Ambiguities in One-dimensional Discrete Phase Retrieval from Fourier Magnitudes

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International Conference on Computational Harmonic Analysis, Shanghai

May 24, 2017

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Outline of the talk

- Phase retrieval: Formulation of the problem
- Trivial and non-trivial ambiguities
- Characterization of solutions
- Ensuring uniqueness (nonnegativity, additional moduli, interference)
- Reconstruction of sparse signals by Prony's method

Problem (Phase retrieval)

Recover the unknown complex-valued signal

$$x := (x[n])_{n \in \mathbb{Z}}$$

with finite support from the FOURIER intensity

$$|\hat{x}(\omega)| \qquad (\omega \in \mathbb{R}).$$

Definition (Discrete-time FOURIER transform)

$$\hat{x}(\omega) := \sum_{n \in \mathbb{Z}} x[n] \, \mathrm{e}^{-\mathrm{i}\omega n} \qquad (\omega \in \mathbb{R})$$

Example

Let x be a complex-valued signal. Then

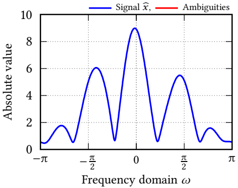
- the rotated signal
$$(y[n]) := \left(\mathrm{e}^{\mathrm{i}\alpha}\, x[n]\right),$$
- the shifted signal
$$(y[n]) := (x[n - n_0]),$$
- the reflected, conjugated signal
$$(y[n]) := \left(\overline{x[-n]}\right)$$

have the same Fourier intensity $|\hat{x}|$.

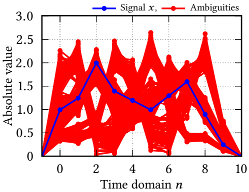
Definition (Trivial and non-trivial ambiguities)

A *trivial ambiguity* is caused by rotation, shift, or reflection and conjugation. All other occurring ambiguities are called *non-trivial.*

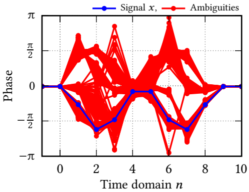
Example



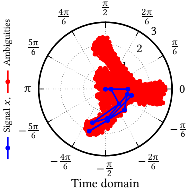
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Definition (Autocorrelation signal)

$$a[n] := \sum_{k\in\mathbb{Z}} \overline{x[k]}\, x[k+n] \qquad (n \in \mathbb{Z}).$$

- The autocorrelation signal is conjugate symmetric, i.e.

$$\overline{a[-n]} = \sum_{k\in\mathbb{Z}} x[k]\, \overline{x[k-n]} = \sum_{k\in\mathbb{Z}} x[k+n]\, \overline{x[k]} = a[n] \quad (n \in \mathbb{Z}).$$

Definition (Autocorrelation function)

$$A(\omega) := \sum_{n\in\mathbb{Z}} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n} = \sum_{n=-N+1}^{N-1} a[n]\, \mathrm{e}^{-\mathrm{i}\omega n}.$$

- The autocorrelation function is a non-negative trigonometric polynomial of degree $N-1$.

Phase retrieval in the frequency domain

- Relationship to the Fourier transform:

$$|\widehat{x}(\omega)|^2 = \Big(\sum_{n\in\mathbb{Z}} x[n]\,\mathrm{e}^{-\mathrm{i}\omega n}\Big)\Big(\sum_{k\in\mathbb{Z}} \overline{x[k]}\,\mathrm{e}^{\mathrm{i}\omega k}\Big)$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x[k+n]\,\overline{x[k]}\,\mathrm{e}^{-\mathrm{i}\omega n} = A(\omega).$$

Equivalent problem

Find a trigonometric polynomial B such that

$$|B(\omega)|^2 = A(\omega).$$

Über trigonometrische Polynome.

Von Herrn *Leopold Fejér* in Budapest.

Einleitung.

Herr *C. Carathéodory* hat in neuerer Zeit bekanntlich wichtige Beziehungen zwischen den Koeffizienten einer unendlichen *Fourier*schen Reihe (oder besser: einer unendlichen harmonischen Entwickelung) und dem Wertevorrat der dargestellten Funktion aufgedeckt.

Neben den scharfen analytischen Lösungen der verschiedenen hierhergehörigen Fragen durch Herrn *Carathéodory* sind dann besonders wesentlich die sehr eleganten algebraischen Lösungen des Herrn *O. Toeplitz.*

Der Inhalt vorliegender Arbeit gehört zu dem eben berührten Gedankenkreise. Es werden in ihr die elementarsten, zu diesem Gebiete gehörigen Fragen erörtert.

Insbesondere ist das Ziel dieser Arbeit darzulegen, wie gewisse Sätze über *unendliche* trigonometrische Entwickelungen im Falle einer *endlichen* trigonometrischen Entwickelung, d. h. im Falle eines trigonometrischen Polynoms gegebener Ordnung *präzisiert* werden können. Im Anhange wird gezeigt, daß gewisse *Tschebyscheff*sche Sätze auch eigentlich hierhergehören.

Die ganz elementare, algebraische Grundlage, die für die Lösung meiner in den §§ 3, 4 behandelten Probleme geeignet ist, und welche in den §§ 1, 2 dieser Arbeit auseinandergesetzt ist, habe ich schon im Jahre 1910 gefunden. Ich habe sogleich verschiedene Anwendungen abgeleitet, aber es fehlte mir noch der Beweis eines wichtigen Umkehrungstheorems. Dieser Beweis wurde durch Herrn *Friedrich Riesz* in einfacher

Definition (Associated polynomial)

$$P_A\left(e^{-i\omega}\right) = e^{-i\omega(N-1)} A(\omega)$$

- The algebraic polynomial P_A is thus defined by

$$P_A(z) := \sum_{n=0}^{2N-2} a[n-N+1]\, z^n \qquad \text{with} \qquad a[-n] = \overline{a[n]}.$$

- Obviously, we have

$$A(\omega) = \left| P_A\left(e^{-i\omega}\right) \right|.$$

- Factorize A with respect to the roots of the polynomial P_A.

- If γ is a root of P_A, then we have

$$P_A(\overline{\gamma}^{-1}) = \sum_{n=0}^{2N-2} a[n-N+1]\,\overline{\gamma}^{-n}$$
$$= \overline{\gamma}^{-2N+2} \sum_{n=0}^{2N-2} \overline{a[N-1-n]}\,\overline{\gamma}^{2N-2-n}$$
$$= \overline{\gamma}^{-2N+2} \sum_{n=0}^{2N-2} \overline{a[n-N+1]}\,\overline{\gamma}^{n}$$
$$= \overline{\gamma}^{-2N+2}\, \overline{P_A(\gamma)}$$

- P_A has the factorization

$$P_A(z) = a[N-1] \prod_{j=1}^{N-1} (z-\gamma_j)(z-\overline{\gamma}_j^{-1}).$$

- For $z := \mathrm{e}^{-\mathrm{i}\omega}$, the absolute value of the linear factors is

$$\left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right)\right| = \left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|\left|\overline{\gamma}_j^{-1}\right|\left|\overline{\gamma}_j - \mathrm{e}^{\mathrm{i}\omega}\right| = \left|\gamma_j\right|^{-1}\left|\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right|^2.$$

- A has the factorization

$$A(\omega) = \left|P_A\left(\mathrm{e}^{-\mathrm{i}\omega}\right)\right| = |a[N-1]| \prod_{j=1}^{N-1} \left|\left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)\left(\mathrm{e}^{-\mathrm{i}\omega} - \overline{\gamma}_j^{-1}\right)\right|$$

$$= |a[N-1]| \prod_{j=1}^{N-1} \left|\beta_j\right|^{-1} \left|\prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right)\right|^2 = |B(\omega)|^2$$

with $\beta_j \in (\gamma_j, \overline{\gamma}_j^{-1})$.

Theorem (Beinert, Plonka [2015])

Let A be a non-negative trigonometric polynomial. Then the problem

$$|B(\omega)|^2 = A(\omega)$$

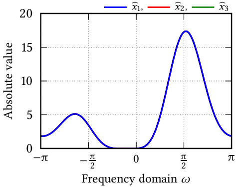
has at least one solution. Every solution has a representation of the form

$$B(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right),$$

where β_j can be chosen from the zero pair $(\gamma_j, \overline{\gamma}_j^{-1})$ of the associated polynomial P_A.

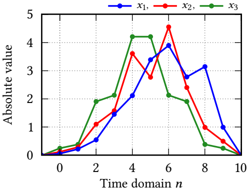
- Each non-trivial solution is completely determined by its *corresponding zero set* $\{\beta_1, \dots, \beta_{N-1}\}$.

Example

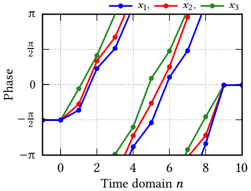


Fourier intensity: $|\widehat{x}(\omega)|$

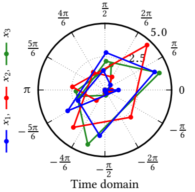
$$|\widehat{x}(\omega)|^2 = \left|\left(e^{-i\omega} + \tfrac{1}{2}\right)\left(e^{-i\omega} + 2\right)\right|^4 \cdot \left|e^{-i\omega} + e^{i\frac{\pi}{10}}\right|^{10}$$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

Corollary

The number of non-trivial ambiguities may vary from 1 *up to* 2^{N-2}.

Proposition (Beinert, Plonka [2015])

Let L *be the number of distinct zero pairs* $(\gamma_\ell, \overline{\gamma}_\ell^{-1})$ *of* P_A *not lying on the unit circle, and let* m_ℓ *be the multiplicity of these zero pairs. The corresponding phase retrieval problem has*

$$\left\lceil \frac{1}{2} \prod_{\ell=1}^{L} (m_\ell + 1) \right\rceil$$

non-trivial ambiguities.

Representation of the ambiguities in time domain

Definition (Convolution of signals)

$$(x_1 * x_2)[n] := \sum_{k \in \mathbb{Z}} x_1[k]\, x_2[n-k].$$

Theorem (Beinert, Plonka [2015])

Let x be a signal with finite support and factorization

$$x = x_1 * x_2.$$

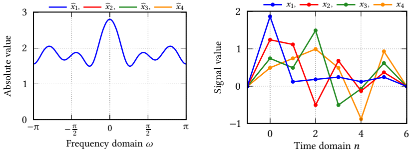
Then the signal

$$y := \mathrm{e}^{\mathrm{i}\alpha} \left(\overline{x_1[-\cdot]}\right) * (x_2[\cdot - n_0])$$

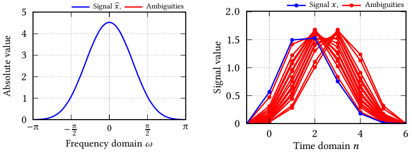
has the same Fourier intensity $|\widehat{x}|$ and all signals with the Fourier intensity $|\widehat{x}|$ can be represented in this manner.

Ensuring uniqueness

Example



Unique non-negative solution



Full non-negative solution set

- The solutions of the phase retrieval problem have the form

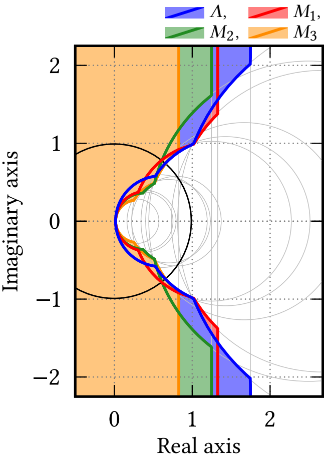
$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\omega n_0} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right).$$

- A solution is non-negative if and only if all coefficients of

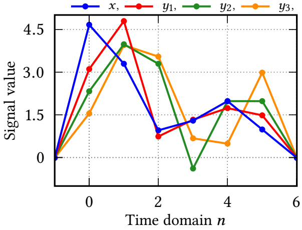
$$Q(z) := \prod_{j=1}^{N-1} (z - \beta_j)$$

are non-negative.

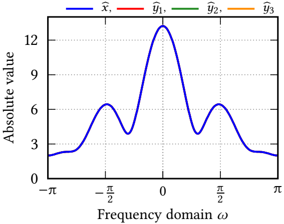
Example



Restriction on $(\beta_{N-2}, \beta_{N-1})$



Signals with $\beta_{N-1} = 3/4 + \mathrm{i}$



Fourier intensity for $\beta_{N-1} = 3/4 + \mathrm{i}$

Phase retrieval of non-negative signals

Theorem (Beinert [2015])

The sets of non-negative signals with support length $N > 3$ that

- *can be recovered uniquely up to reflection*
- *cannot be recovered uniquely up to reflection*

from their Fourier intensities are unbounded sets of infinite Lebesgue measure.

Knowledge of additional moduli

- Recover x from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an ℓ, where x has the support $\{0,\dots,N-1\}$.
- Assume that there exist two non-trivial solutions x and $\widetilde{x}$.
- The Fourier transform of x can be written as

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\alpha} \sqrt{|a[N-1]| \prod_{j=1}^{N-1} |\beta_j|^{-1}} \cdot \prod_{j=1}^{N-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \beta_j\right),$$

  and $\widehat{\widetilde{x}}$ has an analogous representation with $\widetilde{\beta}_j \in \{\beta_j, \overline{\beta}_j^{-1}\}$
- For $|x[N-1-\ell]| = |\widetilde{x}[N-1-\ell]|$, Vieta's formulae yield the condition

$$\prod_{j=1}^{N-1} |\beta_j|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \dots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right| = \prod_{j=1}^{N-1} \left|\widetilde{\beta}_j\right|^{-\frac{1}{2}} \cdot \left| \sum_{1 \le k_1 < \dots < k_\ell \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_\ell} \right|.$$

Knowledge of additional moduli

- Under the assumption $\widetilde{\beta}_j = \overline{\beta}_j^{-1}$ $(j = 1, \ldots, J)$ and $\widetilde{\beta}_j = \beta_j$ otherwise, we obtain

$$\left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \beta_{k_1} \cdots \beta_{k_\ell} \right|^2 = \prod_{j=1}^{J} \left| \overline{\beta}_j \right| \cdot \left| \sum_{1 \le k_1 < \cdots < k_\ell \le N-1} \widetilde{\beta}_{k_1} \cdots \widetilde{\beta}_{k_\ell} \right|^2 .$$

- Identify $\{\beta_1, \ldots, \beta_{N-1}\}$ with the real vector

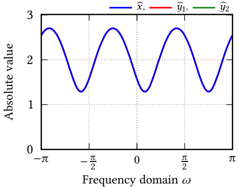
$$(\Re\beta_1, \Im\beta_1, \ldots, \Re\beta_{N-1}, \Im\beta_{N-1})^{\mathrm{T}},$$

and consider the polynomial equation above.

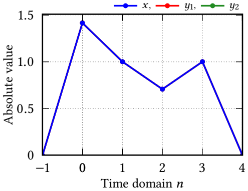
Theorem (Beinert, Plonka [2015])

Almost every signal x can be recovered from $|\widehat{x}|$ and $|x[N-1-\ell]|$ for an arbitrary $\ell \neq {}^{(N-1)}/{}_2$ up to rotations, for $\ell = {}^{(N-1)}/{}_2$ up to reflection/conjugation and rotation.

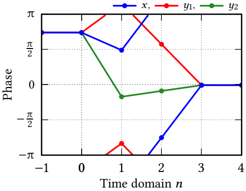
Example



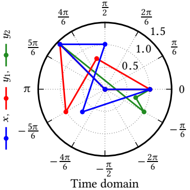
Fourier intensity: $|\widehat{x}(\omega)|$



Absolute value: $|x[n]|$



Phase: $\arg(x[n])$



Polar representation: $x[n]$

- Recover x with the fixed support $\{0, \ldots, N-1\}$ from $|\hat{x}|$, $\arg x[N-1-\ell_1]$, and $\arg x[N-1-\ell_2]$ for $\ell_1 \neq \ell_2$, where x has the support $\{0, \ldots, N-1\}$.
- Assume that there exist two non-trivial solutions x and $\tilde{x}$.
- Rewrite phase conditions into the equation

$$\Re\Big[S_{\ell_1}(B)\Big]\Im\Big[\overline{S_{\ell_2}(\widetilde{B})}\,S_{\ell_2}(B)\,S_{\ell_1}(\widetilde{B})\Big] - \Im\Big[S_{\ell_1}(B)\Big]\Re\Big[\overline{S_{\ell_2}(\widetilde{B})}\,S_{\ell_2}(B)\,S_{\ell_1}(\widetilde{B})\Big] = 0$$

with

$$B := \{\beta_1, \ldots, \beta_{N-1}\} \qquad \text{and} \qquad \widetilde{B} := \{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_{N-1}\}.$$

Theorem (Beinert [2015])

Let ℓ_1 and ℓ_2 two different integers in $\{0, \ldots, N-1\}$. Then almost every signal x can be uniquely recovered from $|\widehat{x}|$ and the phases

$$\arg x[N-1-\ell_1] \quad \textit{and} \quad \arg x[N-1-\ell_2] \qquad (\ell_1 + \ell_2 \neq N-1).$$

For $\ell_1 + \ell_2 = N - 1$, the recovery of the unknown signal is only unique up to reflection/conjugation, except for the case where the phase of both end points is given.

Theorem (Beinert, Plonka [2015])

Let x and h be complex-valued signals with finite support, and assume that the factorization of their symbols

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\, x[N_1 - 1] \prod_{j=1}^{N_1-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \eta_j\right)$$

and

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\, h[N_2 - 1] \prod_{j=1}^{N_2-1} \left(\mathrm{e}^{-\mathrm{i}\omega} - \gamma_j\right)$$

have no common zeros. Then x and h can be uniquely recovered from $|\widehat{x}(\omega)|$, $|\widehat{h}(\omega)|$ and $|\widehat{x}(\omega) + \widehat{h}(\omega)|$ up to common trivial ambiguities.

Sketch of proof

- Assume there are two solutions $x[n]$, $h[n]$ and $\widetilde{x}[n]$, $\widetilde{h}[n]$.
- Use the factorization in the frequency domain:

$$\widehat{x}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_1}\,\widehat{x}_1(\omega)\,\widehat{x}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{x}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\omega k_1}\,\widehat{x}_1(\omega)\,\overline{\widehat{x}_2(\omega)},$$

$$\widehat{h}(\omega) = \mathrm{e}^{\mathrm{i}\omega n_2}\,\widehat{h}_1(\omega)\,\widehat{h}_2(\omega) \quad \text{and} \quad \widehat{\widetilde{h}}(\omega) = \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\omega k_2}\,\widehat{h}_1(\omega)\,\overline{\widehat{h}_2(\omega)}.$$

- Consider the identity

$$\left|\widehat{x}(\omega) + \widehat{h}(\omega)\right|^2 = \left|\widehat{\widetilde{x}}(\omega) + \widehat{\widetilde{h}}(\omega)\right|^2.$$

Theorem (Beinert [2015])

Let x be a discrete-time signal with finite support of length N. If $\mu \neq \frac{2\pi p}{q}$ for all $p \in \mathbb{Z}$ and $q \in \{1, \ldots, N-1\}$, then the signal x can be uniquely recovered up to a rotation from its Fourier intensity $|\widehat{x}|$ and the two interference measurements

$$\left|\mathcal{F}(x + \mathrm{e}^{\mathrm{i}\alpha_1}\mathrm{e}^{\mathrm{i}\mu\cdot}x)\right| \qquad \text{and} \qquad \left|\mathcal{F}(x + \mathrm{e}^{\mathrm{i}\alpha_2}\mathrm{e}^{\mathrm{i}\mu\cdot}x)\right|,$$

where α_1 and α_2 are two real numbers satisfying $\alpha_1 - \alpha_2 \neq \pi k$ for all integer k.

Phase retrieval for sparse spike functions

- Let the unknown signal f be of the form

$$f(t) := \sum_{j=1}^{N} c_j^{(0)}\, \delta(t - T_j) \quad \text{with} \quad \widehat{f}(\omega) = \sum_{j=1}^{N} c_j^{(0)}\, \mathrm{e}^{-\mathrm{i}\omega T_j},$$

where δ denotes the Dirac delta function.

- The (distributional, squared) Fourier intensity of f is given by

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} c_j^{(0)}\, \overline{c}_k^{(0)}\, \mathrm{e}^{-\mathrm{i}\omega(T_j - T_k)} \qquad (\omega \in \mathbb{R}).$$

- Assuming that the differences $T_j - T_k$ with $j \neq k$ are pairwise distinct, we can write the (squared) Fourier intensity as exponential sum

$$\left|\widehat{f}(\omega)\right|^2 = \sum_{\ell=-N(N-1)/2}^{N(N-1)/2} \gamma_\ell\, \mathrm{e}^{-\mathrm{i}\omega\tau_\ell}$$

with $\tau_{-\ell} = -\tau_\ell$ and $\gamma_{-\ell} = \overline{\gamma}_\ell$.

Theorem (Beinert, Plonka [2017])

Let f be a spike function. If

- *the knot differences $T_j - T_k$ differ pairwise for $j, k \in \{1, \dots, N\}, j \neq k$*
- *the coefficients satisfy $|c_1^{(0)}| \neq |c_N^{(0)}|$*
- *the step size $h > 0$ fulfils $h(T_j - T_k) \in (-\pi, \pi)$ for all j, k*

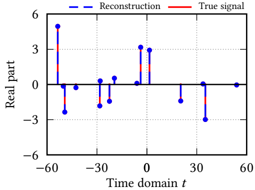
then f can be uniquely recovered from its Fourier intensities $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \dots, 3/2\, N(N-1)$ up to trivial ambiguities.

Corollary

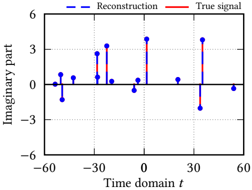
Almost all spike functions f can be uniquely recovered from their Fourier intensities $|\mathcal{F}[f]|$ up to trivial ambiguities.

Numerical example (sparse spike function)

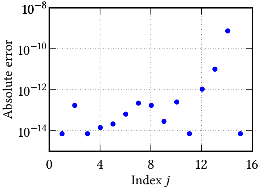
Example



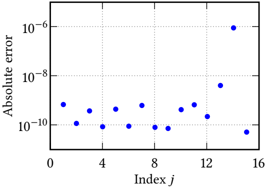
Real part: $\Re f(t)$



Imaginary part: $\Im f(t)$



Absolute error of the knots



Absolute error of the coefficients

Recovery of f from $|\mathcal{F}[f](h\ell)|$ with $\ell = 0, \ldots, 1000$.

Summary/Outlook

- Characterization of the ambiguities in the one-dimensional discrete-time phase retrieval problem.
- Investigation of the quality of different a priori conditions and additional data.

- Phase retrieval in higher dimensions.
- Transferring further results between the discrete-time and continuous-time problem.
- Investigation and development of numerical algorithms.

Thank you for the attention.

- R. Beinert, G. Plonka. *Sparse phase retrieval of one-dimensional signals by Prony's method*, Frontiers in Applied Mathematics and Statistics 3(5), 2017.
- R. Beinert. *Non-negativity constraints in the one-dimensional discrete-time phase retrieval problem.* Information and Inference: A Journal of the IMA, online first (2016), to appear.
- R. Beinert. *One-dimensional phase retrieval with additional interference measurements.* Results in Mathematics, online first (2016), to appear.
- R. Beinert, G. Plonka. *Enforcing uniqueness in one-dimensional phase retrieval by additional signal information in time domain.* Applied and Computational Harmonic Analysis, 2017, online first.
- R. Beinert, G. Plonka. Ambiguities in one-dimensional phase retrieval of structured functions. *PAMM, Proc. Appl. Math. Mech.* **15** (2015), No. 1, pp. 653–654.
- R. Beinert, G. Plonka. Ambiguities in one-dimensional discrete phase retrieval from Fourier magnitudes. *J. Fourier Anal. Appl.* **21** (2015), No. 6, pp. 1169–1198.