Refinement of Vectors of Bernstein Polynomials

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ABSTRACT

For the case of Bernstein polynomials, the refinement mask is calculated recursively, and the refinement matrices are given explicitely. Moreover, the eigenvectors of the transposed refinement matrices are constructed, whereas the eigenvectors of the refinement matrices themselves can be determined by a theorem of Micchelli and Prautzsch.

1. INTRODUCTION

Let $n \in \mathbb{N}$ and let $\boldsymbol{b}^n(t) := (b_0^n(t), \ldots, b_n^n(t))^T$ be a vector of *uniformly refinable* real functions on $[0,1]$, i.e., there are $(n+1) \times (n+1)$ matrices $\boldsymbol{A}_0^n, \ldots, \boldsymbol{A}_{k-1}^n$ ($k \in \mathbb{N}$; $k \geq 2$) such that

$$\boldsymbol{b}^n\left(\frac{t+m}{k}\right) = \boldsymbol{A}_m^n \, \boldsymbol{b}^n(t) \tag{1.1}$$

is satisfied for $m = 0, \ldots, k-1$ and $t \in [0,1]$. These equations are called *refinement equations*, and the matrices $\boldsymbol{A}_m^n$ *refinement matrices* (cf. Micchelli and Prautzsch [5]). It is well–known that the refinement equations (1.1) are closely connected with corresponding subdivision algorithms which provide important techniques for the fast generation of curves (cf. [3, 5]). In [6] and [8], some applications of such equations in the theory of wavelets are discussed.
For polynomials $b_i^n(t)$ $(i = 0, \ldots, n)$ spanning the vector space of all polynomials of degree n, the matrices $\boldsymbol{A}_m^n$ in (1.1) always exist and are uniquely determined.
Here, we consider these matrices in the case of Bernstein polynomials

$$b_i^n(t) = \binom{n}{i} t^i \, (1-t)^{n-i}, \qquad i = 0, \ldots, n, \tag{1.2}$$

and study their spectral properties. In particular, we prove a recursion formula for the *refinement mask* of $\boldsymbol{b}^n$

$$\boldsymbol{A}^n(z) := \frac{1}{k} \sum_{m=0}^{k-1} \boldsymbol{A}_m^n \, z^m, \quad z \in \mathbb{C} \tag{1.3}$$

Furthermore, we derive explicit formulas for the entries of the refinement matrices $\boldsymbol{A}_m^n$ as well as for their eigenvalues and corresponding eigenvectors. Note that for the special case $k = 2$, the corresponding subdivision algorithm is the de Casteljau algorithm (cf. [2]).

2. RECURSIVE COMPUTATION OF THE REFINEMENT MASK

First, we derive a simple recursion formula for the Fourier transform of the vector $\boldsymbol{b}^n(t)$ of Bernstein polynomials $b_i^n(t)$ ($t \in [0,1]$; $i = 0, \ldots, n$). For convenience, outside of $[0,1]$ the polynomials are defined by zero. Denoting the Fourier transform of a function $f \in L^2(\mathbb{R})$ by

$$\hat{f}(u) = \int_{\infty}^{\infty} f(t)\, e^{-iut} \;\mathrm{d}t, \quad u \in \mathbb{R},$$

we have:

Lemma 2.1 *For $n = 0$,*

$$\hat{\boldsymbol{b}}^0(u) = \hat{b}_0^0(u) = \frac{1 - e^{-iu}}{iu}. \tag{2.1}$$

For $n > 0$, the following recursion formula holds:

$$iu\, \hat{\boldsymbol{b}}^n(u) = \boldsymbol{C}_n(e^{-iu}) \begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix}, \qquad u \in \mathbb{R}, \tag{2.2}$$

where the matrix $\boldsymbol{C}_n(z)$ with $z \in \mathbb{C}$ is an $(n+1) \times (n+1)$-matrix of the form

$$\boldsymbol{C}_n(z) := n \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 & -1 \\ -z & 0 & \ldots & 0 & 1 \end{pmatrix}, \quad n \geq 1. \tag{2.3}$$

Proof. We put $b_{-1}^{n-1}(t) := \delta(t)/n$ and $b_n^{n-1}(t) := \delta(t-1)/n$ ($n \geq 1$), where δ denotes the Dirac distribution. Then the known formula for the derivative of Bernstein polynomials $b_i^n(t)$ ($i = 1, \ldots, n-1$)

$$\mathrm{D}b_i^n(t) = n(b_{i-1}^{n-1}(t) - b_i^{n-1}(t))$$

can also be used for $i = 0$ and $i = n$, in view of the jumps of $b_0^n(t)$ at $t = 0$, and $b_n^n(t)$ at $t = 1$. Hence, we obtain for the vector $\boldsymbol{b}^n(t)$

$$\mathrm{D}\boldsymbol{b}^n(t) = n \begin{pmatrix} b_{-1}^{n-1} \\ \boldsymbol{b}^{n-1}(t) \end{pmatrix} - n \begin{pmatrix} \boldsymbol{b}^{n-1}(t) \\ b_n^{n-1} \end{pmatrix}.$$

Taking the Fourier transform, we infer

$$iu\,\hat{\boldsymbol{b}}^n(u) = n\begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix} - n\begin{pmatrix} \hat{\boldsymbol{b}}^{n-1}(u) \\ e^{-iu}/n \end{pmatrix} = \boldsymbol{C}_n(e^{-iu})\begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix}. \quad \blacksquare$$

Remark 2.2 1. Note that

$$\det\, \boldsymbol{C}_n(z) = n^{n+1}\,(1-z).$$

2. The Bernstein polynomials $b_i^n(t)$ $(i = 0, \ldots, n)$ on $[0, 1]$ can also be considered as B–splines defined by the multiple knots

$$\underbrace{0, \ldots, 0}_{n-i+1}, \underbrace{1, \ldots, 1}_{i+1}.$$

Generalizing this definition, we find that b_{-1}^n is determined by the knots $\underbrace{0, \ldots, 0}_{n+2}$ and analogously b_{n+1}^n by $\underbrace{1, \ldots, 1}_{n+2}$. Thus, the above definition of b_{-1}^n and b_{n+1}^n according to the distribution theory makes sense, also from this point of view (cf. [7]).

Example 2.3 For $n = 0$, we have (2.1). For $n = 1$ and $n = 2$, we find

$$\begin{aligned} \hat{\boldsymbol{b}}^1(u) &= \frac{1}{(iu)^2}\begin{pmatrix} 1 & -1 \\ -e^{-iu} & 1 \end{pmatrix}\begin{pmatrix} iu \\ 1-e^{-iu} \end{pmatrix} \\ &= \frac{1}{(iu)^2}\begin{pmatrix} iu - 1 + e^{-iu} \\ 1-(1+iu)e^{-iu} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \hat{\boldsymbol{b}}^2(u) &= \frac{2}{(iu)^3}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -e^{-iu} & 0 & 1 \end{pmatrix}\begin{pmatrix} (iu)^2/2 \\ iu-1+e^{-iu} \\ 1-(1+iu)e^{-iu} \end{pmatrix} \\ &= \frac{1}{(iu)^3}\begin{pmatrix} (iu)^2 - 2iu + 2 - 2e^{-iu} \\ 2(iu-2) + 2(2+iu)e^{-iu} \\ 2 - (2+2iu+(iu)^2)e^{-iu} \end{pmatrix}. \end{aligned}$$

Now, we shall investigate the refinement equations (1.1) for the vector of Bernstein polynomials. Let $k \in \mathbb{N}$, $k \geq 2$ be given. After a substitution, we find that (1.1) is equivalent to the equation

$$\boldsymbol{b}^n(t) = \sum_{m=0}^{k-1} \boldsymbol{A}_m^n\, \boldsymbol{b}^n(kt-m), \quad t \in [0, 1],$$

since at the right–hand side at most one term is different from zero. Fourier transform yields

$$\hat{\boldsymbol{b}}^n(u) = \boldsymbol{A}^n(e^{-iu/k})\,\hat{\boldsymbol{b}}^n(u/k) \tag{2.4}$$

with $\boldsymbol{A}^n$ defined in (1.3). The refinement mask $\boldsymbol{A}^n$ can be characterized in the following way:

Theorem 2.4 *For $n = 0$, we have*

$$\boldsymbol{A}^0(z) = \frac{1}{k} \sum_{m=0}^{k-1} z^m = \frac{1 - z^k}{k\,(1-z)}. \tag{2.5}$$

For $n \geq 1$ and $z \neq 1$, the recursion formula

$$\boldsymbol{A}^n(z) = \frac{1}{k}\, \boldsymbol{C}_n(z^k) \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A}^{n-1}(z) \end{pmatrix} \boldsymbol{C}_n(z)^{-1} \tag{2.6}$$

is satisfied, where the matrices $\boldsymbol{C}_n$ are given in (2.3), and where $\boldsymbol{0}$ is a zero vector of suitable dimension.

Proof. For $n = 0$, we observe that

$$\boldsymbol{b}^0(t) = b_0^0(t) = 1, \quad t \in [0,1],$$

i.e., we have $\boldsymbol{A}_m^0 = 1$ $(m = 0, \dots, k-1)$. Formula (1.3) implies that $\boldsymbol{A}^0(z) = \frac{1}{k} \sum_{m=0}^{k-1} z^m$, so that (2.5) is proved.
Now let $n > 0$. Then from (2.2) and (2.4) we obtain for $u \neq 0$

$$\begin{aligned} \frac{1}{iu}\, \boldsymbol{C}_n(e^{-iu}) \begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix} &= \hat{\boldsymbol{b}}^n(u) = \boldsymbol{A}^n(e^{-iu/k})\, \hat{\boldsymbol{b}}^n(u/k) \\ &= \boldsymbol{A}^n(e^{-iu/k})\, \frac{k}{iu}\, \boldsymbol{C}_n(e^{-iu/k}) \begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u/k) \end{pmatrix}. \end{aligned}$$

Hence, since $\boldsymbol{C}_n(z)$ is regular for $z \neq 1$,

$$\begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix} = k\, \boldsymbol{C}_n(e^{-iu})^{-1}\, \boldsymbol{A}^n(e^{-iu/k})\, \boldsymbol{C}_n(e^{-iu/k}) \begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u/k) \end{pmatrix}.$$

On the other hand, (2.4) with $n-1$ instead of n implies

$$\begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u) \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A}^{n-1}(e^{-iu/k}) \end{pmatrix} \begin{pmatrix} 1/n \\ \hat{\boldsymbol{b}}^{n-1}(u/k) \end{pmatrix}.$$

In these two equations, all entries are rational functions in u and $z = e^{-iu/k}$. Since z is a transcendent function in u, the both equations are identities too, if we consider u and z as independent variables. Moreover, the components of the vectors $(1/n, \hat{\boldsymbol{b}}^{n-1}(u))^T$ and $(1/n, \hat{\boldsymbol{b}}^{n-1}(u/k))^T$ are linearly independent in u, and the entries of the matrices can be considered as constants with respect to u. This implies, that the corresponding matrices are equal, i.e.,

$$\begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{A}^{n-1}(e^{-iu/k}) \end{pmatrix} = k\, \boldsymbol{C}_n(e^{-iu})^{-1}\, \boldsymbol{A}^n(e^{-iu/k})\, \boldsymbol{C}_n(e^{-iu/k}),$$

so that also (2.6) is proved. ∎

Remark 2.5 1. For $z = 1$, the refinement mask $\boldsymbol{A}^n(z)$ can be found by a limiting process $z \to 1$, since the elements of $\boldsymbol{A}^n(z)$ are continuous in z.
2. From the recursion formula in Theorem 2.4 and Remark 2.2, we can easily derive the determinant of $\boldsymbol{A}^n(z)$:

$$\det \boldsymbol{A}^n(z) = \left(\frac{1 - z^k}{k\,(1 - z)} \right)^{n+1} .$$

3. For $z \neq 1$, the inverse matrix $\boldsymbol{C}_n(z)^{-1}$ is explicitly given by

$$\boldsymbol{C}_n(z)^{-1} = \frac{1}{n(1-z)} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ z & 1 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ z & z & \ddots & 1 & 1 \\ z & z & \dots & z & 1 \end{pmatrix} .$$

4. For $n = 1$, (2.6) simplifies to

$$\boldsymbol{A}^1(z) = \frac{1}{k^2\,(1-z)^2} \begin{pmatrix} 1 & -1 \\ -z^k & 1 \end{pmatrix} \begin{pmatrix} k(1-z) & 0 \\ 0 & 1 - z^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix} .$$

This formula can be generalized in the following way. By means of the direct sum of two quadratic matrices $\boldsymbol{A} \oplus \boldsymbol{B} := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$ with a suitable zero matrix $\boldsymbol{0}$, and the ν–dimensional unit matrix $\boldsymbol{I}_\nu$, we define for $\nu = 0, \dots, n$,

$$\boldsymbol{X}_\nu^n(z) := \boldsymbol{I}_{n-\nu} \oplus \frac{1}{\nu}\,\boldsymbol{C}_\nu(z^k), \qquad \boldsymbol{Y}_\nu^n(z) := \boldsymbol{I}_{n-\nu} \oplus \nu(1-z)\,\boldsymbol{C}_\nu(z)^{-1},$$

where $\boldsymbol{I}_0$ is dummy. Then (2.6) immediately implies

$$\boldsymbol{A}^n(z) = \frac{1}{k^{n+1}\,(1-z)^{n+1}}\,\boldsymbol{X}_n^n(z) \dots \boldsymbol{X}_1^n(z)\,\boldsymbol{B}_n(z)\,\boldsymbol{Y}_1^n(z) \dots \boldsymbol{Y}_n^n(z) \qquad (2.7)$$

with

$$\boldsymbol{B}_n(z) := \mathrm{diag}\left(k^n(1-z)^n,\, k^{n-1}(1-z)^{n-1}, \dots, k(1-z),\, 1 - z^k \right).$$

In particular, we have

$$\begin{aligned} \boldsymbol{A}^2(z) &= \frac{1}{k^3(1-z)^3} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 + z^k & -2 \\ -z^k & -z^k & 1 \end{pmatrix} \begin{pmatrix} k^2(1-z)^2 & 0 & 0 \\ 0 & k(1-z) & 0 \\ 0 & 0 & 1 - z^k \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 1 & 1 \\ 2z & z + 1 & 2 \\ z^2 + z & 2z & z + 1 \end{pmatrix} . \end{aligned}$$

For the matrix product $\boldsymbol{X}_n^n(z), \dots, \boldsymbol{X}_1^n(z)$ occuring in (2.7) we easily find

$$\boldsymbol{X}_n^n(z) \dots \boldsymbol{X}_1^n(z) = \left((-1)^{i+j} \binom{j}{i} - (-1)^{i+n} \binom{j}{n-i} \epsilon_{jn} z^k \right)_{i,j=0,\dots,n},$$

where $\epsilon_{jn} := 1 - \delta_{jn}$ with the Kronecker symbol δ_{jn}. The product $\boldsymbol{Y}_1^n(z) \dots \boldsymbol{Y}_n^n(z)$ has not such a simple explicit representation.

3. EXPLICIT REPRESENTATION OF REFINEMENT MATRICES

The determination of the refinement matrices $\boldsymbol{A}_m^n$ $(m = 0, \dots, k-1)$ by means of the refinement mask (1.3) is not quite easy, so that we shall give an explicit representation of them in this section. By

$$\boldsymbol{A}_m^n = (a_{ij}), \quad i, j = 0, \dots, n, \tag{3.1}$$

we introduce the entries a_{ij} of $\boldsymbol{A}_m^n$, which also depend on n, k and m.

Theorem 3.1 *The entries a_{ij} $(i, j = 0, \dots, n)$ of $\boldsymbol{A}_m^n$ have the representation*

$$a_{ij} = \frac{1}{k^n} (k-m)^{n-i} m^i \sum_{\nu=0}^{j} \binom{j}{\nu} \binom{n-j}{\nu+i-j} \left(1 - \frac{1}{k-m}\right)^{\nu} \left(1 + \frac{1}{m}\right)^{j-\nu}. \tag{3.2}$$

Proof. The equations (1.1) and (1.2) imply

$$\binom{n}{i} \left(1 - \frac{t+m}{k}\right)^{n-i} \left(\frac{t+m}{k}\right)^{i} = \sum_{j=0}^{n} a_{ij} \binom{n}{j} (1-t)^{n-j} t^j. \tag{3.3}$$

In view of

$$\begin{aligned} 1 - \frac{t+m}{k} &= \frac{k-m}{k} \left(1 - t + \left(1 - \frac{1}{k-m}\right) t\right), \\ t + m &= m \left(1 - t + \left(1 + \frac{1}{m}\right) t\right), \end{aligned}$$

for $m \neq 0$, the left–hand side of (3.3) can be written as

$$\begin{aligned} &\binom{n}{i} \left(\frac{k-m}{k}\right)^{n-i} \left(\frac{m}{k}\right)^{i} \sum_{\nu=0}^{n-i} \binom{n-i}{\nu} (1-t)^{n-i-\nu} \left(1 - \frac{1}{k-m}\right)^{\nu} t^{\nu} \\ &\times \sum_{\mu=0}^{i} \binom{i}{\mu} (1-t)^{i-\mu} \left(1 + \frac{1}{m}\right)^{\mu} t^{\mu}. \end{aligned}$$

Putting $\nu + \mu = j$, we obtain from (3.3) by a comparison of coefficients

$$\binom{n}{j} a_{ij} = \binom{n}{i} \frac{1}{k^n} (k-m)^{n-i} m^i \sum_{\mu+\nu=j} \binom{n-i}{\nu} \binom{i}{\mu} \left(1 - \frac{1}{k-m}\right)^{\nu} \left(1 + \frac{1}{m}\right)^{\mu},$$

and in view of

$$\binom{n}{i}\binom{n-i}{\nu}\binom{i}{\mu} = \binom{n}{j}\binom{j}{\nu}\binom{n-j}{\nu+i-j},$$

finally (3.2).
According to $\binom{n-j}{\nu+i-j} = 0$ for $\nu < j - i$, formula (3.2) makes also sense for $m = 0$, where only the term with $\nu = j - i$ remains:

$$a_{ij} = \frac{1}{k^j}\binom{j}{i}(k-1)^{j-i} \quad \text{for} \quad m = 0. \qquad \blacksquare \tag{3.4}$$

Analogously, for $m = k - 1$, only the term with $\nu = 0$ remains:

$$a_{ij} = \frac{1}{k^{n-j}}\binom{n-j}{i-j}(k-1)^{i-j} \quad \text{for} \quad m = k-1. \tag{3.5}$$

The formulas (3.4) and (3.5) show that $\boldsymbol{A}_0^n$ is an upper and $\boldsymbol{A}_{k-1}^n$ a lower triangular matrix. Moreover, we only have one single term in (3.2) in the four cases
$i = 0$ with $\nu = j$,

$$a_{0j} = \frac{1}{k^n}(k-m)^{n-j}(k-m-1)^j,$$

$i = n$ with $\nu = 0$,

$$a_{nj} = \frac{1}{k^n} m^{n-j}(m+1)^j,$$

$j = 0$ with $\nu = 0$,

$$a_{i0} = \frac{1}{k^n}\binom{n}{i}(k-m)^{n-i} m^i, \tag{3.6}$$

and $j = n$ with $\nu = n - i$,

$$a_{in} = \frac{1}{k^n}\binom{n}{i}(k-m-1)^{n-i}(m+1)^i. \tag{3.7}$$

Remark 3.2 The equations (3.6) and (3.7) show that the last column of $\boldsymbol{A}_m^n$ equals to the first column of $\boldsymbol{A}_{m+1}^n$ ($m = 0, \ldots, k-2$). This follows also from the statement (a) of Theorem 5.1 in [5], if one uses the fact that $(1, 0, \ldots, 0)^T$ and $(0, \ldots, 0, 1)^T$ are eigenvectors of the matrices $\boldsymbol{A}_0^n$ and $\boldsymbol{A}_{k-1}^n$, respectively, corresponding to the eigenvalue one. The last fact can easily be seen from $a_{00} = 1$ for $m = 0$, $a_{nn} = 1$ for $m = k - 1$, and the triangular structure of $\boldsymbol{A}_0^n$ and $\boldsymbol{A}_{k-1}^n$.

4. SPECTRAL PROPERTIES OF THE REFINEMENT MATRICES

The columns of $\boldsymbol{A}_m^n$ ($m = 0, \ldots, k-1$) possess simple generating functions.

Lemma 4.1 *For an arbitrary parameter λ we have for $j = 0, \ldots, n$,*

$$\sum_{i=0}^{n} a_{ij}\, \lambda^i = \frac{1}{k^n} \Big(k + (\lambda - 1)(m+1)\Big)^j \Big(k + m(\lambda - 1)\Big)^{n-j}. \tag{4.1}$$

Proof. From (3.2) it follows that

$$\begin{aligned} \sum_{i=0}^{n} a_{ij}\, \lambda^i \;&=\; \frac{1}{k^n} \sum_{\nu=0}^{j} \binom{j}{\nu} \left(1 - \frac{1}{k-m}\right)^{\nu} \left(1 + \frac{1}{m}\right)^{j-\nu} \\ &\times \sum_{i=0}^{n} \binom{n-j}{\nu + i - j} (k-m)^{n-i}\, (\lambda m)^i. \end{aligned}$$

Since $\binom{j}{\nu}\binom{n-j}{\nu+i-j} \neq 0$ only for $0 \leq j - \nu \leq i \leq n - \nu \leq n$, the sum over i equals to

$$\sum_{l=0}^{n-j} \binom{n-j}{l} (k-m)^{n-l-j+\nu}\, (\lambda m)^{l+j-\nu} = (k-m)^{\nu}\, (\lambda m)^{j-\nu}\, (k - m + \lambda m)^{n-j}$$

with $l = \nu + i - j$, and the assertion follows from

$$\sum_{\nu=0}^{j} \binom{j}{\nu} (k - m - 1)^{\nu}\, (\lambda(m+1))^{j-\nu} = (k - m - 1 + \lambda(m+1))^j. \qquad \blacksquare$$

Corollary 4.2 . *Equation (4.1) immediately implies that $\boldsymbol{A}_m^{nT}$ has the eigenvalue 1 with the eigenvector $(1, \ldots, 1)^T$, and the eigenvalue k^{-n} with the eigenvector $(1, \lambda, \ldots, \lambda^n)^T$ and $\lambda = 1 + (1-k)/m$, so that $\boldsymbol{A}_m^n$ is a stochastic matrix with respect to the rows.*

By means of (4.1) it is also possible to construct eigenvectors of all eigenvalues k^{-j} ($j = 0, \ldots, n$), but it is easier to derive them from the eigenvectors of $\boldsymbol{A}_m^n$, which were found in [5].

Theorem 4.3 *The matrix $\boldsymbol{A}_m^{nT}$ has the eigenvalues k^{-j} ($j = 0, \ldots, n$) (where $k \geq 2$ is the dilation parameter in the refinement equation (1.1)) with the corresponding eigenvectors*

$$\left(\sum_{\nu=0}^{j} \binom{i}{j-\nu} \binom{n+\nu-j}{\nu} \left(\frac{m}{1-k}\right)^{\nu} \right), \tag{4.2}$$

where $i = 0, \ldots, n$ denotes the row index.

Proof. Let $\boldsymbol{D}_n$ be the diagonal matrix of the eigenvalues

$$\boldsymbol{D}_n := \operatorname{diag}(1, k^{-1}, \ldots, k^{-n}),$$

and $\boldsymbol{G}_n$, $\boldsymbol{U}_n$ matrices of the corresponding eigenvectors of $(\boldsymbol{A}_m^n)^T$ and $\boldsymbol{A}_m^n$, respectively. Then

$$(\boldsymbol{A}_m^n)^T \boldsymbol{G}_m = \boldsymbol{G}_m \boldsymbol{D}_n, \quad \boldsymbol{A}_m^n \boldsymbol{U}_m = \boldsymbol{U}_m \boldsymbol{D}_n$$

and therefore $(\boldsymbol{A}_m^n)^T \boldsymbol{U}_m^{-T} = \boldsymbol{U}_m^{-T} \boldsymbol{D}_n$ with $\boldsymbol{U}_m^{-T} := (\boldsymbol{U}_m^T)^{-1}$. Hence, we find

$$\boldsymbol{G}_m = \boldsymbol{U}_m^{-T} \boldsymbol{F}_m \tag{4.3}$$

with a diagonal matrix $\boldsymbol{F}_m$. According to Theorem 7.1 in [5] we have (after a correction of a misprint)

$$\boldsymbol{U}_m = \boldsymbol{U}_0 \boldsymbol{T}_n \left(\frac{m}{1-k}\right)$$

with

$$\boldsymbol{U}_0 = \left(\binom{n}{j}\binom{j}{i}(-1)^{i-j}\right), \qquad \boldsymbol{T}_n(a) = \left((-a)^{i-j}\binom{i}{j}\right).$$

Here and in the following matrices, we always denote the row index by i and the column index by j $(i, j = 0, \dots, n)$. It is easy to see that $\boldsymbol{T}_n^{-1}(a) = \boldsymbol{T}_n(-a)$ and

$$\boldsymbol{U}_0^{-1} = \left(\frac{\binom{j}{i}}{\binom{n}{i}}\right).$$

In order to obtain a simple result, we choose

$$\boldsymbol{F}_m := \text{diag}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right),$$

so that (4.3) and $\boldsymbol{U}_m^{-T} = \boldsymbol{U}_0^{-T} \boldsymbol{T}_n(-\frac{m}{1-k})^T$ imply

$$\boldsymbol{G}_m = \left(\frac{\binom{i}{j}}{\binom{n}{j}}\right) \cdot \left(\left(\frac{m}{1-k}\right)^{j-i}\binom{j}{i}\binom{n}{j}\right).$$

The entries of the matrix product on the right–hand side are

$$\sum_{l=0}^{n} \frac{\binom{i}{l}}{\binom{n}{l}} \left(\frac{m}{1-k}\right)^{j-l} \binom{j}{l}\binom{n}{j}$$

and in view of

$$\frac{\binom{j}{l}\binom{n}{j}}{\binom{n}{l}} = \binom{n-l}{j-l} = \binom{n+\nu-j}{\nu}$$

with $\nu = j - l$, these are exactly the entries of (4.2). ■

Remark 4.4 1. For $j = n$ the components of the vector (4.2) are

$$\begin{aligned}\sum_{\nu=n-i}^{n} \binom{i}{\nu+i-n}\left(\frac{m}{1-k}\right)^{\nu} &= \left(1+\frac{m}{1-k}\right)^{i}\left(\frac{m}{1-k}\right)^{n-i} \\ &= \left(\frac{m}{1-k}\right)^{n}\left(1+\frac{1-k}{m}\right)^{i}.\end{aligned}$$

Thus, for the eigenvalue k^{-n} we get, up to a constant factor, indeed the same eigenvector as in Corollary 4.2.
2. Eigenvectors for matrices composed by binomial coefficients are also determined in [1] and [4].

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