An Area Preserving Projection from the Regular Octahedron to the Sphere

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Dedicated to Werner Haußmann in memoriam

Abstract. In this paper, we propose an area preserving bijective map from the regular octahedron to the unit sphere $\mathbb{S}^2$, both centered at the origin. The construction scheme consists of two steps. First, each face F_i of the octahedron is mapped to a curved planar triangle $\mathcal{T}_i$ of the same area. Afterwards, each $\mathcal{T}_i$ is mapped onto the sphere using the inverse Lambert azimuthal equal area projection with respect to a certain point of $\mathbb{S}^2$. The proposed map is then used to construct uniform and refinable grids on a sphere, starting from any triangular uniform and refinable grid on the triangular faces of the octahedron.

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1. Introduction

In many applications, especially in geosciences and astronomy, but also in computer vision, one is interested in simple, refinable grids on the sphere. In particular, one requires partitions of $\mathbb{S}^2$ into regions of equal area and small diameter in order to avoid the distortions that often occur in statistical computations and function approximations using non-equal area partitions.

There exist already some constructions of equal area partitions of $\mathbb{S}^2$. Based on the construction by Zhou [12], Leopardi [4] derives a recursive zonal equal area partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ that consists of polar cups and rectilinear regions. This construction has the disadvantage that one has to deal with different kinds of areas. The existence of partitions of $\mathbb{S}^2$ into regions of equal area and small diameter has been already used by Alexander [1], who derived lower bounds for the maximum sum of distances between

points on the sphere. Tegmark [11] considered an icosahedron-based pixeliz-
ing of the sphere. Similarly, the algorithm for binning globally distributed
measurements on the sphere by Teanby [10] is based on repeated subdivi-
sion of the icosahedron. In [8], a subdivision method is proposed using a
spherical triangulation that is obtained by projecting the faces of an icosa-
hedron to the sphere. The HEALPix (Hierarchical Equal Area iso-Latitude
Pixelization) introduced in [3], has been frequently used in recent years for
application of local pixel set operations, multiresolution applications and fast
spherical harmonic transforms.

However, one simple method to construct grids on the sphere is to trans-
fer existing planar grids by a suitable projection. For a survey of known spher-
ical projections from the sphere or parts of the sphere to the plane we refer
to [2, 9]. Therefore we are especially interested in the construction of equal
area partitions being obtained by application of an area-preserving bijective
map from suitable planar domains to (parts of) the sphere.

In [5], one of the authors already suggested a new area preserving pro-
jection method based on a mapping of the square onto a disc in a first step,
followed by a lifting to the sphere by the inverse Lambert projection. This
idea can also be generalized to construct uniform and refinable grids on ellip-
tic domains and on some surfaces of revolution, see [6]. In [7], the authors have
recently constructed an equal area projection from the cube to the sphere.

In this paper we construct an area preserving map from a regular oc-
tahedron to the unit sphere $\mathbb{S}^2$. Thus, *any* grid on the octahedron can be
transported to the sphere. Further, since arbitrary grids on the triangle can
now simply be transported to the sphere, we believe that this construction
may achieve an essential impact for different applications in geosciences. Since
we give explicit formulas both for the map from the octahedron to the sphere,
and from the sphere to the octahedron, the method is easy to implement.

The construction scheme consists of two steps. In the first step, we
construct in Section 3 an area preserving bijection U from each face F_i of the
octahedron, onto a planar domain $\widetilde{F}_i$, bounded by a curved triangle $\mathcal{T}_i$. In the
second step, we combine U with an inverse Lambert azimuthal projection, in
order to map each face F_i of the octahedron onto a subset $\mathcal{F}_i$ of the sphere,
such that $\cup_{i=1}^{8} \overline{\mathcal{F}_i} = \mathbb{S}^2$. A closed form of the inverse area preserving map U^{-1}
is presented in Section 6. Further, we present some examples of the obtained
spherical grids.

2. Preliminaries

Consider the unit sphere $\mathbb{S}^2$ centered at the origin O and the regular octahe-
dron $\mathbb{K}$ of the same area, centered at O and with vertices on the coordinate
axes. Since the area of the sphere is 4π, the area of each face of $\mathbb{K}$ is $\pi/2$ and
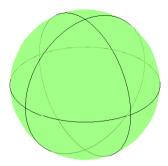
FIGURE 1. The eight spherical triangles obtained as inter-
section of the coordinate planes with the sphere.

the edge of the octahedron has the length

$$\ell = \frac{\sqrt{2\pi}}{\sqrt[4]{3}}. \tag{2.1}$$

We cut the sphere with the coordinate planes $x = 0$, $y = 0$, $z = 0$ and obtain the spherical triangles in Figure 1. Each face F_i, $i = 1, \ldots, 8$, of the octahedron $\mathbb{K}$ is thus situated in one of the following domains

$$
\begin{aligned}
I_1 &= \{(x,y,z),\ x \geq 0, y \geq 0, z \geq 0\},\ I_2 = \{(x,y,z),\ x \geq 0, y \geq 0, z \leq 0\}, \\
I_3 &= \{(x,y,z),\ x \geq 0, y \leq 0, z \geq 0\},\ I_4 = \{(x,y,z),\ x \geq 0, y \leq 0, z \leq 0\}, \\
I_5 &= \{(x,y,z),\ x \leq 0, y \geq 0, z \geq 0\},\ I_6 = \{(x,y,z),\ x \leq 0, y \geq 0, z \leq 0\}, \\
I_7 &= \{(x,y,z),\ x \leq 0, y \leq 0, z \geq 0\},\ I_8 = \{(x,y,z),\ x \leq 0, y \leq 0, z \leq 0\}.
\end{aligned}
$$

More precisely, $F_i \subset I_i$ for $i = 1, \ldots, 8$ and the portion of the sphere situated in I_i will be denoted by $\mathcal{F}_i$, i.e.,

$$\mathcal{F}_i = \{(x, y, z) \in I_i,\ x^2 + y^2 + z^2 = 1\},$$

see Figure 1.

We focus on the portion $\mathcal{F}_1$ of the sphere and consider its center point $G_1 = (\nu, \nu, \nu)$, with $\nu = 1/\sqrt{3}$. The point G_1 is in fact the intersection of the line OG_1' with $\mathbb{S}^2$, where G_1' is the weight center of the face F_1.

We consider the Lambert projection L_{G_1} of the domain $\mathcal{F}_1 \subset \mathbb{S}^2$ with respect to the point G_1. Thus, $\mathcal{F}_1$ will be mapped on a curved triangle that is situated in the plane $x + y + z = \sqrt{3}$ perpendicular to the line OG_1. For calculating this projection, we apply a transformation to the sphere such that G_1 is mapped onto the South Pole $S = (0, 0, -1)$. Such a transformation $\mathcal{R}$ consists in two successive rotations, $\mathcal{R}_1$ and $\mathcal{R}_2$. The first rotation $\mathcal{R}_1$ is taken around the z-axis with angle $-\pi/4$ and maps G_1 onto $P_1 = \left(\frac{\sqrt{2}}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right)$. The second rotation $\mathcal{R}_2$ is around the y-axis and maps P_1 onto $S = (0, 0, -1)$, so its angle is $\beta + \frac{\pi}{2}$, with $\beta = \arcsin \frac{1}{\sqrt{3}}$. The corresponding rotation matrices

of $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$ have the form

$$\mathcal{R}_1 = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{R}_2 = \begin{pmatrix} \cos\left(\beta+\frac{\pi}{2}\right) & 0 & \sin\left(\beta+\frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(\beta+\frac{\pi}{2}\right) & 0 & \cos\left(\beta+\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\mathcal{R} = \mathcal{R}_2 \cdot \mathcal{R}_1 = \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}. \tag{2.2}$$

The matrices $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}$ are orthogonal, therefore $\mathcal{R}^{-1} = \mathcal{R}^T$.

Now, in order to obtain the Lambert projection L_{G_1} with respect to $G_1 \in \mathbb{S}^2$, we use the azimuthal Lambert projection L_S with respect to the South Pole S, as follows:

$$L_{G_1} = \mathcal{R}^{-1} \circ L_S \circ \mathcal{R}. \tag{2.3}$$

It is well known that the image of a point $(x, y, z) \in \mathbb{S}^2$ by L_S onto the tangential plane at the South Pole is

$$\left(\sqrt{\tfrac{2}{1-z}}\, x, \sqrt{\tfrac{2}{1-z}}\, y, -1 \right).$$

For the rotated Lambert projection on $Z_L = -1$ we obtain

$$(L_S \circ \mathcal{R})(x, y, z) = (X_L, Y_L, -1) = \left(\frac{\nu(2z-x-y)}{\sqrt{1+\nu(x+y+z)}}, \frac{-x+y}{\sqrt{1+\nu(x+y+z)}}, -1 \right)$$

with $\nu = 1/\sqrt{3}$. Therefore, for L_{G_1} we find

$$L_{G_1}(x, y, z) = \left(\nu + \frac{\sqrt{2}\nu^2(2x-y-z)}{\sqrt{1+\nu(x+y+z)}}, \nu + \frac{\sqrt{2}\nu^2(2y-x-z)}{\sqrt{1+\nu(x+y+z)}}, \nu + \frac{\sqrt{2}\nu^2(2z-x-y)}{\sqrt{1+\nu(x+y+z)}} \right).$$

Let us denote by l_1, l_2, l_3 the edges of the spherical triangle $\mathcal{F}_1$, situated on the planes $z = 0$, $y = 0$ and $x = 0$, respectively, see Figure 1. These edges are mapped by $L_S \circ \mathcal{R}$ onto the planar curves in the plane $Z_L = -1$, given by the following parametric equations

$$(L_S \circ \mathcal{R})(l_1) : \begin{cases} X_L = \dfrac{(-x-y)}{\sqrt{3+\sqrt{3}(x+y)}} \\ Y_L = \dfrac{\sqrt{3}(-x+y)}{\sqrt{3+\sqrt{3}(x+y)}}, \end{cases} \text{ with } y = \sqrt{1-x^2},\ x \in [0,1], \quad (2.4)$$

$$(L_S \circ \mathcal{R})(l_2) : \begin{cases} X_L = \dfrac{(2z-x)}{\sqrt{3+\sqrt{3}(x+z)}} \\ Y_L = \dfrac{-\sqrt{3}x}{\sqrt{3+\sqrt{3}(x+z)}}, \end{cases} \text{ with } z = \sqrt{1-x^2},\ x \in [0,1],$$

$$(L_S \circ \mathcal{R})(l_3) : \begin{cases} X_L = \dfrac{(2z-y)}{\sqrt{3+\sqrt{3}(y+z)}} \\ Y_L = \dfrac{\sqrt{3}y}{\sqrt{3+\sqrt{3}(y+z)}}, \end{cases} \text{ with } z = \sqrt{1-y^2},\ y \in [0,1].$$

We denote the curved triangle determined by these three curves by $\mathcal{T}_1 = \mathcal{T}$. It can be simply transferred to the plane $x + y + z = \sqrt{3}$ using the rotation map $\mathcal{R}^{-1}$ (and hence finishing the projection L_{G_1}). But we prefer to compare the curved triangle $\mathcal{T}$ with the rotated triangular face $T = \mathcal{R}T_1$ of the octahedron in the plane $Z_L = -1$. A simple calculation shows that the vertices of the equilateral triangle $T = \mathcal{R}T_1$ are $P_1 = (2\alpha, 0, -1)$, $P_2 = (-\alpha, \ell/2, -1)$, and $P_3 = (-\alpha, -\ell/2, -1)$, with

$$\alpha = \sqrt{\frac{\pi}{6\sqrt{3}}} \tag{2.5}$$

and side length ℓ in (2.1). Figure 2 (left) shows the curved triangle $\mathcal{T}$ and the equilateral triangle T of the same area $\pi/2$. With the help of the projection L_{G_1} (resp. $L_S \circ \mathcal{R}$), we have simplified now the problem of finding an area preserving map from the octahedron to the sphere $\mathbb{S}^2$ to a problem of finding a two-dimensional map from an equilateral triangle T, i.e., a face of the octahedron, to the curved triangle $\mathcal{T}$. In our further considerations in Section 3, we will consider only two dimensions while $Z_L = -1$ is fixed.

3. Mapping a triangle onto a curved triangle

In this section we derive an area preserving bijection $U : \mathbb{R}^2 \to \mathbb{R}^2$, which maps the equilateral triangle T with vertices $P_1 = (2\alpha, 0)$, $P_2 = (-\alpha, \ell/2)$, and $P_3 = (-\alpha, -\ell/2)$ onto the curved triangle $\mathcal{T}$ that has been constructed in Section 2. Here, we say that the map U is area-preserving, if it has the property

$$\mathcal{A}(D) = \mathcal{A}(U(D)), \text{ for every domain } D \subseteq \mathbb{R}^2,$$

where $\mathcal{A}(D)$ denotes the area of D.

We consider the three half-lines $h_1 = OP_1$, $h_2 = OP_2$, $h_3 = OP_3$, that form the angles $0, 2\pi/3, 4\pi/3$ around O, see Figure 2(left), and determine three disjoint regions of $\mathbb{R}^2$ defined by

$$\begin{aligned}
\mathcal{Q}_1 &= \{(x, y) \in \mathbb{R}^2, \ \sqrt{3}x \leq y \leq -\sqrt{3}x\}, \\
\mathcal{Q}_2 &= \{(x, y) \in \mathbb{R}^2, \ y < 0, \ y \leq \sqrt{3}x\}, \\
\mathcal{Q}_3 &= \{(x, y) \in \mathbb{R}^2, \ y > 0, \ y \geq -\sqrt{3}x\}
\end{aligned}$$

that are bounded by these half lines. We focus for the moment on the region $\mathcal{Q}_1^+$ defined by

$$\mathcal{Q}_1^+ = \{(x, y) \in \mathbb{R}^2, \ x < 0, \ 0 \leq y \leq -\sqrt{3}x\} \subset \mathcal{Q}_1$$

and take a point $M = (x_M, y_M) = (x_M, mx_M) \in \mathcal{Q}_1^+$, where $m \in [-\sqrt{3}, 0]$ is a parameter, see Figure 2, right. We want to define the map U in such a way that each half-line d_m of equation $y = m\,x$ ($m \in [-\sqrt{3}, 0]$) is mapped onto the half-line $d_{\varphi(m)}$ of equation $y = \varphi(m)x$, where $\varphi : [-\sqrt{3}, 0] \to \mathbb{R}$ is a C^1-function that satisfies

$$\varphi(0) = 0, \ \varphi(-\sqrt{3}) = -\sqrt{3}, \text{ and } -\sqrt{3} \leq \varphi(m) \leq 0 \text{ for } m \in [-\sqrt{3}, 0].$$
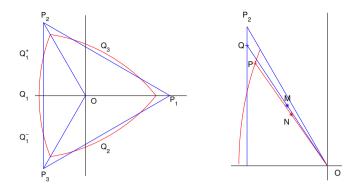
FIGURE 2. Left: the equilateral triangle $T = P_1P_2P_3$, the curved triangle $\mathcal{T}$ and the zones $\mathcal{Q}_1, \mathcal{Q}_1^+, \mathcal{Q}_1^-, \mathcal{Q}_2, \mathcal{Q}_3$; Right: the action of the transform U: points N and P are the images of M and Q, respectively.

We denote by (x_N, y_N) the coordinates of the point $N = U(M)$. Let Q be the intersection of OM with the triangle T (see Figure 2, right). The point Q has the coordinates $(x_Q, y_Q) = (-\alpha, -m\alpha)$, where α is given in (2.5), and the line ON has the equation $y = \varphi(m)x$, i.e., $N = (x_N, \varphi(m)x_N)$. Further, let the point $P = (x_P, y_P) = (x_P, \varphi(m)x_P)$ be the intersection of ON with the curved triangle $\mathcal{T}$. Thus, the coordinates of P satisfy the equations (2.4). Replacing x by the parameter $t = t_P \in [0, 1]$ in (2.4), we obtain

$$x_P = \frac{(-t_P - \sqrt{1-t_P^2})}{\sqrt{3+\sqrt{3}(t_P+\sqrt{1-t_P^2})}}, \tag{3.1}$$

$$y_P = \frac{\sqrt{3}(-t_P+\sqrt{1-t_P^2})}{\sqrt{3+\sqrt{3}(t_P+\sqrt{1-t_P^2})}}, \tag{3.2}$$

and from $y_P = \varphi(m)x_P$ we have

$$\varphi(m) = \frac{y_P}{x_P} = \frac{\sqrt{3}(t_P-\sqrt{1-t_P^2})}{t_P+\sqrt{1-t_P^2}}.$$

From this equality we can determine t_P as

$$t_P = \frac{\varphi(m)+\sqrt{3}}{\sqrt{2}\sqrt{\varphi^2(m)+3}},$$

and further, if we replace t_P in (3.1) and (3.2), we obtain the coordinates of P in the form

$$x_P = -\frac{\sqrt{2}}{\sqrt{3+\varphi^2(m)+\sqrt{2(3+\varphi^2(m))}}},$$

$$y_P = -\frac{\sqrt{2}\,\varphi(m)}{\sqrt{3+\varphi^2(m)+\sqrt{2(3+\varphi^2(m))}}}.$$

Some simple calculations yield the Euklidean distances

$$
\begin{aligned}
OM &= x_M\sqrt{1+m^2}, \\
OQ &= \alpha\sqrt{1+m^2}, \\
ON &= x_N\sqrt{1+\varphi^2(m)}, \\
OP &= \sqrt{2}\left(1 - \frac{\sqrt{2}}{\sqrt{3+\varphi^2(m)}}\right)^{1/2},
\end{aligned}
$$

where α is given in (2.5). In order to simplify the determination of the map U, we suppose that

$$
\frac{ON}{OP} = \frac{OM}{OQ}.
$$

From the above calculations we then obtain $ON = \frac{x_M}{\alpha}OP$ and hence

$$
x_N = \frac{x_M}{\alpha}\frac{\sqrt{2}}{\sqrt{1+\varphi^2(m)}}\sqrt{1 - \frac{\sqrt{2}}{\sqrt{3+\varphi^2(m)}}},
$$

$$
y_N = \varphi(m)x_N = \frac{x_M}{\alpha}\frac{\sqrt{2}\,\varphi(m)}{\sqrt{1+\varphi^2(m)}}\sqrt{1 - \frac{\sqrt{2}}{\sqrt{3+\varphi^2(m)}}}.
$$

Thus, the map U is now completely described by means of the function φ, and we obtain that U maps the point $(x,y) \in \mathcal{Q}_1^+$ onto the point (X,Y) given by

$$
X = \frac{x}{\alpha}\frac{\sqrt{2}}{\sqrt{1+\varphi^2(\frac{y}{x})}}\sqrt{1 - \frac{\sqrt{2}}{\sqrt{3+\varphi^2(\frac{y}{x})}}}, \tag{3.3}
$$

$$
Y = \frac{x}{\alpha}\frac{\sqrt{2}\,\varphi(\frac{y}{x})}{\sqrt{1+\varphi^2(\frac{y}{x})}}\sqrt{1 - \frac{\sqrt{2}}{\sqrt{3+\varphi^2(\frac{y}{x})}}}. \tag{3.4}
$$

Next we have to ensure the area preserving property of U by a suitable determination of φ. For this purpose, we define the function φ such that[1] the Jacobian of U is 1. After simplification, the Jacobian writes as

$$
J(U) = \det\begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix} = \frac{2}{\alpha^2}\frac{\varphi'(\frac{y}{x})}{3+\varphi^2(\frac{y}{x})+\sqrt{2}\sqrt{3+\varphi^2(\frac{y}{x})}}.
$$

For solving the equation $J(U) = 1$ we again substitute $m := \frac{y}{x}$, and thus, in the considered case $0 \le y \le -\sqrt{3}x$ we have $m \in [-\sqrt{3}, 0]$. Hence, with the simplified notation $\varphi = \varphi(m)$, we get

$$
\frac{\varphi'}{3+\varphi^2+\sqrt{2}\sqrt{3+\varphi^2}} = \frac{\alpha^2}{2}. \tag{3.5}
$$

Integration gives

$$
\arctan\varphi - \arctan\frac{\sqrt{2}\varphi}{\sqrt{3+\varphi^2}} = \frac{\alpha^2}{2}m + C.
$$

The condition $\varphi(0) = 0$ yields $C = 0$. Next, in order to determine φ we use the formula

$$
\arctan a - \arctan b = \arctan\frac{a-b}{1+ab} \qquad \forall a,b \in \mathbb{R},\ ab > -1,
$$

---

[1]We recall that the area of a transformed region D is $\mathcal{A}(U(D)) = |J(U)|\mathcal{A}(D)$.

and we further obtain

$$\frac{\varphi\left(\sqrt{3+\varphi^2}-\sqrt{2}\right)}{\sqrt{3+\varphi^2}+\sqrt{2}\varphi^2} = \tan\frac{\alpha^2 m}{2}. \tag{3.6}$$

To simplify this term, we introduce the notation

$$\eta = \tan\frac{\alpha^2 m}{2} = \tan\frac{\pi m}{12\sqrt{3}},$$

and equality (3.6) yields

$$\sqrt{3+\varphi^2}\,(\varphi - \eta) = \sqrt{2}\varphi(\eta\varphi + 1). \tag{3.7}$$

We must have $\varphi > \eta$ and with this condition, (3.7) is equivalent with

$$(3+\varphi^2)(\varphi - \eta)^2 = 2\varphi^2(\eta\varphi + 1)^2,$$
$$(\varphi^2 + 1)((1 - 2\eta^2)\varphi^2 - 6\eta\varphi + 3\eta^2) = 0,$$

which gives

$$\varphi_{1,2} = \varphi_{1,2}(m) = \frac{3\eta \pm \sqrt{6}\eta\sqrt{1+\eta^2}}{1-2\eta^2}.$$

To simplify the formulas, we denote $\delta = \frac{\alpha^2 m}{2} = \frac{\pi m}{12\sqrt{3}} = \frac{\pi y}{12\sqrt{3}x}$ and with $\eta(1+\eta^2)^{-1/2} = \sin\delta$ and $(1+\eta^2)^{-1/2} = \cos\delta$ we further calculate

$$\varphi_{1,2}(m) = \varphi_{1,2}\left(\frac{y}{x}\right) = \frac{\sqrt{3}\eta}{\sqrt{3}\mp\sqrt{2}\sqrt{1+\eta^2}} = \frac{\sqrt{3}\sin\delta}{\sqrt{3}\cos\delta\pm\sqrt{2}}.$$

If we take into account the condition $\varphi(-\sqrt{3}) = -\sqrt{3}$ and the equality

$$\sin\frac{\pi}{12} = \sqrt{3}\cos\frac{\pi}{12} - \sqrt{2},$$

the only convenient solution is

$$\varphi\left(\frac{y}{x}\right) = \frac{\sqrt{3}\sin\delta}{\sqrt{3}\cos\delta - \sqrt{2}}, \qquad \text{with } \delta = \frac{\pi y}{12\sqrt{3}x}.$$

Finally, in order to give the explicit presentation of the area preserving map U, we need to replace φ in the equations (3.3)-(3.4). Some straightforward calculations show that

$$\sqrt{1+\varphi^2} = \frac{\sqrt{5-2\sqrt{6}\cos\delta}}{\sqrt{3}\cos\delta-\sqrt{2}},$$
$$\sqrt{3+\varphi^2} = \frac{\sqrt{3}(\sqrt{3}-\sqrt{2}\cos\delta)}{\sqrt{3}\cos\delta-\sqrt{2}},$$
$$1-\frac{\sqrt{2}}{\sqrt{3+\varphi^2}} = \frac{5-2\sqrt{6}\cos\delta}{3-\sqrt{6}\cos\delta}.$$

Hence, for $(x, y) \in \mathcal{Q}_1^+$, the formulas for the desired area preserving map $U(x, y) = (X, Y)$ are given by

$$X = \frac{2\sqrt{3}\,x}{\sqrt{\pi}}\frac{\sqrt{3}\cos\delta-\sqrt{2}}{\sqrt{\sqrt{3}-\sqrt{2}\cos\delta}} =: f(x, y), \tag{3.8}$$

$$Y = \frac{6x}{\sqrt{\pi}}\frac{\sin\delta}{\sqrt{\sqrt{3}-\sqrt{2}\cos\delta}} =: g(x, y), \text{ with } \delta = \frac{y\pi}{12\sqrt{3}\,x}. \tag{3.9}$$

Similar arguments for the region $\mathcal{Q}_1^- = \{(x, y) \in \mathbb{R}^2, \; x < 0, \; \sqrt{3}x \leq y \leq 0\}$ show that these formulas also hold for $(x, y) \in \mathcal{Q}_1^-$, so we can conclude that the image $U(x, y) = (X, Y)$ of a point (x, y) in the region

$$\mathcal{Q}_1 = \mathcal{Q}_1^+ \cup \mathcal{Q}_1^- = \{(x, y) \in \mathbb{R}^2, \; x < 0, \; \sqrt{3}\,x \leq y \leq -\sqrt{3}\,x\}$$

is given by formulas (3.8)-(3.9).

For the regions $\mathcal{Q}_2$ and $\mathcal{Q}_3$ we may use the formulas (3.8)-(3.9) after performing some rotations. More precisely, we need the rotation matrix

$$\mathcal{A} = \mathcal{A}_{\frac{2\pi}{3}} = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

We denote by U_1, U_2, U_3 the restrictions of U to $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$, respectively. Then, for $(x, y) \in \mathcal{Q}_2$ we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathcal{A}^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{Q}_1,$$

therefore, to (x', y') we can apply U defined by the formulas (3.8)-(3.9) and obtain'$(X', Y') \in \mathcal{Q}_1$,

$$X' = f(x', y'), \quad Y' = g(x', y'), \text{ or } \begin{pmatrix} X' \\ Y' \end{pmatrix} = U_1(x', y').$$

Finally, if we apply to (X', Y') a rotation of $\frac{2\pi}{3}$ we obtain $(X'', Y'') \in \mathcal{Q}_2$ defined by

$$\begin{pmatrix} X'' \\ Y'' \end{pmatrix} := U(x, y) = U_2(x, y) = \mathcal{A} \cdot U_1(x', y') = \mathcal{A} \cdot U_1(\mathcal{A}^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix}).$$

Similarly, for $(x, y) \in \mathcal{Q}_3$ we obtain

$$\begin{pmatrix} X'' \\ Y'' \end{pmatrix} := U(x, y) = U_3(x, y) = \mathcal{A}^{-1} \cdot U_1(x', y') = \mathcal{A}^{-1} U_1(\mathcal{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix}).$$

In conclusion, the area preserving map U which maps equilateral triangles onto curved triangles can be written as follows:

- For $(x, y) \in \mathcal{Q}_1 = \{(x, y) \in \mathbb{R}^2, \ \sqrt{3}x \le y \le -\sqrt{3}x\}$,

$$(x, y) \longmapsto U_1(x, y) = (X, Y) := \left(\frac{2\sqrt{3}x}{\sqrt{\pi}} \sqrt{\frac{\sqrt{3}\cos\delta - \sqrt{2}}{\sqrt{3} - \sqrt{2}\cos\delta}}, \frac{6x}{\sqrt{\pi}} \frac{\sin\delta}{\sqrt{\sqrt{3} - \sqrt{2}\cos\delta}} \right); \tag{3.10}$$

- For $(x, y) \in \mathcal{Q}_2 = \{(x, y) \in \mathbb{R}^2, \ y < 0, \ y \le \sqrt{3}x\}$,

$$(x, y) \longmapsto (X, Y)^T = \mathcal{A} \cdot U_1(\mathcal{A}^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix}); \tag{3.11}$$

- For $(x, y) \in \mathcal{Q}_3 = \{(x, y) \in \mathbb{R}^2, \ y > 0, \ y \ge -\sqrt{3}x\}$,

$$(x, y) \longmapsto (X, Y)^T = \mathcal{A}^{-1} \cdot U_1(\mathcal{A} \cdot \begin{pmatrix} x \\ y \end{pmatrix});$$

with $\delta = \frac{y\pi}{12\sqrt{3}x}$ and $\mathcal{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. Figure 3 shows two examples of planar grids.

4. Mapping F_i onto $\widetilde{F}_i$

The area preserving map $U : \mathbb{R}^2 \to \mathbb{R}^2$ derived in Section 3 depends on two variables. We want to use it for mapping F_i onto $\widetilde{F}_i$, where both F_i and $\widetilde{F}_i$ are included in the same plane of $\mathbb{R}^3$.

We denote by $\mathcal{P}_{-1}$ the plane $z = -1$ and define $\widetilde{U} : \mathcal{P}_{-1} \to \mathcal{P}_{-1}$ by

$$\widetilde{U}(x, y, -1) = (X, Y, -1), \text{ with } (X, Y) = U(x, y), \text{ for all } (x, y) \in \mathbb{R}^2.$$
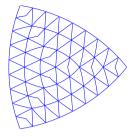
FIGURE 3. Two examples of planar grids of $\mathcal{T}$, images of grids of T constructed with 7 and 8 equidistant parallel lines.

This map allows us to write the formulas for the functions $\mathcal{U}_i : F_i \to \widetilde{F}_i$, for $i = 1, \ldots, 8$, which map triangles (i.e., the faces of the octahedron) onto curved triangles.

As shown in Section 2, we observe that

$$\mathcal{R} \cdot (|x|, |y|, |z|)^T \in \mathcal{P}_{-1},$$

in each of the cases $(x, y, z) \in F_i$, $i = 1, \ldots, 8$, where $\mathcal{R}$ is given in (2.2). Therefore, $\widetilde{U}(\mathcal{R} \cdot (|x|, |y|, |z|)^T) \in \mathcal{P}_{-1}$ and

$$\mathcal{R}^{-1} \cdot \widetilde{U}(\mathcal{R} \cdot (|x|, |y|, |z|)^T) \in \widetilde{F_1}, \text{ for } (x, y, z) \in F_1.$$

So, we define $\mathcal{U}_1 : F_1 \to \widetilde{F_1}$,

$$\mathcal{U}_1(x, y, z) = \mathcal{R}^{-1} \cdot \widetilde{U}(\mathcal{R} \cdot (|x|, |y|, |z|)^T) =: (X, Y, Z), \text{ for } (x, y, z) \in F_1, \quad (4.1)$$

and further, with the notations in (4.1), the required applications $\mathcal{U}_i : F_i \to \widetilde{F}_i$ are defined as follows:

$$\begin{aligned}
\mathcal{U}_2(x, y, z) &= (X, Y, -Z), & \mathcal{U}_3(x, y, z) &= (X, -Y, Z), \\
\mathcal{U}_4(x, y, z) &= (X, -Y, -Z), & \mathcal{U}_5(x, y, z) &= (-X, Y, Z), \\
\mathcal{U}_6(x, y, z) &= (-X, Y, -Z), & \mathcal{U}_7(x, y, z) &= (-X, -Y, Z), \\
\mathcal{U}_8(x, y, z) &= (-X, -Y, -Z).
\end{aligned}$$

Finally, we denote by $\mathcal{U} : \mathbb{K} \to \cup_{i=1}^8 \widetilde{F}_i$ the map satisfying the condition $\mathcal{U} = \mathcal{U}_i$ on F_i.

5. Mapping the curved triangles onto the sphere

The complete mapping from the regular octahedron $\mathbb{K}$ to the sphere $\mathbb{S}^2$ is now described in two steps. In the first step, each face F_i of $\mathbb{K}$ will be mapped onto a planar domain $\widetilde{F}_i$, bounded by a curved triangle $\mathcal{T}_i$, using the transformation $\mathcal{U}$ constructed in the previous section. In the second step, each $\widetilde{F}_i$ will be

mapped onto $\mathcal{F}_i \subseteq \mathbb{S}^2$ by the inverse Lambert azimuthal projection, with respect to $G_i = U(W_i)$, where W_i is the weight center of F_i. Obviously

$$\bigcap_{i=1}^{8} \text{int } \mathcal{F}_i = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{8} \mathcal{F}_i = \mathbb{S}^2.$$

Applying the inverse Lambert projection L_S^{-1}, the point $(X, Y, -1)$, situated in the tangent plane to $\mathbb{S}^2$ at the South Pole S, maps onto $(x_L, y_L, z_L) \in \mathbb{S}^2$ given by

$$x_L = \sqrt{1 - \tfrac{X^2+Y^2}{4}} \, X, \tag{5.1}$$

$$y_L = \sqrt{1 - \tfrac{X^2+Y^2}{4}} \, Y, \tag{5.2}$$

$$z_L = \tfrac{X^2+Y^2}{2} - 1. \tag{5.3}$$

Thus, the application $L_{G_i}^{-1} \circ \mathcal{U}$ maps the face F_i of the octahedron onto $\mathcal{F}_i \subset \mathbb{S}^2$, for $i = 1, \dots, 8$. From (2.3) we have $L_{G_1}^{-1} = \mathcal{R}^{-1} \circ L_S^{-1} \circ \mathcal{R}$, and for obtaining the other projections $L_{G_i}^{-1}$ we proceed as in the previous section. Again, $\mathcal{R} \cdot (|x|, |y|, |z|)^T \in \mathcal{P}_{-1}$, therefore we use formulas (5.1)-(5.3) and we have

$$(x_L, y_L, z_L) = L_S^{-1}(\mathcal{R} \cdot (|x|, |y|, |z|)^T).$$

Further,

$$(X, Y, Z)^T := \mathcal{R}^{-1} \cdot (x_L, y_L, z_L)^T \in \mathcal{F}_1,$$

and then, with these notations, for the other cases we obtain the following expressions for $L_{G_i}^{-1} : \widetilde{F}_i \to \mathcal{F}_i$:

$$L_{G_2}^{-1}(x, y, z) = (X, Y, -Z), \qquad L_{G_3}^{-1}(x, y) = (X, -Y, Z),$$
$$L_{G_4}^{-1}(x, y, z) = (X, -Y, -Z), \qquad L_{G_5}^{-1}(x, y) = (-X, Y, Z),$$
$$L_{G_6}^{-1}(x, y, z) = (-X, Y, -Z), \qquad L_{G_7}^{-1}(x, y) = (-X, -Y, Z),$$
$$L_{G_8}^{-1}(x, y, z) = (-X, -Y, -Z).$$

Figure 4 shows some grids of the sphere, where a regular partition of the faces of the octahedron into 4, 16, 25 and 64 equal area triangles has been applied.

6. The inverse map

To make the area preserving map $\mathcal{U} : \mathbb{K} \to \cup_{i=1}^{8} \widetilde{F}_i$ applicable in practice, we also need to derive a closed simple form for the inverse mapping $\mathcal{U}^{-1}$. In view of the above considerations we can restrict to the calculation of the inverse U^{-1} of the area-preserving map proposed in Section 3, since the mapping of the portions $\mathcal{F}_i$ to $\tilde{F}_i$ is obtained by the azimuthal Lambert projection, and the area preserving mapping of the curved triangle $\tilde{F}_i$ to the faces F_i of the octahedron is completely understood, if we find $U^{-1} : T \to \mathcal{T}$ with the two-dimensional triangles considered in Section 3, see Figure 2. Further, we
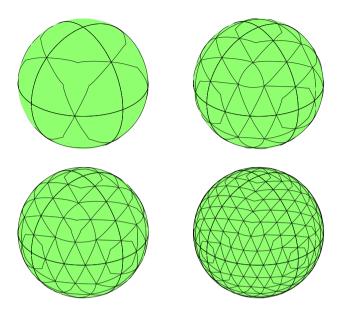
FIGURE 4. Grids on the sphere.

can restrict the calculations again to the region $\mathcal{Q}_1$, and the complete map U^{-1} is then derived by rotation.

With the notation

$$\delta = \frac{y\pi}{12\sqrt{3}x},$$

(excluding the case $x = 0$), from (3.8) and (3.9) we have

$$X^2 = \frac{12x^2}{\pi} \frac{(3\cos^2\delta + 2 - 2\sqrt{6}\cos\delta)}{(\sqrt{3} - \sqrt{2}\cos\delta)}, \qquad Y^2 = \frac{36x^2}{\pi} \frac{(\sin^2\delta)}{(\sqrt{3} - \sqrt{2}\cos\delta)}.$$

In particular, we obtain

$$
\begin{aligned}
X^2 + Y^2 &= \frac{12x^2}{\pi} \frac{(5 - 2\sqrt{6}\cos\delta)}{(\sqrt{3} - \sqrt{2}\cos\delta)}, \\
X^2 + \tfrac{1}{3}Y^2 &= \frac{12x^2}{\pi}(\sqrt{3} - \sqrt{2}\cos\delta).
\end{aligned}
\tag{6.1}
$$

We define

$$B := \frac{X^2 + \frac{1}{3}Y^2}{X^2 + Y^2} = \frac{(3 + 2\cos^2\delta - 2\sqrt{6}\cos\delta)}{(5 - 2\sqrt{6}\cos\delta)}$$

and derive the relation

$$\cos^2\delta + \sqrt{6}(B - 1)\cos\delta + \frac{(3 - 5B)}{2} = 0$$

with the solutions

$$(\cos\delta)_{1,2} = \frac{-\sqrt{3}(B-1) \pm \sqrt{B(3B-1)}}{\sqrt{2}}.$$

Since $B \in (\frac{1}{3}, 1)$, the only convenient solution is $\cos \delta = \frac{\sqrt{3}(1-B)+\sqrt{B(3B-1)}}{\sqrt{2}}$.
Then, from

$$1 - B = \frac{2Y^2}{3(X^2+Y^2)} \quad \text{and} \quad 3B - 1 = \frac{2X^2}{X^2+Y^2}$$

it follows that

$$\cos \delta = \frac{\sqrt{2}Y^2+X\sqrt{3X^2+Y^2}}{\sqrt{3}(X^2+Y^2)}. \tag{6.2}$$

For simplicity, we introduce the notation $w := Y/X$. Since $(X, Y) \in \mathcal{Q}_1$, we have $w \in [-\sqrt{3}, \sqrt{3}]$ and

$$\delta = \arccos \left(\frac{\sqrt{2}w^2+\sqrt{3+w^2}}{\sqrt{3}(1+w^2)} \right) = \arccos \left(\frac{\sqrt{2}w^2}{\sqrt{3}(w^2+1)} + \frac{\sqrt{w^2+3}}{\sqrt{3}(w^2+1)} \right).$$

Now we use the identity

$$\arccos(ab + \sqrt{1-a^2}\sqrt{1-b^2}) = \begin{cases} \arccos a - \arccos b & \text{for } a < b, \\ \arccos b - \arccos a & \text{for } a \geq b, \end{cases}$$

where $a = \frac{\sqrt{2}w}{\sqrt{3(1+w^2)}}$ and $b = \frac{w}{\sqrt{1+w^2}}$, and we obtain

$$\delta = \frac{\pi}{12\sqrt{3}} \frac{y}{x} = \begin{cases} \arccos \left(\frac{\sqrt{2}w}{\sqrt{3(1+w^2)}} \right) - \arccos \left(\frac{w}{\sqrt{1+w^2}} \right) & \text{for } w \in (0, \sqrt{3}], \\ -\arccos \left(\frac{\sqrt{2}w}{\sqrt{3(1+w^2)}} \right) + \arccos \left(\frac{w}{\sqrt{1+w^2}} \right) & \text{for } w \in [-\sqrt{3}, 0]. \end{cases}$$
$$\tag{6.3}$$

For the calculation of x from X and Y, we use the equalities in (6.1) and (6.2) yielding

$$\begin{aligned} x^2 &= \frac{\pi}{12} \frac{(X^2+\frac{1}{3}Y^2)}{(\sqrt{3}-\sqrt{2}\cos \delta)} \\ &= \frac{\pi}{12\sqrt{3}} \frac{(3X^2+Y^2)(X^2+Y^2)}{3(X^2+Y^2)-\sqrt{2}(\sqrt{2}Y^2+X\sqrt{3X^2+Y^2})} \\ &= \frac{\pi}{12\sqrt{3}} \frac{\sqrt{3X^2+Y^2}(X^2+Y^2)}{\sqrt{3X^2+Y^2}-\sqrt{2}X} \\ &= \frac{\pi}{12\sqrt{3}} \sqrt{3X^2+Y^2}(\sqrt{3X^2+Y^2}+\sqrt{2}X). \end{aligned}$$

Finally, from (6.3) we find $y = |y|(\text{sign } Y)$ with

$$\begin{aligned} |y| &= \frac{12\sqrt{3}}{\pi}|x||\delta| = \frac{12\sqrt{3}}{\pi}|x| \left| \arccos \left(\frac{\sqrt{2}w^2}{\sqrt{3}(w^2+1)} + \frac{\sqrt{w^2+3}}{\sqrt{3}(w^2+1)} \right) \right| \\ &= \frac{12\sqrt{3}}{\pi}|x| \left| \arccos \frac{\sqrt{2}Y}{\sqrt{3(X^2+Y^2)}} - \arccos \frac{Y}{\sqrt{X^2+Y^2}} \right|. \end{aligned}$$

The obtained formulas for x and y form an explicit representation of the inverse map U^{-1} in the region $\mathcal{Q}_1$. For the other two regions, the map is directly obtained using the rotation matrix $\mathcal{A}$ in Section 3.

Acknowledgement

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