therewith, we find for $\widehat{x}$ and $\widehat{h}$ representations of the form

$$\widehat{x_1}(\omega)=x_1[m_1]\prod_{j=1}^{L_1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\eta_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega}-\frac{1}{\overline{\eta}_j}\right)\prod_{j=1}^{m_1-2L_1}\left(\mathrm{e}^{-\mathrm{i}\omega}-\mathrm{e}^{-\mathrm{i}\nu_j}\right)$$

and

$$\widehat{h_1}(\omega)=h_1[m_2]\prod_{\ell=1}^{L_2}\left(\mathrm{e}^{-\mathrm{i}\omega}-\gamma_\ell\right)\left(\mathrm{e}^{-\mathrm{i}\omega}-\frac{1}{\overline{\gamma}_\ell}\right)\prod_{\ell=1}^{m_2-2L_2}\left(\mathrm{e}^{-\mathrm{i}\omega}-\mathrm{e}^{-\mathrm{i}\mu_\ell}\right).$$

In particular, $\mathbf{x}_1$ and $\mathbf{h}_1$ are invariant under reflection, see Corollary 3.3, i.e.,

$$\widehat{x_1}(\omega)=\mathrm{e}^{-\mathrm{i}\omega m_1}\,\overline{\widehat{x_1}(\omega)} \qquad \text{and} \qquad \widehat{h_1}(\omega)=\mathrm{e}^{-\mathrm{i}\omega m_2}\,\overline{\widehat{h_1}(\omega)}.$$

Together with (17) and (18), it follows that

$$\widehat{x}(\omega) = \widehat{x_1}(\omega)\widehat{x_2}(\omega), \qquad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\,\mathrm{e}^{-\mathrm{i}\omega m_1}\,\overline{\widehat{x_1}(\omega)}\,\overline{\widehat{x_2}(\omega)}$$

$$\widehat{h}(\omega) = \mathrm{e}^{-\mathrm{i}k_1\omega}\widehat{h_1}(\omega)\widehat{h_2}(\omega), \qquad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\,\mathrm{e}^{-\mathrm{i}\omega(m_2+k_2)}\,\overline{\widehat{h_1}(\omega)}\,\overline{\widehat{h_2}(\omega)}$$

and thus $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{h}}$ are trivial ambiguities of $\mathbf{x}$ and $\mathbf{h}$. We obtain by (22)

$$\begin{aligned}
\widehat{y}(\omega) &= \widehat{x_1}(\omega)\widehat{x_2}(\omega) + \mathrm{e}^{-\mathrm{i}k_1\omega}\widehat{h_1}(\omega)\widehat{h_2}(\omega), \\
\widehat{\widetilde{y}}(\omega) &= \left(\mathrm{e}^{\mathrm{i}\alpha_1}\,\overline{\widehat{x_1}(\omega)}\,\overline{\widehat{x_2}(\omega)} + \mathrm{e}^{\mathrm{i}\alpha_2}\,\mathrm{e}^{\mathrm{i}k_1\omega}\,\overline{\widehat{h_1}(\omega)}\,\overline{\widehat{h_2}(\omega)}\right)\mathrm{e}^{-\mathrm{i}\omega m_1} \\
&= \overline{\left(\mathrm{e}^{-\mathrm{i}\alpha_1}\,\widehat{x_1}(\omega)\,\widehat{x_2}(\omega) + \mathrm{e}^{-\mathrm{i}\alpha_2}\,\mathrm{e}^{-\mathrm{i}k_1\omega}\,\widehat{h_1}(\omega)\,\widehat{h_2}(\omega)\right)}\,\mathrm{e}^{-\mathrm{i}\omega m_1}.
\end{aligned}$$

For $\alpha_1 = \alpha_2$, the signal sums $\widetilde{\mathbf{y}}$ and $\mathbf{y}$ are also trivial ambiguities of each other.

3. If the condition (20) holds true then a similar procedure leads to the identities

$$\widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\,\mathrm{e}^{\mathrm{i}\omega(N_1-m_1-1)}\,\widehat{x}(\omega) \qquad \text{and} \qquad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\,\mathrm{e}^{\mathrm{i}\omega(-k_2+N_2-m_2-1)}\,\widehat{h}(\omega).$$

and to the condition $S := N_2 - m_2 - 1 - k_2 + k_1 = N_1 - m_1 - 1$ such that

$$\begin{aligned}
\widehat{y}(\omega) &= \widehat{x_1}(\omega)\widehat{x_2}(\omega) + \mathrm{e}^{-\mathrm{i}\omega k_1}\,\widehat{h_1}(\omega)\widehat{h_2}(\omega), \\
\widehat{\widetilde{y}}(\omega) &= \left(\mathrm{e}^{\mathrm{i}\alpha_1}\,\widehat{x_1}(\omega)\,\widehat{x_2}(\omega) + \mathrm{e}^{\mathrm{i}\alpha_2}\,\mathrm{e}^{-\mathrm{i}\omega k_1}\,\widehat{h_1}(\omega)\,\widehat{h_2}(\omega)\right)\mathrm{e}^{\mathrm{i}\omega S}.
\end{aligned}$$

Hence, we can always recover $\mathbf{x}$ and $\mathbf{h}$ uniquely up to trivial ambiguities in both cases. □

Remark 5.5. (i) If all signals are real, see [24], then the prefactor C is also real, i.e., $C = \overline{C}$. Further the real zeros of $\widehat{x_1}$ and $\widehat{h_1}$ occur in pairs

$$\left(\eta_j, \overline{\eta}_j^{-1}\right) \qquad \text{and} \qquad \left(\gamma_\ell, \overline{\gamma}_\ell^{-1}\right)$$

but the complex zeros occur in quads

$$\left(\eta_j, \overline{\eta}_j, \eta_j^{-1}, \overline{\eta}_j^{-1}\right) \qquad \text{and} \qquad \left(\gamma_\ell, \overline{\gamma}_\ell, \gamma_\ell^{-1}, \overline{\gamma}_\ell^{-1}\right).$$

That implies

$$\prod_{j=1}^{N_1-m_1-1}\left(-\overline{\eta}_j\right) = 1 \qquad \text{and} \qquad \prod_{\ell=1}^{N_2-m_2-1}\left(-\overline{\gamma}_\ell\right) = 1$$

and therefore $\alpha_1 = \alpha_2 \in \{0, \pi\}$, i.e., $\widetilde{\mathbf{y}}$ and $\mathbf{y}$ always are trivial ambiguities of each other. The same can be observed in the case when (20) holds true. Hence, we can recover $\mathbf{x}$, $\mathbf{h}$ and $\mathbf{y}$ uniquely up to trivial ambiguities from the given autocorrelations functions.

(ii) Note that in the complex setting, the rotations α_1 and α_2 can be different. For example, if $\widehat{x}$ is a real function and $\widehat{h}$ is an imaginary function, then $\widehat{\widetilde{y}} = \widehat{x} + \mathrm{e}^{\mathrm{i}\pi}\,\widehat{h}$ solve the given phase retrieval problem with $\alpha_1 = 0$ and $\alpha_2 = \pi$.

(iii) In [34], the same complex phase retrieval problem is considered, but here a fourth autocorrelation function

$$\left|\widehat{x}(\omega) + \mathrm{i}\,\widehat{h}(\omega)\right|^2$$

is employed to ensure the unique reconstruction up to trivial ambiguities. The advantage of a fourth measurement set is that the signals **x** and **h** can be recovered easily by using the complex polarization formula and comparing the roots of the different polynomials.

(iv) In [8], beside the intensity $|\widehat{x}(\omega)|^2$ also the intensities of interferences of $\widehat{x}(\omega)$ with shifted versions of itself,

$$\left|\widehat{x}(\omega) + \widehat{x}\left(\omega - \tfrac{2\pi s}{N}\right)\right|^2 \qquad \text{and} \qquad \left|\widehat{x}(\omega) - \mathrm{i}\,\widehat{x}\left(\omega - \tfrac{2\pi s}{N}\right)\right|^2$$

are applied. With the notation $\widehat{x}\left(\frac{2\pi k}{N}\right) = \left|\widehat{x}\left(\frac{2\pi k}{N}\right)\right| \mathrm{e}^{i\phi_k}$ a comparison of

$$\left|\widehat{x}(\omega) + \widehat{x}\left(\omega - \tfrac{2\pi s}{N}\right)\right|^2 \qquad \text{and} \qquad \left|\widehat{x}(\omega) - \mathrm{i}\,\widehat{x}\left(\omega - \tfrac{2\pi s}{N}\right)\right|^2$$

for $\omega = \frac{2\pi k}{N}$, $k = 0, \ldots, N-1$ then yields the values for the phase differences $\phi_{k-s} - \phi_k$, $k = 0, \ldots, N-1$. Therefore, if $s \in \mathbb{N}$ is prime with N, the signal **x** can be uniquely reconstructed from these intensities up to the trivial rotation ambiguity. ○

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