A Generalized Prony Method for Reconstruction of Sparse Sums of Eigenfunctions of Linear Operators

Thomas Peter[*] Gerlind Plonka[†]

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Abstract

We derive a new generalization of Prony's method to reconstruct M-sparse expansions of (generalized) eigenfunctions of linear operators from only $\mathcal{O}(M)$ suitable values in a deterministic way. The proposed method covers the well-known reconstruction methods for M-sparse sums of exponentials as well as for the interpolation of M-sparse polynomials by using special linear operators in $C(\mathbb{R})$. Further, we can derive new reconstruction formulas for M-sparse expansions of orthogonal polynomials using the Sturm-Liouville operator. The method is also applied to the recovery of M-sparse vectors in finite-dimensional vector spaces.

Key words: Sparse expansions of eigenfunctions, sparse sums of exponentials, sparse polynomials, sparse expansions of orthogonal polynomials, sparse vectors, annihilating filters, nonlinear approximation.

Mathematics Subject Classification: 34L10, 65D15, 65T40, 15A18, 47A70

1 Introduction

In signal analysis, we often have some a priori knowledge about the underlying structure of the wanted signal that we need to exploit suitably. Using this structure, we are faced with the problem of determining a certain number of parameters from the given signal measurements. Considering for example a structured function of the form

$$f(\omega) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\omega T_j} \tag{1.1}$$

with (unknown) complex parameters c_j and T_j, $j = 1, \ldots, M$, and assuming that $-\pi < \operatorname{Im} T_1 < \ldots < \operatorname{Im} T_M < \pi$, we aim to reconstruct c_j and T_j from a given small amount of (possibly noisy) measurement values $f(\ell)$. Using Prony's method [27] or one of its stabilized variants, we are able to reconstruct f with only $2M$ function values $f(\ell)$, $\ell = 0, \ldots, 2M-1$, see [5, 17, 24, 25, 28]. The solution of this problem involves the determination of a so-called **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} (z - \lambda_j)$$

[*]University of Göttingen, Institute for Numerical and Applied Mathematics, Lotzestr. 16-18, 37083 Göttingen, Germany. Email: t.peter@math.uni-goettingen.de

[†]University of Göttingen, Institute for Numerical and Applied Mathematics, Lotzestr. 16-18, 37083 Göttingen, Germany. Email: plonka@math.uni-goettingen.de

with $\lambda_j := \mathrm{e}^{T_j}$. Assuming that $P(z)$ has the monomial presentation

$$P(z) = \sum_{k=0}^{M} p_k z^k,$$

and using the structure of f, a short computation yields for $m = 0, \ldots, M-1$,

$$\begin{aligned}\sum_{k=0}^{M} p_k f(k+m) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \mathrm{e}^{(k+m)T_j} = \sum_{j=1}^{M} c_j \lambda_j^m \left(\sum_{k=0}^{M} p_k \lambda_j^k\right) \\ &= \sum_{j=1}^{M} c_j \lambda_j^m \underbrace{P(\lambda_j)}_{=0} = 0.\end{aligned}$$

With $p_M = 1$ we obtain the linear Hankel system

$$\sum_{k=0}^{M-1} p_k\, f(k+m) = -f(M+m), \quad k = 0, \ldots, M-1, \tag{1.2}$$

providing the coefficients p_k of the Prony polynomial $P(z)$. Now, the unknown parameters T_j can be extracted from the zeros $\lambda_j = \mathrm{e}^{T_j}$ of $P(z)$. Afterwards, the coefficients c_j are obtained by solving the overdetermined linear system

$$f(\ell) = \sum_{j=1}^{M} c_j \mathrm{e}^{\mathrm{i}\ell T_j}, \quad \ell = 0, \ldots, 2M-1.$$

In recent years, the Prony method has been successfully applied to different inverse problems as e.g. for approximation of Green functions in quantum chemistry [32] or fluid dynamics [4], for localization of particles in inverse scattering [16], for parameter estimation of dispersion curves of guided waves [29], and for analysis of ultrasonic signals [7]. The renaissance of Prony's method originates from some modifications of the algorithm described above that considerably stabilize the original approach, as e.g. the ESPRIT method, the matrix pencil method or the approximate Prony method, [28, 17, 25]. These techniques can also be applied if the number M of relevant frequencies is not known beforehand, provided that a sufficiently large number of measurements $f(\ell)$ is given, and the applications in practice show that they work well even in case of noisy measurements. Error estimates for the performance of Prony-like methods with noisy measurements are derived in [25, 11, 2]. Just recently, the reconstruction of functions of the form (1.1) (with Re $T_j = 0$) using a total variation minimization has been proposed in [8]. To tackle this minimization problem, a semidefinite program is applied to solve the dual problem in a first step. The obtained vector is used to define a special polynomial that possesses exactly M zeros on the unit circle which are related to the wanted frequencies Im T_j. The exact connections between the minimization approach in the context of super-resolution and the direct algorithms for the Prony method will be subject of further research.

Searching the literature, one finds different further reconstruction methods that are closely related to Prony's method at second glance.

In spectral analysis the annihilating filter method is frequently applied. This idea has been used already long ago for the construction of cyclic codes, [30]. For a given signal $S[n]$, the FIR filter $A[n]$ is called annihilating filter of $S[n]$, if

$$(A * S)[n] = \sum_{j\in\mathbb{Z}} A[j]\, S[n-j] = \sum_{j\in\mathbb{Z}} A[-j] S[n+j] = 0.$$

Using the z-transform $A(z) = \sum_{n=0}^{M} A[-n] z^n$ and comparing this convolution equation to (1.2), we observe that $A(z)$ undertakes the task of the Prony-polynomial. Vetterli et al. [31]

introduced signals with finite rate of innovation, i.e., signals with a special structure having only a finite number of degrees of freedom per time unit. Using the annihilating filter method, these signals can be completely reconstructed, see also [20, 6].

In some applications, one needs to apply an integral transform in order to obtain functions of the form (1.1), then the Prony method can be applied in transform domain, see e.g. [14, 10, 22, 1, 23].

In computer algebra, one is faced with the computation and processing of multivariate polynomials of high order. But if the polynomial f is M-sparse, i.e.,

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{M} c_j x_1^{d_{j_1}} x_2^{d_{j_2}} \cdots x_n^{d_{j_n}}$$

with $c_1, \ldots, c_M \in \mathbb{C}$ and with M pairwise different vectors $(d_{j_1}, \ldots, d_{j_n}) \in \mathbb{N}^n$, f can be completely recovered using only $2M$ suitably chosen function values. Here again, the number of needed evaluations does not depend on the degree of the polynomial f but on the number M of active terms. A stochastic approach to the problem was given in [33]. A deterministic algorithm goes back to Ben-Or and Tiwari [3], and has been shown to be closely related to the Prony method, see e.g. [18, 13].

Only a few papers have been concerned with the question whether the above interpolation problem can be also solved efficiently if the polynomial is sparse in a different polynomial basis. This problem turned out to be difficult to solve and suitable methods were only given for the Pochhammer basis and the basis of Chebyshev polynomials of first kind, [19, 12].

Recently, we considered in [21] the function reconstruction problem for sparse Legendre expansions of order N of the form

$$f(x) = \sum_{j=1}^{M} c_j P_{n_j}(x)$$

with $0 \le n_1 < n_2 \ldots < n_M = N$, where $M \ll N$, aiming at a generalization of Prony's method for this case. We derived a reconstruction algorithm involving the function and derivative values $f^{(\ell)}(1)$, $\ell = 0, \ldots, 2M-1$. The reconstruction in [21] is based on special properties of Legendre polynomials, but it does not provide a direct approach for further generalization of the method to other sparse orthogonal polynomial expansions or to other function systems apart from exponentials and monomials.

In [9], the idea of efficient sparse polynomial interpolation has been transferred to the more general case of M-term sums of characters of abelian monoids. This approach has been also used in [15] for the reconstruction of functions being linear combinations of eigenfunctions of linear operators on suitable algebras on integral domains. This last paper can be seen as one starting point of our considerations.

In this paper, we want to present a new very general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators. This new insight provides us with a tool to unify all Prony-like methods on the one hand and to essentially generalize the Prony approach on the other hand. Thus it will establish a much broader field of applications of the method. In particular, we will show that all well-known Prony-like reconstruction methods for exponentials and polynomials known so far, can be seen as special cases of this approach. Moreover, the new insight into Prony-like methods enables us to derive new reconstruction algorithms for orthogonal polynomial expansions including Jacobi, Laguerre, and Hermite polynomials. The approach also applies to finite dimensional vector spaces, and we derive a deterministic reconstruction method for M-sparse vectors from only $2M$ measurements.

We point out that the provided generalized Prony method can also be applied to other operators and eigenfunctions. In particular, one may apply it e.g. to Bessel or Hermite functions that are eigenfunctions of special differential operators, to orthogonal polynomials that are eigenfunctions of difference equations, and to eigenfunctions of suitable integral operators. However, in this paper we restrict ourselves to the examples described above.

We will concentrate on the representation of the generalized Prony method for recovering expansions of eigenfunctions, where the number M of terms in the sum is known in advance. Of course, our ideas can also be generalized to the the case of unknown number of terms in the considered expansion. The construction and analysis of stabilized algorithms for the generalized Prony approach will be subject of further research.

The paper is organized as follows. In Section 2, we derive the general reconstruction method for sparse expansions of eigenfunctions of suitable linear operators. We also consider the case of generalized eigenfunctions. In Section 3, we revisit the well-known reconstruction methods for sparse sums of exponentials and for sparse polynomials, where the exponentials can be regarded as eigenfunctions of the translation operator, and the monomials are eigenfunctions of the dilation operator in $C(\mathbb{R})$. Moreover, we show that also sparse linear combinations of $w_k(x)\mathrm{e}^{T_j x}$ can be recovered using the general approach with generalized eigenfunctions, where $w_k(x)$ is a polynomial of order k. Further, exponentials as well as monomials can also be seen as eigenfunctions of suitable differential operators yielding new types of Prony-like reconstructions. Section 4 is dedicated to the reconstruction of sparse expansions of orthogonal polynomials that can be seen as eigenfunctions of the Sturm-Liouville operator. In particular, we show that the reconstruction algorithm for Legendre expansions derived in [21] can be seen as a special example of this general method. Finally, we consider the recovery of M-sparse vectors $\mathbf{x} \in \mathbb{C}^N$ with $N \ll M$ from $\mathbf{y} = \mathbf{F}_{N,2M}\mathbf{x}$, where $\mathbf{F}_{N,2M} \in \mathbb{C}^{2M \times N}$ contains the first $2M$ rows of the Fourier matrix $\mathbf{F}_N$.

2 Prony's method for sparse expansions of eigenfunctions

We want to generalize the Prony method to sparse expansions of eigenfunctions of certain linear operators. Let V be a normed vector space over $\mathbb{C}$, and let $\mathcal{A} : V \to V$ be a linear operator.

Assume that $\mathcal{A}$ possesses eigenvalues, and let $\Lambda := \{\lambda_j : j \in I\}$ be a (sub)set of pairwise distinct eigenvalues of $\mathcal{A}$, where I is a suitable index set. We consider the eigenspaces $\mathcal{V}_j = \{v : \mathcal{A}v = \lambda_j v\}$ to the eigenvalues λ_j, and for each $j \in I$, we predetermine a one-dimensional subspace $\tilde{\mathcal{V}}_j$ of $\mathcal{V}_j$ that is spanned by the normalized eigenfunction v_j. In particular, we assume that there is a unique correspondence between λ_j and v_j, $j \in I$.

An expansion f of eigenfunctions of the operator $\mathcal{A}$ is called M-sparse if its representation consists of only M non-vanishing terms, i.e. if

$$f = \sum_{j \in J} c_j v_j, \qquad \text{with } J \subset I \text{ and } |J| = M. \tag{2.1}$$

Due to the linearity of the operator $\mathcal{A}$, the k-fold application of $\mathcal{A}$ to f yields

$$\mathcal{A}^k f = \sum_{j \in J} c_j \mathcal{A}^k v_j = \sum_{j \in J} c_j \lambda_j^k v_j. \tag{2.2}$$

Further, let $F : V \to \mathbb{C}$ be a linear functional with the property $F v_j \neq 0$ for all $j \in I$. We show that the expansion f in (2.1) can be reconstructed using only the $2M$ values $F(\mathcal{A}^k f)$, $k = 0, \ldots, 2M-1$.

Theorem 2.1 *With the above assumptions, the expansion f in* (2.1) *of eigenfunctions $v_j \in \tilde{\mathcal{V}}_j$, $j \in J \subset I$, of the linear operator $\mathcal{A}$ can be uniquely reconstructed from the values $F(\mathcal{A}^k f)$, $(k = 0, \ldots, 2M-1)$, i.e., the "active" eigenfunctions v_j as well as the coefficients $c_j \in \mathbb{C}$, $j \in J$, in* (2.1) *can be determined uniquely.*

Proof. We give a constructive proof.

1. We define the so-called Prony polynomial

$$P(z) := \prod_{j \in J} (z - \lambda_j),$$

where the roots λ_j, $j \in J$, are the eigenvalues corresponding to the (unknown) active eigenfunctions v_j in the representation of f. Further, let $P(z) = \sum_{k=0}^{M} p_k z^k$, with $p_M = 1$, be the monomial representation of the Prony polynomial. Combining the unknown coefficients p_k with the given values $F(\mathcal{A}^k f)$, $k = 0, \ldots, 2M-1$, and using (2.2) we observe the following relation for $m = 0, 1, \ldots,$

$$\begin{aligned}\sum_{k=0}^{M} p_k F(\mathcal{A}^{k+m} f) &= \sum_{k=0}^{M} p_k F\left(\sum_{j\in J} c_j \lambda_j^{k+m} v_j\right) = \sum_{j\in J} c_j \lambda_j^m \left(\sum_{k=0}^{M} p_k \lambda_j^k\right) F v_j \\ &= \sum_{j\in J} c_j \lambda_j^m \underbrace{P(\lambda_j)}_{=0} F v_j = 0.\end{aligned}$$

Together with $p_M = 1$, the coefficients p_k, $k = 0, \ldots, M-1$, of the Prony polynomial can now be determined via the linear system

$$\sum_{k=0}^{M-1} p_k F(\mathcal{A}^{k+m} f) = -F(\mathcal{A}^{M+m} f), \quad m = 0, \ldots, M-1.$$

Indeed, the coefficient matrix $\mathbf{G} := (F(\mathcal{A}^{k+m} f))_{k,m=0}^{M-1,M-1}$ is an invertible Hankel matrix since (2.2) yields

$$\mathbf{G} = \mathbf{V}_\lambda \cdot \mathrm{diag}(\mathbf{c}) \cdot \mathrm{diag}(F\mathbf{v}) \cdot \mathbf{V}_\lambda^{\mathrm{T}}$$

with the Vandermonde matrix

$$\mathbf{V}_\lambda := \left(\lambda_j^k\right)_{k=0, j\in J}^{M-1}$$

and with the diagonal matrices $\mathrm{diag}(\mathbf{c}) = \mathrm{diag}(c_j)_{j\in J}$, $\mathrm{diag}(F\mathbf{v}) = \mathrm{diag}(F v_j)_{j\in J}$, where the indices $j \in J$ are assumed to be given in a fixed order. By assumption, $\mathbf{V}_\lambda$ as well as the diagonal matrices $\mathrm{diag}(\mathbf{c})$ and $\mathrm{diag}(F\mathbf{v})$ have full rank yielding the invertibility of $\mathbf{G}$.

2. Having determined the Prony polynomial

$$P(z) = \sum_{k=0}^{M} p_k z^k = \prod_{j\in J} (z - \lambda_j),$$

we can evaluate the eigenvalues λ_j, $j \in J$, that are the zeros of $P(z)$. Since the eigenspaces $\tilde{\mathcal{V}}_j$ are assumed to be one-dimensional we can uniquely determine the corresponding eigenfunctions v_j, $j \in J$.

3. In the last step we compute the coefficients c_j, $j \in J$, of the expansion (2.1) by solving the overdetermined linear system

$$F(\mathcal{A}^k f) = \sum_{j\in J} c_j \lambda_j^k v_j, \quad k = 0, \ldots, 2M-1,$$

using the eigenvalues λ_j and eigenfunctions v_j found in the previous step. ∎

This general approach to the Prony method enables us to derive reconstruction algorithms for a variety of different systems of eigenfunctions. We summarize the algorithm as follows.

Algorithm 2.2 **(Reconstruction of the sparse expansion (2.1))**

Input: *M, $F(\mathcal{A}^k f)$, $k = 0, \ldots, 2M-1$.*

1. *Solve the linear system*

$$\sum_{k=0}^{M-1} p_k F(\mathcal{A}^{k+m} f) = -F(\mathcal{A}^{M+m} f), \qquad m = 0, \ldots, M-1.$$

2. *Form the Prony polynomial* $P(z) = \sum_{k=0}^{M} p_k z^k$ *using the obtained values* p_k, $k = 0, \ldots, M-1$ *from step 1 and* $p_M = 1$. *Compute the zeros* λ_j, $j \in J$, *of* $P(z)$ *and determine the corresponding (normalized) eigenfunctions* v_j, $j \in J$.

3. *Compute the coefficients* c_j *by solving the overdetermined system*

$$F(\mathcal{A}^k f) = \sum_{j \in J} c_j \lambda_j^k v_j \qquad k = 0, \ldots, 2M-1.$$

Output: c_j, v_j, $j \in J$, *determining* f *in* (2.1).

Let us also consider the case of generalized eigenfunctions. Let $r \geq 1$ be a fixed integer. Analogously as for linear operators in finite-dimensional vector spaces, we say that $\tilde{v}_\ell$, $\ell = 1, \ldots, r$, are generalized eigenfunctions of multiplicity ℓ of a linear operator $\mathcal{A} : V \to V$ to the eigenvalue λ, if

$$(\mathcal{A} - \lambda I)^\ell \tilde{v}_\ell = 0, \qquad \ell = 1, \ldots, r,$$

and

$$\mathcal{A} \tilde{v}_\ell = \lambda \tilde{v}_\ell + \sum_{s=1}^{\ell-1} \alpha_{\ell,s} \tilde{v}_s, \qquad \ell = 1, \ldots, r, \tag{2.3}$$

with some constants $\alpha_{\ell,s} \in \mathbb{C}$. Again, let $\Lambda = \{\lambda_j,\, j \in I\}$ be a (sub)set of pairwise distinct eigenvalues of $\mathcal{A}$, and for each $j \in I$, let $\{\tilde{v}_{j,\ell} : \ell = 1, \ldots, r\}$ be a predetermined set of linearly independent generalized eigenfunctions to the eigenvalue λ_j. Further, let $F : V \to \mathbb{C}$ be a functional with $F(\tilde{v}_{j,\ell}) \neq 0$ for $j \in I, \ell = 1, \ldots, r$.

Theorem 2.3 *With the above assumptions, the expansion*

$$f = \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \tilde{v}_{j,\ell}, \qquad J \subset I,\ |J| = M,\ r \geq 1,$$

of generalized eigenfunctions of the linear operator $\mathcal{A}$ *to the eigenvalues* λ_j, $j \in J \subset I$, *can be uniquely reconstructed from the values* $F(\mathcal{A}^k f)$, $k = 0, \ldots, 2rM-1$, *supposed that the matrix* $(F(\mathcal{A}^{k+m} f))_{k,m=0}^{rM-1,rM-1}$ *is invertible.*

Proof. 1. Using an induction argument, equation (2.3) implies that there exist coefficients $\beta_{j,\ell,s}^k$ such that the generalized eigenfunctions $\tilde{v}_{j,\ell}$ (of multiplicity ℓ) of $\mathcal{A}$ to the eigenvalue λ_j satisfy

$$\mathcal{A}^k \tilde{v}_{j,\ell} = \sum_{s=0}^{\ell-1} \beta_{j,\ell,s}^k \, \lambda_j^{\max\{k-s,0\}} \tilde{v}_{j,\ell-s},$$

where $\beta_{j,\ell,0}^k = 1$.

We consider now a generalized Prony polynomial of the form

$$P(z) := \prod_{j \in J} (z - \lambda_j)^r = \sum_{k=0}^{Mr} p_k z^k, \tag{2.4}$$

where again λ_j denote the unknown eigenvalues of $\mathcal{A}$ that determine the active sets of (generalized) eigenfunctions. Then we obtain the following relation for $m = 0, \ldots, Mr - 1$,

$$\begin{aligned}
\sum_{k=0}^{Mr} p_k F(\mathcal{A}^{k+m} f) &= \sum_{k=0}^{Mr} p_k F\left(\sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \mathcal{A}^{k+m} \tilde{v}_{j,\ell} \right) \\
&= \sum_{k=0}^{Mr} p_k F\left(\sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \sum_{s=0}^{\ell-1} \beta_{j,\ell,s}^{k+m} \lambda_j^{\max\{k+m-s,0\}} \tilde{v}_{j,\ell-s} \right) \\
&= \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \sum_{s=0}^{\ell-1} \left(\sum_{k=0}^{Mr} p_k \beta_{j,\ell,s}^{k+m} \lambda_j^{\max\{k+m-s,0\}} \right) F(\tilde{v}_{j,\ell-s}) = 0,
\end{aligned}$$

where the observation is used that the term $\sum\limits_{k=0}^{Mr} p_k \beta_{j,\ell,s}^{k+m} \lambda_j^{\max\{k+m-s,0\}}$ can be written as a linear combination of the Prony polynomial $P(z)$ and the first $r-1$ derivatives of $P(z)$ evaluated at λ_j and hence vanishes for all $j \in J, \ell = 1, \ldots, r$, and $s = 0, \ldots, \ell - 1$. In this way, we obtain again a linear Hankel system of the form

$$\sum_{k=0}^{rM-1} p_k F(\mathcal{A}^{k+m} f) = -F(\mathcal{A}^{rM+m} f), \qquad m = 0, \ldots, rM - 1$$

in order to determine the coefficients p_k of the Prony polynomial. Having determined the zeros λ_j of the Prony polynomial, we obtain the corresponding eigenfunctions $\tilde{v}_{j,1}, \ldots, \tilde{v}_{j,r}$, and afterwards compute the complex coefficients $c_{j,\ell}$ by solving the overdetermined system

$$F(\mathcal{A}^k f) = \sum_{j \in J} \sum_{\ell=1}^{r} c_{j,\ell} \mathcal{A}^k \tilde{v}_{j,\ell}, \qquad k = 0, \ldots, 2rM - 1.$$

∎

Remark 2.4 *Generalized Prony polynomials of the form* (2.4) *have already been used for the reconstruction of spline functions in* [31] *and* [23].

3 Sparse sums of exponentials and monomials revisited

Using the general Prony approach introduced in Section 2, we would like to revisit the well-known methods for reconstruction of sparse exponential sums and sparse monomial sums.

3.1 Sparse exponential sums

Let us consider the vector space $C(\mathbb{R})$ of continuous functions, and let $S_a : C(\mathbb{R}) \to C(\mathbb{R})$ with

$$S_a f := f(\cdot + a), \quad a \in \mathbb{R} \backslash \{0\}$$

be the shift operator on $C(\mathbb{R})$. We observe that $\{\mathrm{e}^{Ta} : T \in \mathbb{C}, \ \mathrm{Im}\, T \in [-\frac{\pi}{a}, \frac{\pi}{a})\}$ is a set of pairwise distinct eigenvalues of S_a, and by

$$S_a \mathrm{e}^{x(T + \frac{2\pi \mathrm{i} k}{a})} = \mathrm{e}^{(x+a)(T + \frac{2\pi \mathrm{i} k}{a})} = \mathrm{e}^{Ta} \mathrm{e}^{x(T + \frac{2\pi \mathrm{i} k}{a})}, \qquad x \in \mathbb{R}, \ k \in \mathbb{Z},$$

we find for each eigenvalue $\lambda_T := \mathrm{e}^{Ta}$, $T \in I := \{T \in \mathbb{C}, \ \mathrm{Im}\, T \in [-\frac{\pi}{a}, \frac{\pi}{a})\}$, the eigenspace $\mathcal{V}_T := \mathrm{span}\, \{\mathrm{e}^{x(T + \frac{2\pi \mathrm{i} k}{a})} : k \in \mathbb{Z}\}$. In order to obtain a unique correspondence between λ_T and its eigenfunction, we consider only the subeigenspaces $\tilde{\mathcal{V}}_T = \mathrm{span}\, \{\mathrm{e}^{Tx}\}$. Further, let the functional $F : C(\mathbb{R}) \to \mathbb{C}$ be given by

$$F(f) := f(x_0), \quad \forall f \in C(\mathbb{R}),$$

with an arbitrarily fixed $x_0 \in \mathbb{R}$. Hence $F(\mathrm{e}^{T\cdot}) = \mathrm{e}^{Tx_0} \neq 0$ for all $T \in I$. Applying Theorem 2.1 yields that the sparse sum of exponentials

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{T_j x} \tag{3.1}$$

with pairwise different $T_j \in \mathbb{C}$ and Im $T_j \in [-\frac{\pi}{a}, \frac{\pi}{a})$ can be uniquely reconstructed from the values

$$F(S_a^k f) = F(f(\cdot + ka)) = f(x_0 + ka), \quad k = 0, \ldots, 2M-1,$$

e.g., from $2M$ equidistant sampling points with sampling distance a, starting at point x_0.

Moreover, Theorem 2.3 also admits efficient recovery of sums of the form

$$f(x) = \sum_{j=1}^{M} \sum_{\ell=0}^{r} c_{j,\ell} w_\ell(x) \mathrm{e}^{T_j x}$$

with $r \geq 0$, where w_ℓ, $\ell = 0, \ldots, r$ denote algebraic polynomials of exact degree ℓ. Indeed we easily observe, that $w_\ell(x)\mathrm{e}^{Tx}$, $\ell = 0, \ldots, r$ are linearly independent generalized eigenfunctions of multiplicity $\ell + 1$ of the shift operator S_a to the eigenvalue e^{Ta}. Therefore, we can apply Theorem 2.3 for reconstruction of f using the $2M(r+1)$ function values $f(x_0 + ka)$, $k = 0, \ldots, 2M(r+1) - 1$.

Remarks 3.1 *1. There are other operators that also possess eigenfunctions of the form* e^{Tx}. *For example, the shift operator* S_a *in the above considerations can be replaced by the difference operator* $\Delta_a f(x) = f(x+a) - f(x)$ *or even by an* m*-fold difference operator* $\Delta_a^m f := \Delta_a^{m-1}(\Delta_a f)$, $(m \in \mathbb{N})$. *Using again* $Ff := f(x_0)$, *the reconstruction then involves the values*

$$F(\Delta_a^{mk} f) = \Delta_a^{mk} f(x_0) = \sum_{\ell=0}^{mk} \binom{mk}{\ell} (-1)^\ell f(x_0 + \ell a).$$

2. Instead of using the functional $Ff = f(x_0)$ *for some fixed* $x_0 \in \mathbb{R}$ *one can use also a different functional. The functional* $Ff := \int_{x_0}^{x_0+a} f(x)\mathrm{d}x$ *leads to a reconstruction method, where* f *in* (3.1) *can be reconstructed from the values*

$$\int_{x_0}^{x_0+a} S_a^k f(x)\mathrm{d}x = \int_{x_0}^{x_0+a} f(x+ka)\mathrm{d}x = \int_{x_0+ka}^{x_0+(k+1)a} f(x)\mathrm{d}x, \qquad k = 0, \ldots, 2M-1.$$

3. The reconstruction method also applies to the multivariate case. Let $S_{\mathbf{a}} : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ *be the shift operator with*

$$S_{\mathbf{a}} f(x_1, \ldots, x_d) = f(x_1 + a_1, x_2 + a_2, \ldots, x_d + a_d),$$

with the set of eigenfunctions $\{\mathrm{e}^{\mathbf{T}\cdot\mathbf{x}} = \mathrm{e}^{T_1 x_1 + \cdots + T_d x_d} : T_\ell \in \mathbb{C}, \mathrm{Im}\, T_\ell \in [-\frac{\pi}{a}, \frac{\pi}{a}), \ell = 1, \ldots, d\}$. *The corresponding eigenvalues* $\mathrm{e}^{\mathbf{T}\cdot\mathbf{a}}$ *allow a unique conclusion to the corresponding eigenfunction* $\mathrm{e}^{\mathbf{T}\cdot\mathbf{x}}$ *if there exists an injective linear mapping that maps* $\mathbf{T}$ *to* $\mathbf{T}\cdot\mathbf{a}$. *This condition can be satisfied by a suitable restriction of* $\mathbf{T}$ *and a special choice of* $\mathbf{a}$. *Let* $f(\mathbf{x})$ *be of the form*

$$f(\mathbf{x}) = \sum_{j=1}^{M} c_j \mathrm{e}^{\mathbf{T}_j \cdot \mathbf{x}}$$

with $\mathbf{T}_j = (T_{j,1}, \ldots, T_{j,d})^{\mathrm{T}}$, *where* $T_{j,\ell} \in \mathbb{N}$ *for* $j = 1, \ldots, M$, $\ell = 1, \ldots, d$. *Choose now pairwise relatively prime numbers* $p_1, \ldots, p_d$ *with* $p_\ell > \max_{j=1,\ldots,M} T_{j,\ell}$ *for* $\ell = 1, \ldots, d$. *Further, let* $N = p_1 p_2 \cdots p_d$, *and* $\mathbf{a} := (\frac{N}{p_1}, \ldots, \frac{N}{p_d})$. *Then each variable* $\mathbf{T}_j \in \mathbb{N}^d$ *can be uniquely* *determined from* $\tau_j := \mathbf{T}_j \cdot \mathbf{a}$ *using the reverse steps of the Chinese remainder theorem. We have*

$$\tau_j = \sum_{\ell=1}^{d} \frac{T_{j,\ell} N}{p_\ell}.$$

Hence, $\tau_j \equiv T_{j,\ell}$ *mod* p_ℓ *for* $1 \leq \ell \leq d$, *and we can recover* $\mathbf{T}_j$ *from* τ_j *by* $T_{j,\ell} = \tau_j - p_\ell \left\lfloor \frac{\tau_j}{p_\ell} \right\rfloor$, *see* [13]. *Unfortunately, this procedure is highly unstable.*

Another procedure for recovery of multivariate exponential sums is based on the determination of $\mathbf{T}_j \in \mathbb{C}^d$, $\mathrm{Im}\, \mathbf{T}_j \in [-\frac{\pi}{a}, \frac{\pi}{a})^d$, *from different scalar products* $\mathbf{T}_j \cdot \mathbf{a}_1$, $\mathbf{T}_j \cdot \mathbf{a}_2$, *etc., see* [26, 23].

Finally, we want to mention that the exponentials can also be seen as eigenfunctions of differential operators. Let us consider the vector space $C^\infty(\mathbb{R})$ of infinitely differentiable functions, and let $\frac{\mathrm{d}}{\mathrm{d}x} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the differentiation operator. We observe that $\{T : T \in \mathbb{C}\}$ is a set of pairwise distinct eigenvalues of $\frac{\mathrm{d}}{\mathrm{d}x}$ and by

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{Tx} = T \mathrm{e}^{Tx}$$

we can uniquely relate the eigenfunction e^{Tx} to the eigenvalue T. Let x_0 be a fixed real number. Then, with $F(f) := f(x_0)$, $\forall f \in \mathbb{C}^\infty(\mathbb{R})$, we can again apply Theorem 2.1 for recovering the sparse sum of exponentials

$$f(x) = \sum_{j=1}^{M} c_j \mathrm{e}^{T_j x}$$

with pairwise different $T_j \in \mathbb{C}$. The reconstruction of f can be uniquely performed using the values

$$F\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} f\right) = f^{(k)}(x_0), \quad k = 0, \ldots, 2M-1,$$

where $x_0 \in \mathbb{R}$ can be chosen arbitrarily.
Moreover, let $\{w_\ell\}_{\ell=0}^{r}$ be a basis of the space of polynomials of degree at most r and $\deg(w_\ell) = \ell$, $\ell = 0, \ldots, r$. Then we easily check that the functions $w_\ell(x)\mathrm{e}^{Tx}$, $\ell = 0, \ldots, r$, form linearly independent generalized eigenfunctions of multiplicity $\ell + 1$ of the linear operator $\frac{\mathrm{d}}{\mathrm{d}x}$, and Theorem 2.3 applies for the recovery of sparse expansions of the form

$$f(x) = \sum_{j=1}^{M} \sum_{\ell=0}^{r} c_{j,\ell} w_\ell(x) \mathrm{e}^{T_j x}$$

using the derivative values $f^{(k)}(x_0)$, $k = 0, \ldots, 2M(r+1) - 1$, for $r \geq 0$.

3.2 Sparse monomial sums

Let us consider the Banach space $C(\mathbb{R})$ of continuous functions and let $D_a : C(\mathbb{R}) \to C(\mathbb{R})$ with

$$D_a f(x) := f(ax), \quad a \in \mathbb{R}_+ \backslash \{1\}$$

be the dilatation operator on $C(\mathbb{R})$. By

$$D_a x^p = (ax)^p = a^p x^p$$

we observe that $\{x^p : p \in \mathbb{C}\}$ is a set of eigenfunctions to D_a with corresponding eigenvalues a^p. In order to ensure that the eigenvalues a^p are pairwise distinct, we assume that $\operatorname{Im} p \in [-\frac{\pi}{\ln a}, \frac{\pi}{\ln a})$. Therefore, we consider only the "admissible" set of eigenfunctions $\{x^p : p \in \mathbb{C}, \operatorname{Im} p \in [-\frac{\pi}{\ln a}, \frac{\pi}{\ln a})\}$. Further, let the functional $F : C(\mathbb{R}) \to \mathbb{C}$ be given by $F(f) := f(x_0)$ for all $f \in C(\mathbb{R})$, where $x_0 \in \mathbb{R}\backslash\{0\}$ is arbitrary but fixed, such that $F(\cdot)^p = x_0^p \neq 0$.

With Theorem 2.1, we can uniquely reconstruct the sparse sum of monomials

$$f(x) = \sum_{j=1}^{M} c_j x^{p_j}$$

with $c_j \in \mathbb{C}\backslash\{0\}$ and pairwise different $p_j \in \mathbb{C}$ which satisfy $\operatorname{Im} p_j \in [-\frac{\pi}{\ln a}, \frac{\pi}{\ln a})$, using the $2M$ values $F(D_a^k f) = f(a^k x_0)$, $(k = 0, \ldots, 2M-1)$.

Remarks 3.2 *1. The above approach generalizes the Ben-Or and Tiwari algorithm* [3] *for interpolating sparse polynomials in the sense that we are not restricted to integer exponents p_j. It can be applied to multivariate sums, where some restrictions to the set of admissible eigenfunctions are needed in order to ensure an injective mapping from the eigenvalues to the eigenfunctions, see Remarks* 3.1 *for the similar case of multivariate exponential sums.*

2. By using a different functional F, a further generalization of the Ben-Or and Tiwari reconstruction method is possible. As before, for example $Ff := \int_0^1 f(x)\mathrm{d}x$ leads to a reconstruction method using the values

$$F(D_a^k f) = \int_0^1 f(a^k x)\mathrm{d}x = \frac{1}{a^k}\int_0^{a^k} f(x)\mathrm{d}x, \quad k = 0, \ldots, 2M-1.$$

We may admit also generalized eigenfunctions of the dilation operator $D_a : C((0,\infty)) \to C((0,\infty))$. Assuming that $a > 0$, $a \neq 1$, we observe that functions of the form $(\ln x)^\ell x^p$, $\ell = 0, \ldots, r$ are generalized eigenfunctions of multiplicity $\ell + 1$ of D_a (as defined in Section 2), since

$$\begin{aligned} D_a\left((\ln x)^\ell x^p\right) &= (\ln ax)^\ell (ax)^p = (\ln a + \ln x)^\ell a^p x^p \\ &= a^p (\ln x)^\ell x^p + \sum_{s=0}^{\ell-1} \binom{\ell}{s} (\ln a)^{\ell-s} a^p (\ln x)^s x^p. \end{aligned}$$

Thus, by Theorem 2.3, we are able to recover also expansions of the form

$$f(x) = \sum_{j=1}^{M}\sum_{\ell=0}^{r} c_{j,\ell} (\ln x)^\ell x^{p_j}$$

from the measurements $f(a^k x_0)$, $k = 0, \ldots, 2(r+1)M - 1$ for $r \geq 0$, where $x_0 > 0$ is fixed.

Eigenfunctions of monomial form can also be obtained using suitable differential operators. Let $d_x : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the differential operator of the form

$$d_x f(x) := \frac{\mathrm{d}}{\mathrm{d}x}(x f(x)) = f(x) + x f'(x).$$

Then we have

$$d_x(x^p) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{p+1}) = (p+1)x^p, \qquad p \in \mathbb{R},$$

and the operator d_x possesses the set $\{p+1 : p \in \mathbb{R}\}$ of pairwise different eigenvalues with corresponding eigenfunctions x^p. We consider now a sparse monomial expansion of the form

$$f(x) = \sum_{j=1}^{M} c_j x^{p_j}$$

with $c_j \in \mathbb{C} \setminus \{0\}$ and pairwise different $p_j \in \mathbb{R}$. Using Theorem 2.1, this expansion can be completely recovered from $F((d_x)^k f) = (d_x)^k f(x_0)$, $k = 0, \ldots 2M-1$. A simple induction argument shows that these values can be obtained recursively from the derivative values $f^{(\ell)}(x_0)$, $\ell = 0, \ldots, 2M-1$, where $x_0 \in \mathbb{R} \setminus \{0\}$ can be arbitrarily chosen.

4 Sparse sums of orthogonal polynomials

Let us again consider the Banach space $C^\infty(\mathbb{R})$ of infinitely differentiable functions. Let $L_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the Sturm-Liouville differential operator of the form

$$L_{p,q} f(x) := p(x) f''(x) + q(x) f'(x), \quad f \in C^\infty(\mathbb{R}), \tag{4.1}$$

where $p(x)$ and $q(x)$ are polynomials of degree 2 and 1, respectively. It is well-known, that suitably defined orthogonal polynomials Q_n are eigenfunctions of this differential operator for special sets of eigenvalues $\lambda_n, n \in \mathbb{N}_0$, i.e., $L_{p,q} Q_n = \lambda_n Q_n$. For convenience, we list the most prominent orthogonal polynomials with their corresponding $p(x), q(x)$ and their eigenvalues λ_n, $n \in \mathbb{N}$ in Table 1.

$p(x)$	$q(x)$	λ_n	name	symbol
$(1-x^2)$	$(\beta - \alpha - (\alpha+\beta+2)x)$	$-n(n+\alpha+\beta+1)$	Jacobi	$P_n^{(\alpha,\beta)}$
$(1-x^2)$	$-(2\alpha+1)x$	$-n(n+2\alpha)$	Gegenbauer	$C_n^{(\alpha)}$
$(1-x^2)$	$-2x$	$-n(n+1)$	Legendre	P_n
$(1-x^2)$	$-x$	$-n^2$	Chebyshev 1. kind	T_n
$(1-x^2)$	$-3x$	$-n(n+2)$	Chebyshev 2. kind	U_n
1	$-2x$	$-2n$	Hermite	H_n
x	$(\alpha+1-x)$	$-n$	Laguerre	$L_n^{(\alpha)}$

Table 1. Polynomials $p(x)$ and $q(x)$ defining the Sturm-Liouville operator, corresponding eigenvalues λ_n and eigenfunctions.

Obviously, Gegenbauer, Legendre, and Chebyshev polynomials are special cases of Jacobi polynomials, where we have $C_n^{(\alpha)} := P_n^{(\alpha-1/2,\alpha-1/2)}$, $P_n := P_n^{(0,0)}$, $T_n := P_n^{(-\frac{1}{2},-\frac{1}{2})}$ and $U_n := P_n^{(\frac{1}{2},\frac{1}{2})}$.

We observe easily that for a set of eigenfunctions $\{Q_n : n \in \mathbb{N}_0\}$, the corresponding eigenvalues are pairwise different and well separated, i.e. $\lambda_n \neq \lambda_m$ for $n \neq m$. Further, we choose the functional $F : C^\infty(\mathbb{R}) \to \mathbb{C}$ that returns f at a fixed value $x_0 \in \mathbb{R}$, i.e., $F(f) := f(x_0)$, with the condition that $Q_n(x_0) \neq 0$ for all $n \in \mathbb{N}_0$.

The polynomial f is now an M-sparse expansion of orthogonal polynomials Q_n, $n \geq 0$, if it has the form

$$f(x) = \sum_{j=1}^{M} c_{n_j} Q_{n_j}(x) \tag{4.2}$$

where $c_{n_j} \in \mathbb{C} \setminus \{0\}$, $0 \leq n_1 < \ldots, < n_M = N$ are the indices of the "active" basis polynomials Q_{n_j} in the expansion, and $n_M = N \gg M$ is the polynomial degree of f.

Now Theorem 2.1 yields that $f(x)$ can be uniquely recovered using the values

$$F(L_{p,q}^k f) = L_{p,q}^k f(x_0) = \sum_{j=1}^{M} c_{n_j} \lambda_{n_j}^k Q_{n_j}(x_0), \quad k = 0, \ldots, 2M-1.$$

We will show that these values $L_{p,q}^k f(x_0)$, $(k = 0, \ldots, 2M-1)$ can be determined uniquely by the derivative values $f^{(m)}(x_0)$ for $m = 0, \ldots, 4M-2$, and this assertion holds not only for sparse but for all expansions of orthogonal polynomials f.

Theorem 4.1 *Let $f \in C^\infty(\mathbb{R})$ be an arbitrary polynomial of degree $N \in \mathbb{N}$ and let $L_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be the Sturm-Liouville differential operator as given in (4.1). Then, for each fixed $x \in \mathbb{R}$, the values $L_{p,q}^k f(x)$, $k = 0, \ldots, 2M-1$, can be determined uniquely by the derivative values $f^{(m)}(x)$, $m = 0, \ldots, 4M-2$, and we have*

$$L_{p,q}^k f(x) = \sum_{\ell=1}^{2k} g_{\ell,k}(x)\, f^{(\ell)}(x)$$

for $k \geq 1$. Here $g_{1,1}(x) = q(x)$, $g_{2,1}(x) = p(x)$, and for $k \geq 2$, $g_{\ell,k}(x)$ satisfies the recursion

$$\begin{aligned} g_{\ell,k}(x) = &\, \ell \left(\frac{\ell-1}{2} p''(x) + q'(x) \right) g_{\ell,k-1}(x) \\ &+ ((\ell-1)p'(x) + q(x)) g_{\ell-1,k-1}(x) + p(x) g_{\ell-2,k-1}(x), \qquad \ell = 1, \ldots, 2k, \end{aligned} \tag{4.3}$$

with the convention $g_{\ell,k}(x) = 0$ for $k \geq 1$, $\ell \notin \{1, \ldots, 2k\}$.

Proof. 1. Let $\{Q_n : n \in \mathbb{N}_0\}$ be the set of orthogonal polynomials that are eigenfunctions of the operator $L_{p,q}$ to the eigenvalues λ_n, i.e.

$$L_{p,q} Q_n = \lambda_n Q_n.$$

Since $\{Q_n : n \in \mathbb{N}_0\}$ forms a basis of the space of polynomials we find a unique expansion of f,

$$f(x) = \sum_{n=0}^{N} \alpha_n Q_n(x).$$

2. For $k = 0$, we observe that $L_{p,q}^0 f(x) = f(x)$, i.e. $L_{p,q}^0$ is the identity operator. Since the operator $L_{p,q}$ is a differential operator of order 2, we can use for $k \geq 1$ the ansatz

$$L_{p,q}^k f(x) = \sum_{\ell=1}^{2k} g_{\ell,k}(x) f^{(\ell)}(x), \quad k \geq 1, \tag{4.4}$$

with polynomials $g_{\ell,k}$, $k \in \mathbb{N}$, $\ell = 1, \ldots, 2k$. In particular, for $k = 1$ we have

$$L_{p,q} f(x) = p(x) f''(x) + q(x) f'(x),$$

i.e., $g_{1,1}(x) = q(x)$ and $g_{2,1}(x) = p(x)$. We now prove by induction on k that the coefficients $g_{\ell,k}(x)$ in (4.4) satisfy the recursion (4.3) for $k \geq 2$ and $\ell = 1, \ldots, 2k$. Using (4.4) and the general Leibniz rule, we find for all $n \in \mathbb{N}_0$,

$$\begin{aligned} L_{p,q}^k Q_n(x) = L_{p,q}^{k-1}(\lambda_n Q_n(x)) &= \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) \lambda_n Q_n^{(\ell)}(x) \\ &= \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) [p(x) Q_n''(x) + q(x) Q_n'(x)]^{(\ell)} \\ &= \sum_{\ell=1}^{2k-2} g_{\ell,k-1}(x) [p(x) Q_n^{(\ell+2)}(x) + \ell p'(x) Q_n^{(\ell+1)}(x) \\ &\quad + \binom{\ell}{2} p''(x) Q_n^{(\ell)}(x) + q(x) Q_n^{(\ell+1)}(x) + \ell q'(x) Q_n^{(\ell)}(x)] \\ &= \sum_{\ell=1}^{2k} g_{\ell,k}(x) Q_n^{(\ell)}(x) \end{aligned}$$

due to the vanishing higher derivatives of $p(x), q(x)$. A comparison of coefficients leads to the recursion formulas for $g_{\ell,k}$ in (4.3). Hence, we finally obtain

$$L_{p,q}^k f(x) = \sum_{n=0}^{N} \alpha_n L_{p,q}^k Q_n(x) = \sum_{n=0}^{N} \alpha_n \sum_{\ell=0}^{2k} g_{\ell,k}(x) Q_n^{(\ell)}(x) = \sum_{\ell=0}^{2k} g_{\ell,k}(x) f^{(\ell)}(x)$$

with $g_{\ell,k}(x)$ as given in the theorem. ■

Corollary 4.2 *If $x_0 \in \mathbb{R}$ is a zero of the polynomial $p(x)$ in the definition (4.1) of the Sturm-Liouville operator, i.e., if $p(x_0) = 0$, then $L_{p,q} f(x_0)$ reduces to $L_{p,q} f(x_0) = q(x_0) f'(x_0)$, and the values $L_{p,q}^k f(x_0)$, $k = 0, \ldots, 2M-1$ can be determined by $f^{(m)}(x_0)$, $m = 0, \ldots, 2M-1$ only. More precisely, we have*

$$L_{p,q}^k f(x_0) = \sum_{\ell=1}^{k} g_{\ell,k}(x_0) f^{(\ell)}(x_0)$$

with $g_{1,1}(x_0) = q(x_0)$ and

$$g_{\ell,k}(x_0) = \ell \left(\frac{\ell - 1}{2} p''(x_0) + q'(x_0) \right) g_{\ell,k-1}(x_0) + ((\ell - 1)p'(x_0) + q(x_0)) g_{\ell-1,k-1}(x_0)$$

for $k \geq 2$, $\ell \in \{1, \ldots, k\}$, where we assume that $g_{\ell,k}(x_0) = 0$ for $k \geq 1$, $\ell \notin \{1, \ldots, k\}$. In particular, for the Sturm-Liouville operator for Jacobi polynomials with $p(x) = (1 - x^2)$, we need only the values $f^{(m)}(1)$ (respectively $f^{(m)}(-1)$), $m = 0, \ldots, 2M-1$, in order to reconstruct $L_{p,q}^k f(1)$, (respectively $L_{p,q}^k f(-1)$), $k = 0, \ldots, 2M-1$.

The proof of Corollary 4.2 is similar to that of Theorem 4.1 and is therefore omitted.

We summarize the algorithm for reconstructing orthogonal polynomial expansions as follows.

Algorithm 4.3 **(Reconstruction of f in (4.2))**

Input: *Sturm Liouville operator with $p(x), q(x)$ and λ_n as well as the basis $\{Q_n : n \in \mathbb{N}_0\}$, M, $x_0 \in \mathbb{R}$, $f^{(m)}(x_0)$, $m = 0, \ldots, 4M-2$.*

Preprocessing: Construct $\mathbf{G} = (g_{\ell,k})_{k,\ell=1}^{2M-1,4M-2} \in \mathbb{R}^{(2M-1)\times(4M-2)}$ with $g_{1,1} := q(x_0)$, $g_{2,1} := p(x_0)$, $g_{\ell,1} := 0$ for $\ell \notin \{1, 2\}$, and

$$g_{\ell,k} := \begin{cases} \ell \left(\frac{\ell-1}{2} p''(x_0) + q'(x_0) \right) g_{\ell,k-1} & \\ +((\ell-1)p'(x_0) + q(x_0)) g_{\ell-1,k-1} + p(x_0) g_{\ell-2,k-1}, & k > 1,\ \ell \in \{1, \ldots, 2k\} \\ 0, & k > 1,\ \ell \notin \{1, \ldots, 2k\} \end{cases}$$

Observe that the construction of $\mathbf{G}$ only depends on the chosen basis and not on the given data $f^{(m)}(x_0)$.

1. *Calculate $\mathbf{h}_1 := \mathbf{G}\mathbf{f}_1$, where $\mathbf{f}_1 := (f^{(m)}(x_0))_{m=1}^{4M-2}$. Put now*

$$\mathbf{h} := \begin{pmatrix} f(x_0) \\ \mathbf{h_1} \end{pmatrix}$$

   such that $\mathbf{h} = (h_\ell)_{\ell=0}^{2M-1} = (L_{p,q}^\ell f(x_0))_{\ell=0}^{2M-1} \in \mathbb{C}^{2M}$.

2. *Solve the Hankel system*

$$\begin{pmatrix} h_0 & h_1 & \ldots & h_{M-1} \\ h_1 & h_2 & \ldots & h_M \\ \vdots & & & \vdots \\ h_{M-1} & h_M & \ldots & h_{2M-2} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} h_M \\ h_{M+1} \\ \vdots \\ h_{2M-1} \end{pmatrix}.$$

3. Put $p_M = 1$ and compute the zeros λ_{n_j}, $j = 1, \ldots, M$, of the Prony-polynomial

$$P(z) = \sum_{k=0}^{M} p_k z^k.$$

4. Extract the indices n_j, $j = 1, \ldots, M$, from the obtained eigenvalues λ_{n_j} and compute the coefficients c_{n_j} of the sparse orthogonal polynomial expansion in (4.2) by solving the overdetermined Vandermonde-type system

$$\sum_{j=1}^{M} c_{n_j} Q_{n_j}^{(\ell)}(x_0) = f^{(\ell)}(x_0), \quad \ell = 0, \ldots, 2M - 1.$$

Output: *n_j, c_{n_j}, $j = 1, \ldots, M$, determining f in (4.2).*

Example 4.4 Sparse Laguerre expansions

The Laguerre polynomials with parameter α are solutions of the second order differential equation

$$x(L_n^{(\alpha)})''(x) + (\alpha + 1 - x)(L_n^{(\alpha)})'(x) = -n\, L_n^{(\alpha)}(x),$$

with eigenvalues $\lambda_n = -n$. Using Theorems 2.1 and 4.1 a sparse Laguerre expansion of the form

$$f(x) = \sum_{j=1}^{M} c_{n_j} L_{n_j}^{(\alpha)}(x)$$

with $c_{n_j} \in \mathbb{C}\backslash\{0\}$ and active indices $0 \le n_1 < \cdots < n_M = N$ can be reconstructed from $f^{(m)}(x_0)$, $m = 0, \ldots, 4M - 2$. Here, x_0 has to satisfy $L_n^{(\alpha)}(x_0) \neq 0$ for all $n \in \mathbb{N}_0$. If we choose $x_0 = 0$, formula (4.3) simplifies to

$$\begin{aligned}
&g_{1,1}(0) = \alpha + 1, && \\
&g_{\ell,k}(0) = (\ell + \alpha) g_{\ell-1,k-1}(0) - \ell g_{\ell,k-1}(0), && k > 1, \ell = 1, \ldots, k, \\
&g_{\ell,k}(0) = 0, && k \ge 1, \ell \notin \{1, \ldots, k\}.
\end{aligned}$$

For example, for $M = 2$, this leads to the triangular matrix

$$\mathbf{G} = \begin{pmatrix} (1+\alpha) & 0 & 0 \\ -(1+\alpha) & (1+\alpha)(2+\alpha) & 0 \\ (1+\alpha) & -3(1+\alpha)(2+\alpha) & (1+\alpha)(2+\alpha)(3+\alpha) \end{pmatrix}$$

in the preprocessing step of Algorithm 4.3. Let us give a small numerical example. For $\alpha = 0$ and given values $f(0), f'(0), \ldots, f^{(11)}(0)$ of the function

$$f(x) = \sum_{j=1}^{6} c_{n_j} L_{n_j}(x),$$

we use Algorithm 4.3 to calculate approximations $\tilde{n}_j, \tilde{c}_{n_j}$ of the original parameters n_j, c_{n_j} for $j = 1, \ldots, 6$, as shown in Table 2.

j	n_j	c_{n_j}	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	142	-3	142.0000000018223	-2.999999999999987
2	125	-1	125.0000000494359	-1.000000000000034
3	91	2	90.9999998114290	2.000000000000063
4	69	-3	69.0000003316075	-3.000000000000058
5	53	-1	53.0000003445395	-0.999999999999988
6	11	2	10.9999999973030	2.000000000000004

Table 2. Numerical evaluation of indices of active basis polynomials and coefficients of a sparse Laguerre expansion using Algorithm 4.3.

Here, since we know that the orders n_j of the polynomials are integers, we have rounded the values $\tilde{n}_j$ to the next integer before proceeding with the last step of Algorithm 4.3. While the degree of the polynomial $f(x)$ is 142, the 12 function and derivative values $f^{(m)}$, $m = 0, \ldots, 11$, are sufficient for reconstruction of the sparse expansion.

Example 4.5 Sparse Legendre expansions
We consider sparse Legendre expansions that have already been studied in [21]. The n-th Legendre polynomial P_n satisfies the operator equation

$$(1-x^2)P_n''(x) - 2xP_n'(x) = -n(n+1)P_n(x). \tag{4.5}$$

Hence, a sparse Legendre expansion of the form

$$f(x) = \sum_{j=1}^{M} c_{n_j} P_{n_j}(x)$$

with $c_{n_j} \in \mathbb{C}\backslash\{0\}$ and active indices $0 \le n_1 < \cdots < n_M = N$ can be reconstructed from the values $f^{(m)}(x_0)$, $m = 0, \ldots, 4M-2$, for arbitrarily chosen $x_0 \in \mathbb{R}$ satisfying $P_n(x_0) \neq 0$ for all $n \in \mathbb{N}_0$. In particular, for $x_0 = 1$ (or $x_0 = -1$) we need only the values from $f^{(m)}(1)$ (resp.$f^{(m)}(1)$), $m = 0, \ldots, 2M-1$, for the unique reconstruction of f. Multiplying (4.5) with a constant $\alpha \neq 0$ does not change the solutions. Thus we can consider

$$L^k_{p,q,\alpha} P_n(x) := \alpha(1-x^2)P_n''(x) - 2\alpha x P_n'(x) = -n(n+1)\alpha P_n(x),$$

where $p_\alpha(x) = \alpha(1-x^2)$, $q_\alpha(x) = -2\alpha x$ and $\lambda_{n,\alpha} = -n(n+1)\alpha$. Hence

$$L^k_{p,q,\alpha} f(1) = \sum_{\ell=1}^{k} g^\alpha_{\ell,k}(1) f^{(\ell)}(1)$$

with

$$\begin{aligned}
&g^\alpha_{1,1}(1) = -2\alpha, \\
&g^\alpha_{\ell,k}(1) = -\ell(\ell+1)\alpha g^\alpha_{\ell,k-1}(1) - 2\alpha\ell g^\alpha_{\ell-1,k-1}(1), && k > 1, \ell = 1, \ldots, k, \\
&g^\alpha_{\ell,k}(1) = 0, && k \ge 1, \ell \notin \{1, \ldots, k\}.
\end{aligned}$$

The constant α can be chosen suitably in order to improve the condition of the matrix $\mathbf{G}$. In particular, for $\alpha = -\frac{1}{2}$ we obtain

$$g^{-1/2}_{\ell,k}(1) = \frac{\ell(\ell+1)}{2} g^{-1/2}_{\ell,k-1}(1) + \ell g^{-1/2}_{\ell-1,k-1}(1)$$

yielding for $M = 3$,

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 8 & 6 & 0 & 0 \\ 1 & 26 & 60 & 24 & 0 \\ 1 & 80 & 438 & 480 & 120 \end{pmatrix},$$

see also [21]. We use Algorithm 4.3 in order to recover the sparse Legendre expansion

$$f(x) = -3P_{5492}(x) - P_{465}(x) + 2P_{54}(x)$$

of degree 5492 from the given values $f(1)$, $f'(1), \ldots, f^{(5)}(1)$. Table 3 contains the computed approximations $\tilde{n}_j, \tilde{c}_{n_j}$ of the original parameters n_j, c_{n_j} for $j = 1, 2, 3$.

j	n_j	c_{n_j}	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	54	2	53.983951125658	2.000000000000048
2	465	-1	465.000054039331	-1.000000000000048
3	5492	-3	5491.999999999999	-3.000000000000000

Table 3. Numerical evaluation of indices of active basis polynomials and coefficients of a sparse Legendre expansion using Algorithm 4.3.

Here again, since we know that the orders n_j of the polynomials are integers, we have rounded the values $\tilde{n}_j$ to the next integer before proceeding with the last step of Algorithm 4.3.

5 Recovery of sparse vectors

The generalized Prony method considered in Section 2 can also be applied to finite dimensional vector spaces. Let $\mathbf{x} \in \mathbb{C}^N$ be M-sparse, i.e., only M components of $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ are different from zero.

We want to recover $\mathbf{x}$ from only $2M$ samples $y_k = \mathbf{a}_k^{\mathrm{T}}\mathbf{x}$, $(k = 0, \ldots, 2M-1)$, where the vectors $\mathbf{a}_k \in \mathbb{C}^N$ need to be chosen suitably. The problem of reconstructing sparse vectors using only a small amount of measurements has been heavily studied in the research field of compressed sensing, where the recovery algorithms are usually based on ℓ^1-minimization or greedy methods. In this regard, often a stochastic matrix $\mathbf{A} \in \mathbb{C}^{M_1 \times N}$ is used in order to recover $\mathbf{x}$ from $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{C}^{M_1}$, $(M_1 \geq 2M)$ with high probability.

Here we want to derive a deterministic method to recover $\mathbf{x}$, where $\mathbf{A} \in \mathbb{C}^{2M \times N}$ with rows $\mathbf{a}_k^{\mathrm{T}} \in \mathbb{C}^N$ is explicitly given and of minimal dimension. For this purpose, we use a linear operator $\mathbf{D} : \mathbb{C}^N \to \mathbb{C}^N$ that can be represented by a diagonal matrix $\mathbf{D} = \mathrm{diag}(d_0, \ldots, d_{N-1})$ with pairwise different entries d_j, $(j = 0, \ldots, N-1)$. Obviously, the unit vectors $\mathbf{e}_j := (\delta_{j,\ell})_{\ell=0}^{N-1}$ form a system of eigenvectors of $\mathbf{D}$ with $\mathbf{D}\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 0, \ldots, N-1$. Further we choose a linear functional $F : \mathbb{C}^N \to \mathbb{C}$ of the form $F\mathbf{x} = \mathbb{1}^{\mathrm{T}}\mathbf{x} := \sum_{j=1}^{N} x_j$. Hence, $F\mathbf{e}_j = \mathbb{1}^{\mathrm{T}}\mathbf{e}_j = 1 \neq 0$ holds.

Using Theorem 2.1, we can now reconstruct a sparse vector $\mathbf{x}$ of the form

$$\mathbf{x} = \sum_{j=1}^{M} c_{n_j} \mathbf{e}_{n_j}$$

with $0 \leq n_1 < \cdots < n_M \leq N-1$ from the values

$$F(\mathbf{D}^k\mathbf{x}) = \mathbb{1}^{\mathrm{T}} \cdot \mathbf{D}^k\mathbf{x} = \mathbf{a}_k^{\mathrm{T}}\mathbf{x},$$

where $\mathbf{a}_k^{\mathrm{T}} = (d_0^k, \ldots, d_{N-1}^k)$, $k = 0, \ldots, 2M-1$.

Algorithm 5.1 **(Reconstruction of a sparse vector)**

Input: *M, $y_k = \mathbf{a}_k^{\mathrm{T}}\mathbf{x}$, $k = 0, \ldots, 2M-1$.*

1. *Solve the Hankel system*

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{M-1} \\ y_1 & y_2 & \cdots & y_M \\ \vdots & & & \vdots \\ y_{M-1} & y_M & \cdots & y_{2M-2} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} y_M \\ y_{M+1} \\ \vdots \\ y_{2M-1} \end{pmatrix}.$$

2. *Set $p_M = 1$ and compute the zeros λ_j, $j = 1, \ldots, M$, of the Prony polynomial*

$$P(z) = \sum_{k=0}^{M} p_k z^k.$$

3. The set of zeros $\{\lambda_1, \dots, \lambda_M\}$ of $P(z)$ is a subset of the set of eigenvalues $\{d_0, \dots, d_{N-1}\}$ of $\mathbf{D}$. Determine the eigenvectors $\mathbf{e}_{n_j}$, $j = 1, \dots, M$, respectively the indices n_j, that correspond to the zeros/eigenvalues $\lambda_1, \dots, \lambda_M$.

4. Compute the coefficients c_{n_j}, $j = 1, \dots, M$, from the overdetermined system

$$y_k = \sum_{j=1}^{M} c_{n_j} d_{n_j}^k, \quad k = 0, \dots, 2M-1.$$

Output: *n_j, c_{n_j}, $j = 1, \dots, M$, determining $\mathbf{x}$.*

To demonstrate this approach we want to present a small numerical example. Let $\mathbf{x} \in \mathbb{R}^{128}$ be a 3-sparse vector with $x_{28} = 3, x_{71} = -1, x_{99} = 4$, and let $\mathbf{D} = \mathrm{diag}(k/32)_{k=-63}^{64} \in \mathbb{R}^{128\times 128}$. For a given vector of values $\mathbf{f} = (f_k)_{k=0}^{5}$ with $f_k = \mathbb{1} \cdot \mathbf{D}^k \mathbf{x}$ we compute approximations $\tilde{n}_j$ and $\tilde{x}_{n_j}$ according to Algorithm 5.1. The results are shown in the Table 4.

n_j	x_{n_j}	$\tilde{n}_j$	$\tilde{x}_{n_j}$
28	3	27.99999999999999	3
71	-1	71.00000000000001	-1
99	4	99.00000000000000	4

Table 4. Numerical evaluation of the indices and the coefficients of a sparse vector $\mathbf{x}$ using Algorithm 5.1.

Remarks 5.2 *1. In order to obtain a stable algorithm, the operator $\mathbf{D}$ may for example be chosen as*

$$\mathbf{D} = \mathrm{diag}(\omega_N^0, \omega_N^1, \dots, \omega_N^{N-1}),$$

where $\omega_N := \mathrm{e}^{-2\pi \mathrm{i}/N}$ denotes the N-th root of unity. For this choice of $\mathbf{D}$, the vector $\mathbf{y} = (y_k)_{k=0}^{2M-1}$ of needed input values for Algorithm 5.1 is given by

$$\mathbf{y} = \mathbf{F}_{N,2M}\mathbf{x},$$

where $F_{N,2M} = (\omega_N^{k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$ contains the first $2M$ rows of the Fouriermatrix of order N.

2. In the above considerations, the canonical basis can be replaced by any other basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_N\}$ of $\mathbb{C}^N$. Choose a diagonal matrix $\mathbf{D}$ with pairwise different (complex) entries $\lambda_1, \dots, \lambda_N$. Then the operator $\mathbf{A} := \mathbf{B}\mathbf{D}\mathbf{B}^{-1} : \mathbb{C}^N \to \mathbb{C}^N$, where $\mathbf{B} = (\mathbf{b}_1 \dots \mathbf{b}_N) \in \mathbb{C}^{N\times N}$ contains the columns $\mathbf{b}_j$, possesses the eigenvalues $\lambda_1, \dots, \lambda_N$ with corresponding eigenvectors $\mathbf{b}_1, \dots, \mathbf{b}_N$ by construction. Further, we define a functional $F : \mathbb{C}^N \to \mathbb{C}$ satisfying $F\mathbf{b}_\ell \neq 0$ for $\ell = 1, \dots, N$. We can e.g. choose $F\mathbf{x} := \mathbf{a}^T\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^N$, where $\mathbf{a}$ is taken suitably. Hence, a sparse expansion

$$\mathbf{x} = \sum_{j=1}^{M} c_{n_j} \mathbf{b}_{n_j}$$

in the basis B can by Theorem 2.1 be recovered by

$$F(\mathbf{A}^k\mathbf{x}) = \mathbf{a}^T\mathbf{A}^k\mathbf{x}, \qquad k = 0, \dots, 2M-1.$$

3. Using Theorem 2.3, we can apply the recovery procedure also when the given operator possesses eigenvalues with higher multiplicity, where also generalized eigenvectors can be incorporated.

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