

Reconstruction of non-stationary signals by the generalized Prony method

Gerlind Plonka

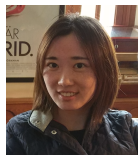
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- Original Prony method: Reconstruction of Sparse Exponential Sums
- Prony's Method Based on the Shift Operator
- Prony's Method Based on the Differential Operator
- Generalized Operator Based Prony Method

Collaborations



Thomas Peter, Manfred Tasche, Marius Wischerhoff, Ingeborg Keller, Kilian Stampfer, Ran Zhang

Prony Method: Reconstruction of Sparse Exponential Sums

Function $f(x) = \sum_{j=1}^M c_j e^{\alpha_j x}$

We have $M, f(\ell), \ell = 0, \dots, 2M - 1$

We want $c_j, \alpha_j \in \mathbb{C}$, where $-\pi \leq \operatorname{Im} \alpha_j < \pi, j = 1, \dots, M$.

Prony Method: Reconstruction of Sparse Exponential Sums

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Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^M (z - e^{\alpha_j}) = \sum_{\ell=0}^M p_\ell z^\ell$$

with unknown parameters α_j and $p_M = 1$.

$$\begin{aligned} \sum_{\ell=0}^M p_\ell f(\ell + m) &= \sum_{\ell=0}^M p_\ell \sum_{j=1}^M c_j e^{\alpha_j(\ell+m)} = \sum_{j=1}^M c_j e^{\alpha_j m} \sum_{\ell=0}^M p_\ell e^{\alpha_j \ell} \\ &= \sum_{j=1}^M c_j e^{\alpha_j m} P(e^{\alpha_j}) = 0, \quad m = 0, \dots, M - 1. \end{aligned}$$

Reconstruction Algorithm

Input: $f(\ell)$, $\ell = 0, \dots, 2M - 1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \dots & f(M-1) \\ f(1) & f(2) & \dots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \dots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^M p_{\ell} z^{\ell}$ and extract the parameters α_j from its zeros $z_j = e^{\alpha_j}$, $j = 1, \dots, M$.
- Compute c_j solving the linear system

$$f(\ell) = \sum_{j=1}^M c_j e^{\alpha_j \ell}, \quad \ell = 0, \dots, 2M - 1.$$

Output: Parameters α_j and c_j , $j = 1, \dots, M$.

(Almost) Equivalent Models

If we can reconstruct

$$f(x) = \sum_{j=1}^M c_j e^{\alpha_j x},$$

then we can also reconstruct

$$g(t) = \sum_{j=1}^M c_j \delta(t - t_j) \quad \Rightarrow \quad \widehat{g}(x) = \sum_{j=1}^M c_j e^{-it_j x}$$

$$g(t) = \sum_{j=1}^M c_j \phi(t - t_j) \quad \Rightarrow \quad \widehat{g}(x) = \left(\sum_{j=1}^M c_j e^{-it_j x} \right) \widehat{\phi}(x)$$

$$g(t) = \sum_{j=1}^M \frac{c_j}{t - \alpha_j} \quad \Rightarrow \quad \mathcal{L}^{-1}(g)(x) = \sum_{j=1}^M c_j e^{\alpha_j x}$$

Literature

[Prony] (1795):

[Schmidt] (1979):

[Roy, Kailath] (1989):

[Hua, Sakar] (1990):

[Stoica, Moses] (2000):

[Vetterli, Marziliano, Blu (2002):

[Potts, Tasche] (2010, 2011):

[Peter, Plonka] (2013):

Reconstruction of difference equations

MUSIC (Multiple Signal Classification)

ESPRIT (Estimation of signal parameters
via rotational invariance techniques)

Matrix-pencil method

Annihilating filters

Finite rate of innovation signals

Approximate Prony method

Generalized Prony Method

Sidi ('75,'82,'85); Golub, Milanfar, Varah ('99); Maravić, Vetterli ('04);

Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10);

Andersson, Carlsson, de Hoop ('10), Berent, Dragotti, Blu ('10),

Batenkov, Sarg, Yomdin ('12,'13); Filbir, Mhaskar, Prestin ('12);

Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13);

Plonka, Tasche ('14); Kunis, Peter, Römer, von der Ohe ('16);

Wei, Dragotti ('16); Sauer ('17); Cuyt, Lee ('17), Mourrain ('17), ...

Very incomplete list !!!

Prony's Method for sparse approximation

Let $\mathbf{y} = (y_k)_{k=0}^L \in \mathbb{C}^{L+1}$ be a given.

Goal: Approximate \mathbf{y} by $\mathbf{f} = (f_k)_{k=0}^L \in \mathbb{C}^{L+1}$ where

$$f_k = \sum_{j=1}^M c_j e^{\alpha_j k} = \sum_{j=1}^M c_j z_j^k, \quad k = 0, \dots, L, \quad M \leq L/2$$

with $c_j, z_j = e^{\alpha_j} \in \mathbb{C}, j = 1, \dots, M$, such that

$$\|\mathbf{y} - \mathbf{f}\|_2^2 = \sum_{k=0}^L |y_k - f_k|^2$$

is minimal.

Survey: Modification of Prony's method (Zhang & Plonka '19)

Towards a Generalization of Prony's Method

Revisiting Prony's Method Using the Shift Operator

Let $S_h f := f(\cdot + h)$, $h \in \mathbb{R} \setminus \{0\}$. Then

$$(S_h e^{\alpha \cdot})(x) = e^{\alpha(h+x)} = e^{\alpha h} e^{\alpha x} \quad (\text{eigenfunction}).$$

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For

$$f(x) = \sum_{j=1}^M c_j e^{\alpha_j x} \quad \text{with} \quad P(z) := \prod_{j=1}^M (z - e^{\alpha_j h}) = \sum_{\ell=0}^M p_\ell z^\ell$$

we have

$$P(S_h)f = \sum_{\ell=0}^M p_\ell (S_h^\ell f) = \sum_{\ell=0}^M p_\ell S_{h\ell} \sum_{j=1}^M c_j e^{\alpha_j \cdot} = \sum_{\ell=0}^M p_\ell \sum_{j=1}^M c_j S_{h\ell} e^{\alpha_j \cdot}$$

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Revisiting Prony's Method Using the Shift Operator

Thus, $f(x) = \sum_{j=1}^M c_j e^{\alpha_j x}$ solves the difference equation $P(S_h)f = 0$.

Moreover,

$$S_h^k P(S_h)f = P(S_h)S_h^k f = \sum_{\ell=0}^M p_\ell S_h^{\ell+k} f = 0, \quad k \in \mathbb{Z}.$$

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With the point evaluation functional $Gf = G_0f := f(0)$,

$$G_0(S_h^k P(S_h)f) = \sum_{\ell=0}^M p_\ell G_0(S_h^{\ell+k} f) = \sum_{\ell=0}^M p_\ell f(h(\ell+k)) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients p_ℓ of the Prony polynomial from M of these equations, i.e., using $f(hk)$, $k = 0, \dots, 2M - 1$.

Change the Sampling Scheme

Thus, $f(x) = \sum_{j=1}^M c_j e^{\alpha_j x}$ solves the difference equation $P(S_h)f = 0$.

Moreover, for each linear operator $T : C^\infty(\mathbb{R}) \mapsto C^\infty(\mathbb{R})$

$$T^k P(S_h)f = T^k P(S_h)f = \sum_{\ell=0}^M p_\ell T^k S_h^\ell f = 0, \quad k \in \mathbb{Z}.$$

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With the linear functional $G : C^\infty(\mathbb{R}) \mapsto \mathbb{C}$

$$G(T^k P(S_h)f) = \sum_{\ell=0}^M p_\ell G(T^k S_h^\ell f) = \sum_{\ell=0}^M p_\ell G(T^k f(h(\ell+\cdot))) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients p_ℓ of the Prony polynomial from M of these equations, i.e., using $G(T^k S_h^\ell f)$, $\ell = 0, \dots, M$, $k = 1, \dots, M$.

Revisiting Prony's Method Using the Shift Operator

Example: Choose $h = 1$, $T = S_{h/2} = S_{1/2}$, $G = G_0$, then the linear system reads (for even M)

$$\begin{pmatrix} f(0) & f(1) & \dots & f(M-1) \\ f(\frac{1}{2}) & f(\frac{3}{2}) & \dots & f(\frac{M-1}{2}) \\ f(1) & f(2) & \dots & f(M) \\ \vdots & \vdots & & \vdots \\ f(\frac{M}{2}-1) & f(\frac{M}{2}) & \dots & f(\frac{3M}{2}-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+\frac{1}{2}) \\ f(M+1) \\ \vdots \\ f(\frac{3M}{2}-1) \end{pmatrix}$$

This matrix is not longer of Hankel form.

Revisiting Prony's Method Using the Differential Operator

Let $\frac{d}{dx} : C^\infty(\mathbb{R}) \mapsto C^\infty(\mathbb{R})$, $\frac{d}{dx} f := f'$. Then

$$\left(\frac{d}{dx} e^{\alpha \cdot}\right)(x) = \alpha e^{\alpha x} \quad (\text{eigenfunction}).$$

Revisiting Prony's Method Using the Differential Operator

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For

$$f(x) = \sum_{j=1}^M c_j e^{\alpha_j x} \quad \text{with} \quad \tilde{P}(z) := \prod_{j=1}^M (z - \alpha_j) = \sum_{\ell=0}^M \tilde{p}_\ell z^\ell$$

we have

$$\tilde{P}\left(\frac{d}{dx}\right)f = \sum_{\ell=0}^M \tilde{p}_\ell \left(\frac{d}{dx}\right)^\ell f = \sum_{\ell=0}^M \tilde{p}_\ell \left(\frac{d}{dx}\right)^\ell \sum_{j=1}^M c_j e^{\alpha_j \cdot} = \sum_{\ell=0}^M \tilde{p}_\ell \sum_{j=1}^M c_j \left(\frac{d}{dx}\right)^\ell e^{\alpha_j \cdot}$$

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we have

$$\begin{aligned} \tilde{P}\left(\frac{d}{dx}\right)f &= \sum_{\ell=0}^M \tilde{p}_\ell \left(\frac{d}{dx}\right)^\ell f = \sum_{\ell=0}^M \tilde{p}_\ell \left(\frac{d}{dx}\right)^\ell \sum_{j=1}^M c_j e^{\alpha_j \cdot} = \sum_{\ell=0}^M \tilde{p}_\ell \sum_{j=1}^M c_j \left(\frac{d}{dx}\right)^\ell e^{\alpha_j \cdot} \\ &= \sum_{\ell=0}^M \tilde{p}_\ell \sum_{j=1}^M c_j \alpha_j^\ell e^{\alpha_j \cdot} = \sum_{j=1}^M c_j e^{\alpha_j \cdot} \sum_{\ell=0}^M \tilde{p}_\ell \alpha_j^\ell = 0. \end{aligned}$$

Revisiting Prony's Method Using the Differential Operator

Thus, $f(x) = \sum_{j=1}^M c_j e^{\alpha_j x}$ solves the differential equation $\tilde{P}\left(\frac{d}{dx}\right)f = 0$.

Moreover,

$$\left(\frac{d}{dx}\right)^k \tilde{P}\left(\frac{d}{dx}\right)f = \sum_{\ell=0}^M \tilde{p}_\ell \left(\frac{d}{dx}\right)^{\ell+k} f = 0, \quad k \in \mathbb{Z}.$$

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With the point evaluation functional $G_0 f := f(0)$,

$$G_0\left(\left(\frac{d}{dx}\right)^k \tilde{P}\left(\frac{d}{dx}\right)f\right) = \sum_{\ell=0}^M \tilde{p}_\ell G_0\left(\left(\frac{d}{dx}\right)^{\ell+k} f\right) = \sum_{\ell=0}^M \tilde{p}_\ell f^{(\ell+k)}(0) = 0, \quad k \in \mathbb{Z}.$$

We can compute the coefficients \tilde{p}_ℓ of the Prony polynomial from M of these equations, i.e., using $f^{(k)}(0)$, $k = 0, \dots, 2M - 1$.

Switch Between Operators with the Same Eigenfunctions

What is the connection between Prony's method using the shift operator or the differential operator?

We have

$$\frac{d}{dx}e^{\alpha \cdot} = \alpha e^{\alpha \cdot}, \quad S_h e^{\alpha \cdot} = e^{\alpha h} e^{\alpha \cdot}.$$

Obviously, the spectra are connected by the map $\exp(h \cdot) : \alpha \mapsto e^{h\alpha}$.

Switch Between Operators with the Same Eigenfunctions

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Obviously, the spectra are connected by the map $\exp(h \cdot) : \alpha \mapsto e^{h\alpha}$. Moreover, for all eigenfunctions $e^{\alpha x}$,

$$\begin{aligned} \exp\left(h \frac{d}{dx}\right) e^{\alpha x} &= \sum_{k=0}^{\infty} \frac{h^k}{k!} \left(\frac{d}{dx}\right)^k e^{\alpha x} = \sum_{k=0}^{\infty} \frac{h^k}{k!} \alpha^k e^{\alpha x} \\ &= e^{\alpha(x+h)} = S_h e^{\alpha x}. \end{aligned}$$

Thus,

$$\exp\left(h \frac{d}{dx}\right) f(x) = S_h f(x).$$

Generalized Prony method (Peter & Plonka '13)

V : normed vector space

$\mathcal{A} : V \rightarrow V$ linear operator

$\{v_\lambda : \lambda \in \sigma(\mathcal{A})\}$ set of eigenfunctions of \mathcal{A} to **pairwise different** eigenvalues $\lambda \in \sigma(\mathcal{A}) \subset \mathbb{C}$,

$$\mathcal{A} v_\lambda = \lambda v_\lambda.$$

Let

$$f = \sum_{\lambda \in \Lambda_f} c_\lambda v_\lambda, \quad \Lambda_f \subset \sigma(\mathcal{A}) \quad \text{with} \quad |\Lambda_f| = M, \quad c_\lambda \in \mathbb{C}.$$

Let $G : V \rightarrow \mathbb{C}$ be a linear functional with $G(v_\lambda) \neq 0$ for all $\lambda \in \sigma(\mathcal{A})$.

We have $M, G(\mathcal{A}^\ell f)$ for $\ell = 0, \dots, 2M - 1$

We want $\Lambda_f \subset \sigma(\mathcal{A}), c_\lambda \in \mathbb{C}$ for $\lambda \in \Lambda_f$

Generalized Prony Method

Theorem (Peter & Plonka '13)

The expansion

$$f = \sum_{\lambda \in \Lambda_f} c_\lambda v_\lambda, \quad \Lambda_f \subset \sigma(\mathcal{A}) \quad \text{with} \quad |\Lambda_f| = M, \quad c_\lambda \in \mathbb{C}.$$

of eigenfunctions v_λ of the linear operator \mathcal{A} can be uniquely recovered by $G(\mathcal{A}^\ell f)$, $\ell = 0, \dots, 2M - 1$, where $G : V \rightarrow \mathbb{C}$ is a linear functional with $G(v_\lambda) \neq 0$ for all $\lambda \in \sigma(\mathcal{A})$.

Generalized Prony Method

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Example: Let $V = C^\infty$, $\mathcal{A} := S_h$ with $S_h f = f(h + \cdot)$, and

$$f = \sum_{\lambda \in \Lambda_f} c_\lambda v_\lambda = \sum_{\lambda \in \Lambda_f} c_\lambda e^{\lambda \cdot},$$

where $\Lambda_f \subset \mathbb{R} + i[-\pi, \pi)$. Choose the functional $G_0 f := f(0)$, then f can be recovered from the samples

$$G_0(\mathcal{A}^\ell f) = S_h^\ell f(0) = f(h\ell), \quad \ell = 0, \dots, 2M - 1.$$

Generalized Operator Based Prony Method (Stampfer & Plonka '19)

Assume, you want to recover a sparse expansion

$$f = \sum_{j=1}^M c_j v_j, \quad v_j \in V, c_j \in \mathbb{C}.$$

Idea:

- Find a linear operator \mathcal{A} such that v_j are eigenfunctions of \mathcal{A} to pairwise different eigenvalues.
- Check, whether $G(\mathcal{A}^\ell f)$, $\ell = 0, \dots, 2M - 1$ can be computed from the given information.
- If not, transfer to a different operator $\mathcal{B} = \varphi(\mathcal{A})$ and suitable functionals G_k such that $G_k(\mathcal{B}^\ell f)$, $\ell = 0, \dots, M$, $k = 1, \dots, M$ that can be computed from the given information.
- Apply the generalized Prony method to recover v_j and c_j , $j = 1, \dots, M$.

Differential Operators of First Order and Generalized Shifts

Reconstruct

$$f(x) = \sum_{j=1}^M c_j e^{\alpha_j G(x)},$$

i.e., find $c_j, \alpha_j, j = 1, \dots, M$, where G is differentiable and strictly monotone on $[a, b]$.

Differential Operators of First Order and Generalized Shifts

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Find a linear operator with eigenfunctions $e^{\alpha_j G(x)}$:

Let $g(x) := 1/G'(x)$ and

$$\mathcal{A}f(x) := g(x) \frac{d}{dx} f(x).$$

Then

$$\mathcal{A}e^{\alpha G(x)} = g(x) \frac{d}{dx} e^{\alpha G(x)} = \alpha e^{\alpha G(x)}.$$

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Then

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Using the generalized Prony method, f can be recovered using

$$F\left(\left(g(\cdot) \frac{d}{dx}\right)^k f\right), \quad k = 0, \dots, 2M - 1.$$

However, these may be difficult to provide.

Change the operator:

$$\begin{aligned}\exp(\tau \mathcal{A})f(x) &= \exp\left(\tau g(x) \frac{d}{dx}\right) f(x) \\ &= \sum_{\ell=0}^{\infty} \frac{\tau^{\ell}}{\ell!} \left(g(x) \frac{d}{dx}\right)^{\ell} \left(\sum_{j=1}^M c_j e^{\alpha_j G(x)}\right) \\ &= \sum_{j=1}^M c_j \left(\sum_{\ell=0}^{\infty} \frac{\tau^{\ell}}{\ell!} \alpha_j^{\ell}\right) e^{\alpha_j G(x)} \\ &= \sum_{j=1}^M c_j e^{\alpha_j \tau} e^{\alpha_j G(x)} \\ &= \sum_{j=1}^M c_j e^{\alpha_j G(G^{-1}(\tau + G(x)))} = f(G^{-1}(\tau + G(x))).\end{aligned}$$

Theorem (Stampfer & Plonka '19)

Let

$$f(x) = \sum_{j=1}^M c_j e^{\alpha_j G(x)},$$

where $G(x)$ is continuous and monotone on an interval $[a, b]$. Let $\tau k + G(x_0) \in G([a, b])$ for $k = 0, \dots, 2M - 1$. Then $f(x)$ can be uniquely reconstructed from the function samples

$$f(G^{-1}(\tau k + G(x_0))), \quad k = 0, \dots, 2M - 1.$$

Differential Operators of First Order and Generalized Shifts

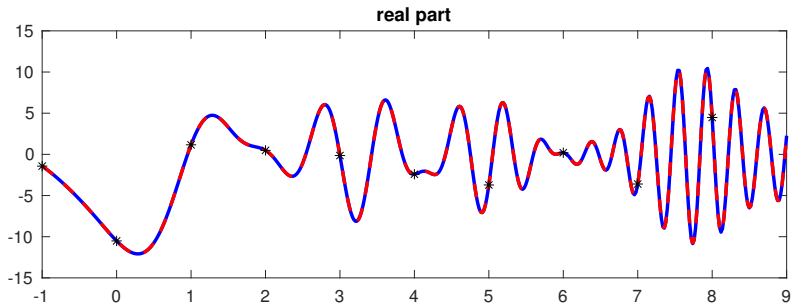
$g(x)$	$G(x)$	eigenfunctions	sampling values
$1/x$	$-\frac{1}{2}x^2$	$\exp(-\frac{\alpha}{2}x^2)$	$f(\sqrt{x_0 - k\tau})$
1	x	$\exp(\alpha x)$	$f(\tau k + x_0)$
x	$\log(x)$	x^α	$f(e^{\tau k} x_0)$
$-\sqrt{1-x^2}$	$\arccos x$	$\exp(\alpha \arccos x)$	$f(\cos(k\tau + \arccos(x_0)))$
$\sqrt{1-x^2}$	$\arcsin x$	$\exp(\alpha \arcsin x)$	$f(\sin(k\tau + \arcsin(x_0)))$
$\frac{1}{\cos(x)}$	$\sin x$	$\exp(\alpha \sin x)$	$f(\arcsin(k\tau + \sin(x_0)))$
$-\frac{1}{\sin(x)}$	$\cos x$	$\exp(\alpha \cos x)$	$f(\arccos(k\tau + \cos(x_0)))$

Examples of operators $A = g(\cdot) \frac{d}{dx}$, corresponding eigenfunctions $\exp(\alpha G(\cdot))$ and sampling values for $k = 0, \dots, 2M - 1$ with sampling parameter τ to recover expansions f .

Example: Recovery of shifts of Gaussians (Plonka, Stampfer, Keller '19)

$$f(x) = \sum_{j=1}^5 c_j e^{i(x-\alpha_j)^2}$$

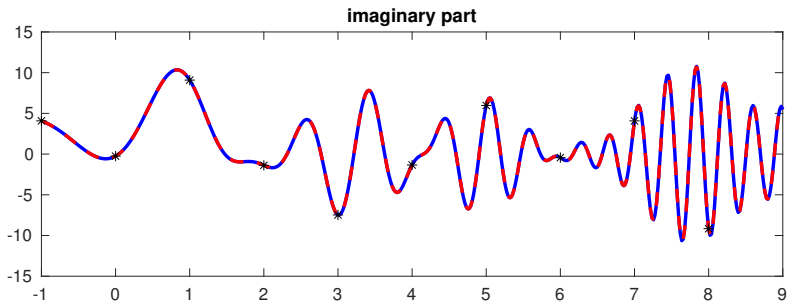
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
Re c_j	-2.37854	-4.55545	2.54933	-2.57214	-0.57597
Im c_j	0.75118	-0.56308	0.94536	0.42117	0.73366
α_j	0.64103	-0.18125	-1.50929	-0.53137	-0.23778



Example: Recovery of shifts of Gaussians

$$f(x) = \sum_{j=1}^5 c_j e^{i(x-\alpha_j)^2}$$

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\operatorname{Re} c_j$	-2.37854	-4.55545	2.54933	-2.57214	-0.57597
$\operatorname{Im} c_j$	0.75118	-0.56308	0.94536	0.42117	0.73366
α_j	0.64103	-0.18125	-1.50929	-0.53137	-0.23778



Summary

- Prony's method can be used in many contexts, since sparse representations can be often transformed to the form of exponential sums, e.g.,

$$g(t) = \sum_{j=1}^M c_j \phi(t - t_j) \quad \Rightarrow \quad \hat{g}(x) = \left(\sum_{j=1}^M c_j e^{-it_j x} \right) \hat{\phi}(x)$$

- For noisy samples, one should use the modified Prony method.
- Prony's method can be generalized to recover sparse expansions of eigenfunctions of linear operators.
- One can use different operators with the same (sub)set of eigenfunctions.
- One can employ the generalized Prony method to find new more general sampling schemes.



K. Stampfer, Gerlind Plonka: The generalized operator-based Prony method. preprint, arXiv:1901.08778.



R. Zhang, G. Plonka: Optimal approximation with exponential sums by maximum likelihood modification of Prony's method. *Adv. Comput. Math.*, **45**(3) (2019), 1657-1687.



G. Plonka, V. Pototskaia: Computation of adaptive Fourier series by sparse approximation of exponential sums. *J. Fourier Anal. Appl.* **25**(4) (2019), 1580-1608.



G. Plonka, K. Stampfer and I. Keller: Reconstruction of stationary and non-stationary signals by the generalized Prony method. *Anal. Appl.* **17**(2) (2019), 179-210.



R. Beinert, G. Plonka: Sparse phase retrieval of one-dimensional signals by Prony's method. *Frontiers Appl. Math. Statist.* **3**(5) (2017), open access, doi: 10.3389/fams.2017.00005.



M. Wischerhoff, G. Plonka: Reconstruction of polygonal shapes from sparse Fourier samples. *J. Comput. Appl. Math.* **297** (2016), 117-131.



G. Plonka, M. Tasche: Prony methods for recovery of structured functions. *GAMM-Mitt.* **37**(2) (2014) 239-258.



T. Peter, G. Plonka: A generalized Prony method for reconstruction of sparse sums of eigenfunctions of linear operators. *Inverse Problems* **29** (2013), 025001.



G. Plonka, M. Wischerhoff: How many Fourier samples are needed for real function reconstruction? *J. Appl. Math. Comput.* **42** (2013), 117-137.



T. Peter, G. Plonka, D. Rosca: Representation of sparse Legendre expansions. *J. Symbolic Comput.* **50** (2013), 159-169.