Spline Wavelets with Higher Defect

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Abstract. In this paper a generalized multiresolution analysis, generated by cardinal B-splines of degree m and defect r, is considered. Using the cardinal Hermite fundamental splines of degree $2m+1$ and defect r new spline wavelets with defect r are represented. In contrast with other papers dealing with wavelets with higher defect (cf. [3, 4]) the two–scale symbol $\mathbf{Q}_m^r$ of the wavelet vector can explicitly be given.

§1. Introduction

The subject of this paper is a natural generalization of the concept of interpolatory spline wavelets introduced in [2, pp. 177]. Let $\{V_j^m\}$ $(j \in \mathbb{Z})$ be the multiresolution analysis of $L_2(\mathbb{R})$ generated by the cardinal B-spline N_m of degree m. Further, with $\{W_j^m\}$ $(j \in \mathbb{Z})$ we denote the sequence of wavelet spaces, in the sense that

$$V_{j+1}^m = V_j^m \oplus W_j^m,$$

where $\oplus$ indicates the orthogonal summation.

Let $\mathcal{T} := \{z \in \mathbb{C},\ |z| = 1\}$. With the help of the Euler–Frobenius polynomial of degree $2m+1$

$$\Phi_{2m+1}^1(z) := \sum_{l=\infty}^{\infty} N_{2m+1}(l) z^l \quad (z \in \mathcal{T})$$

we introduce the cardinal fundamental spline L_{2m+1} of degree $2m+1$

$$L_{2m+1} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{N}_{2m+1}(u)}{\Phi_{2m+1}^1(e^{-iu})} e^{-iu\cdot}\, \mathrm{d}u$$

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satisfying

$$L_{2m+1}(n) = \delta_{0n} \quad (n \in \mathbb{Z})$$

with the Kronecker symbol δ. For $m \in \mathbb{N}$ the *interpolatory wavelet* $\psi_{I,m}$ is defined by

$$\psi_{I,m} := \mathrm{D}^{m+1} L_{2m+1}(2 \cdot -1),$$

where D denotes the differential operator. Then $\psi_{I,m}$ generates the wavelet spaces W_j^m $(j \in \mathbb{Z})$

$$W_j^m := \mathrm{clos}_{L_2}\big(\mathrm{span}\{\psi_{I,m}(2^j \cdot -l);\ l \in \mathbb{Z}\}\big)$$

(cf. [2], p. 178).

We want to generalize this concept in the following way.
Let $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ be given integers. We consider equidistant knots of multiplicity r

$$x_l^r := \left\lfloor \frac{l}{r} \right\rfloor, \tag{1.1}$$

where $\lfloor x \rfloor$ means the integer part of $x \in \mathbb{R}$.

Let $N_k^{m,r} \in C^{m-r}(\mathbb{R})$ $(r \le m,\ k \in \mathbb{Z})$ denote the normalized B-splines of degree m and defect r with the knots $x_k, \ldots, x_{k+m+1}$. For $r = m+1$ the B-splines $N_k^{m,m+1}$ $(k = 0, \ldots, m)$ coincide with the well-known Bernstein polynomials. According to the distribution theory, let $N_k^{m,r}$ be defined for $r > m+1$ and $k = 0, \ldots, r-m-2$ as follows

$$N_k^{m,r} := \frac{1}{r-1-k}\, \mathrm{D}^{r-m-2-k}\, \delta, \tag{1.2}$$

where δ denotes the Dirac distribution.

Using the ideas in [3, 4] in Section 2, we shall consider the *generalized multiresolution analysis* $\{V_j^{m,r}\}$ $(j \in \mathbb{Z})$ *of multiplicity* r of $L_2(\mathbb{R})$ generated by the linearly independent scaling functions $N_k^{m,r}$ $(k = 0, \ldots, r-1)$, that is

$$V_j^{m,r} := \mathrm{clos}_{L_2}\left(\mathrm{span}\left\{N_k^{m,r}(2^j \cdot -l);\ k = 0, \ldots, r-1\right\}\right). \tag{1.3}$$

In particular, an explicit formula for the two–scale symbol $\mathbf{P}_m^r$ of the B-spline vector $\mathbf{N}_m^r := (N_k^{m,r})_{k=0}^{r-1}$ can be given (cf. [7]).

Let $\{W_j^{m,r}\}$ $(j \in \mathbb{Z})$ denote the sequence of wavelet spaces determined by

$$V_{j+1}^{m,r} = V_j^{m,r} \oplus W_j^{m,r}.$$

In Section 3 we shall introduce the cardinal Hermite fundamental splines $L_k^{2m+1,r}$ $(k = 0, \ldots, r-1)$ satisfying for $n \in \mathbb{Z}$ the interpolation conditions

$$\mathrm{D}^\nu\, L_k^{2m+1,r}(n) = \delta_{0n}\, \delta_{\nu k} \quad (\nu, k = 0, \ldots, r-1). \tag{1.4}$$

We put

$$\psi_k^{m,r} := \mathrm{D}^{m+1} L_k^{2m+1,r}(2\cdot -1) \quad (k=0,\ldots,r-1).$$

Contrary to [3, 4] we can firstly give an explicit formula for the two–scale symbol $\mathbf{Q}_m^r$ of the wavelet vector $\mathbf{\Psi}_m^r := (\psi_k^{m,r})_{k=0}^{r-1}$. This two–scale symbol $\mathbf{Q}_m^r$ can be used for the computation of the *wavelets* $\psi_k^{m,r}$ $(k=0,\ldots,r-1)$ *with defect* r as well as for deriving the Riesz basis property in the wavelet space $W_0^{m,r}$. For $r \geq m+1$ the wavelets $\psi_k^{m,r}$ $(k=0,\ldots,r-1)$ are compactly supported, for $r \leq m$ they have exponential decay.

Note that in [3] other wavelets are constructed, which are derived from special compactly supported splines, firstly introduced in [9].

In Section 4 we show the close connection between the wavelet space $W_0^{m,r}$ and the subspace $V_{1,0}^{2m+1,r} \subset V_1^{2m+1,r}$, which contains splines with degree $2m+1$ and defect r satisfying some interpolation conditions. Analogous assertions for the simple case $r=1$ can be found in [2].

Finally, in Section 5 the obtained formulas are applied to the case of cubic spline wavelets $(m=3)$ with defect $r=2$.

§2. Multiresolution Analysis of Multiplicity r

For a summary of basic properties of B-splines with multiple knots we refer to [1, 7]. Here we recall only the following important relations.

Let $\hat{\mathbf{N}}_m^r := (\hat{N}_k^{m,r})_{k=0}^{r-1}$ be the vector of Fourier transformed B-splines

$$\hat{N}_k^{m,r} := \int_{-\infty}^{\infty} N_k^{m,r}(x)\, e^{-i\cdot x}\, \mathrm{d}x.$$

For the Fourier transformed B-spline vector $\hat{\mathbf{N}}_m^r$ of length $r > m+1 \geq 1$ we find by (1.2)

$$\hat{\mathbf{N}}_m^r(u) = \Big(\frac{(iu)^{r-m-2}}{r-1}, \ldots, \frac{(iu)^0}{m+1}, \hat{\mathbf{N}}_m^{m+1}(u)^{\mathrm{T}}\Big)^{\mathrm{T}},$$

where $\mathbf{N}_m^{m+1}$ denotes the vector of the $m+1$ Bernstein polynomials of degree m. Further, we put

$$\hat{\mathbf{N}}_{-1}^r(u) := \Big(\frac{(iu)^{r-1}}{r-1}, \ldots, \frac{(iu)^1}{1}, 1\Big)^{\mathrm{T}} \quad (u \in \mathbb{R}). \tag{2.1}$$

Then for $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ the following recursion relation can be found:

$$(iu)\, \hat{\mathbf{N}}_m^r(u) = \mathbf{A}_m^r(e^{-iu})\, \hat{\mathbf{N}}_{m-1}^r(u) \quad (u \in \mathbb{R}). \tag{2.2}$$

The (r,r)-matrices $\mathbf{A}_m^r(z)$ $(z \in \mathcal{T})$ are defined for $m > r-1$ by

$$\mathbf{A}_m^r(z) := m \begin{pmatrix} \frac{1}{x_m^r} & -\frac{1}{x_{m+1}^r} & \dots & 0 & 0 \\ 0 & \frac{1}{x_{m+1}^r} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{x_{m+r-2}^r} & -\frac{1}{x_{m+r-1}^r} \\ -\frac{z}{x_m^r} & 0 & \dots & 0 & \frac{1}{x_{m+r-1}^r} \end{pmatrix}, \tag{2.3}$$

where x_{m+k}^r $(k = 0, \dots, r-1)$ are given in (1.1). For $m = r-1 > 0$ let

$$\mathbf{A}_m^{m+1}(z) := m \begin{pmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ -z & 0 & \dots & 0 & 1 \end{pmatrix} \tag{2.4}$$

and for $0 \le m < r-1$

$$\mathbf{A}_m^r(z) := \begin{pmatrix} \mathbf{I}_{r-m-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_m^{m+1}(z) \end{pmatrix}, \tag{2.5}$$

where $\mathbf{A}_0^1(z) := 1 - z$. Further, $\mathbf{I}_{r-m-1}$ denotes the $(r-m-1)$-th unit matrix and $\mathbf{0}$ a zero matrix (cf. [7]).

Note that $\det \mathbf{A}_m^r(z) = c_m^r\,(1-z)$. The Fourier transformed *two–scale relation of* $\mathbf{N}_m^r$ is given by

$$\hat{\mathbf{N}}_m^r = \mathbf{P}_m^r(e^{-i\cdot/2})\,\hat{\mathbf{N}}_m^r(\cdot/2) \quad (m \in \mathbb{N}_0,\ r \in \mathbb{N}).$$

The *two–scale symbol (or refinement mask) of* $\mathbf{N}_m^r$

$$\mathbf{P}_m^r(z) := \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathbf{P}_n\, z^n \quad (z \in \mathcal{T}) \tag{2.6}$$

is a finite sum and satisfies for $m \ge 0$ the following recursion formula

$$\begin{aligned} \mathbf{P}_m^r(z) &= \frac{1}{2}\mathbf{A}_m^r(z^2)\,\mathbf{P}_{m-1}^r(z)\,\mathbf{A}_m^r(z)^{-1} \quad (z \in \mathcal{T},\ z \ne 1) \\ \mathbf{P}_m^r(1) &= \frac{1}{2}\lim_{u\to 0} \mathbf{A}_m^r(e^{-2iu})\,\mathbf{P}_{m-1}^r(e^{-iu})\,\mathbf{A}_m^r(e^{-iu})^{-1} \quad (u \in \mathbb{R}) \end{aligned} \tag{2.7}$$

with $\mathbf{A}_m^r(z)$ defined in (2.3) – (2.5) and

$$\mathbf{P}_{-1}^r(z) := \mathrm{diag}\,(2^{r-1}, \dots, 2^0)^{\mathrm{T}} \tag{2.8}$$

(cf. [7]). In particular, the two–scale symbol $\mathbf{P}_m^r(z)$ is a matrix polynomial in z with

$$\det\,\mathbf{P}_m^r(z) = 2^{-rm+r(r-3)/2}\,(1+z)^{m+1} \quad (z \in \mathcal{T}).$$

The functions $N_k^{m,r}(\cdot - l)$ form a Riesz basis (or $L_2(\mathbb{R})$-stable basis) of $V_0^{m,r}$ (cf. [1]). The Riesz basis property is equivalent to the assertion that the *autocorrelation symbol* $\mathbf{\Phi}_m^r$, defined by

$$\mathbf{\Phi}_m^r(e^{-iu}) := \sum_{n=-\infty}^{\infty} \hat{\mathbf{N}}_m^r(u+2\pi n)\,\hat{\mathbf{N}}_m^r(u+2\pi n)^\star \tag{2.9}$$

with $\hat{\mathbf{N}}_m^r(u)^\star := \overline{\hat{\mathbf{N}}_m^r(u)^{\mathrm{T}}}$ is positive definite (cf. [4, 5, 7]).

Further, we introduce the following *Euler–Frobenius matrix*

$$\mathbf{H}_{2m+1}^r := (H_k^\nu)_{\nu,k=0}^{r-1} \tag{2.10}$$

with

$$H_k^\nu(z) := \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu N_k^{2m+1,r}(l) z^l \quad (k,\, \nu = 0, 1, \ldots, r-1,\, z \in \mathcal{T}). \tag{2.11}$$

For $2m+1-\nu \le r-1$, the functions $\mathrm{D}^\nu N_k^{2m+1,r}$ are understood according to the distribution theory. For $r=1$ we obtain the well-known Euler–Frobenius polynomial

$$\mathbf{H}_{2m+1}^1(z) = H_{2m+1}^0(z) = \sum_{l=-\infty}^{\infty} N_{2m+1}(l)\, z^l.$$

By the Poisson summation formula the matrix $\mathbf{H}_{2m+1}^r$ reads for $z = e^{-iu}$ as follows

$$\mathbf{H}_{2m+1}^r(e^{-iu}) = \sum_{l=-\infty}^{\infty} \Big((i(u+2\pi l))^k\Big)_{k=0}^{r-1} \hat{\mathbf{N}}_{2m+1}^r(u+2\pi l)^{\mathrm{T}} \quad (u \in \mathbb{R}). \tag{2.12}$$

For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ we have the following relationship:

$$\begin{aligned} \mathbf{\Phi}_m^r(z) &= \mathbf{D}_{m,0}^r(z)\,\mathbf{D}_r\,\overline{\mathbf{H}_{2m+1}^r(z)}\,(\mathbf{D}_{m,1}^r(z)^\star)^{-1} \quad (z \in \mathcal{T},\, z \ne 1),\\ \mathbf{\Phi}_m^r(1) &= \lim_{u\to 0} \mathbf{D}_{m,0}^r(e^{-iu})\,\mathbf{D}_r\,\mathbf{H}_{2m+1}^r(e^{iu})\,(\mathbf{D}_{m,1}^r(e^{-iu})^\star)^{-1} \quad (u \in \mathbb{R}) \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} \mathbf{D}_{m,0}^r(z) &:= \mathbf{A}_m^r(z)\,\mathbf{A}_{m-1}^r(z)\ldots\mathbf{A}_0^r(z),\\ \mathbf{D}_{m,1}^r(z) &:= \mathbf{A}_{2m+1}^r(z)\,\mathbf{A}_{2m}^r(z)\ldots\mathbf{A}_{m+1}^r(z), \end{aligned} \tag{2.14}$$

where the (r,r)-matrices $\mathbf{A}_k^r$ $(k = 0, \ldots, 2m+1)$ are defined in (2.3) – (2.5),

$$\mathbf{D}_r := (-1)^{m+1} \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{(-1)^{r-1}}{r-1} \\ 0 & 0 & \ldots & \frac{(-1)^{r-2}}{r-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & (-1)^1 & \ldots & 0 & 0 \\ (-1)^0 & 0 & \ldots & 0 & 0 \end{pmatrix} \quad (r > 1)$$

and $\mathbf{D}_1 := (-1)^{m+1}$. In particular, the invertibility of the autocorrelation matrix $\mathbf{\Phi}_m^r(z)$ for $z \in \mathcal{T}$ causes the invertibility of the Euler–Frobenius matrix $\mathbf{H}_{2m+1}^r(z)$ for $z \in \mathcal{T}$ (cf. [7]).

Since $V_j^{m,r}$ contains the space $V_j^{m,1}$ generated by the cardinal B-spline N_m (cf. [2]), it follows that

$$\operatorname{clos}_{L_2} \bigcup_{j=-\infty}^{\infty} V_j^{m,r} = L_2(\mathbb{R}).$$

The Riesz basis property and the partition of unity property for the B-splines $N_k^{m,r}(\cdot - l)$ $(k = 0, \ldots, r-1,\ l \in \mathbb{Z})$ also lead to

$$\bigcap_{j=-\infty}^{\infty} V_j^{m,r} = \{0\}.$$

Thus, the sequence $\{V_j^{m,r}\}$ $(j \in \mathbb{Z})$ generates a multiresolution analysis of $L_2(\mathbb{R})$ with multiplicity r (cf. [3,4]).

§3. The Wavelet Space $W_j^{m,r}$

In this section we want to find spline wavelets with defect r

$$\psi_k^{m,r} \qquad (k = 0, \ldots, r-1), \tag{3.1}$$

such that the integer translates of (3.1) form a Riesz basis of the wavelet space $W_0^{m,r} := V_1^{m,r} \ominus V_0^{m,r}$.

First we want to introduce cardinal Hermite fundamental splines. Let $\hat{\mathbf{L}}_{2m+1}^r := (\hat{L}_k^{2m+1,r})_{k=0}^{r-1}$ be the Fourier transformed vector of spline functions $L_k^{2m+1,r}$ $(k = 0, \ldots, r-1)$ defined by

$$\hat{\mathbf{L}}_{2m+1}^r(u) := (\mathbf{H}_{2m+1}^r(e^{-iu})^{\mathrm{T}})^{-1}\, \hat{\mathbf{N}}_{2m+1}^r(u), \tag{3.2}$$

and $\mathbf{L}_{2m+1}^r := (L_k^{m,r})_{k=0}^{r-1}$. Then we have:

Theorem 3.1. *The spline functions $L_k^{2m+1,r}$ $(k = 0, \ldots, r-1)$ are cardinal Hermite fundamental splines in $V_0^{2m+1,r}$, i.e., for $n \in \mathbb{Z}$ the interpolation conditions*

$$\mathrm{D}^\nu L_k^{2m+1,r}(n) = \delta_{0n}\, \delta_{\nu k} \quad (\nu, k = 0, \ldots, r-1) \tag{3.3}$$

are satisfied.

Proof: By $\mathcal{W}$ we define the Wiener class. Since $\det \mathbf{H}_{2m+1}^r(z) \in \mathcal{W}$, it follows that there exists a representation

$$[\mathbf{H}_{2m+1}^r(e^{-iu})]^{-1} = \sum_{n=-\infty}^{\infty} \mathbf{H}_n\, e^{-iun},$$

where the elements of the (r,r)-matrices $\mathbf{H}_n$ $(n \in \mathbb{Z})$ lie in l_1. Thus, (3.2) implies that the functions $L_k^{2m+1,r}$ $(k=0,\dots,r-1)$ are contained in $V_0^{2m+1,r}$.

It remains to show that the interpolation conditions (3.3) hold. Putting

$$[\mathrm{D}^\nu \mathbf{L}^r_{2m+1}]^\sim := \sum_{n=-\infty}^{\infty} [\mathrm{D}^\nu \mathbf{L}^r_{2m+1}]^\wedge(\cdot + 2\pi n)$$

it follows by the Poisson summation formula that

$$[\mathrm{D}^\nu \mathbf{L}^r_{2m+1}]^\sim = \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu \mathbf{L}^r_{2m+1}(l)\, e^{-i\cdot l}.$$

Therefore we have to show that $[\mathrm{D}^\nu \mathbf{L}^r_{2m+1}]^\sim \equiv \mathbf{e}_\nu$, where $\mathbf{e}_\nu := (\delta_{k\nu})_{k=0}^{r-1}$ are the unit vectors. By

$$[\mathrm{D}^\nu \mathbf{N}^r_{2m+1}]^\sim = \sum_{l=-\infty}^{\infty} \mathrm{D}^\nu \mathbf{N}^r_{2m+1}(l)\, e^{-i\cdot l} = (H_k^\nu)_{k=0}^{r-1}$$

the relation (3.2) leads to

$$\begin{aligned}
&\big([\mathrm{D}^0 \mathbf{L}^r_{2m+1}]^\sim(u), \dots, [\mathrm{D}^{r-1} \mathbf{L}^r_{2m+1}]^\sim(u)\big)\\
&= [\mathbf{H}^r_{2m+1}(e^{-iu})^{\mathrm{T}}]^{-1} \big((H_k^0(e^{-iu}))_{k=0}^{r-1}, \dots, (H_k^{r-1}(e^{-iu}))_{k=0}^{r-1}\big)\\
&= [\mathbf{H}^r_{2m+1}(e^{-iu})^{\mathrm{T}}]^{-1}\, \mathbf{H}^r_{2m+1}(e^{-iu})^{\mathrm{T}}\\
&= \mathbf{I}. \qquad \blacksquare
\end{aligned}$$

Now let

$$\psi_k^{m,r} := \mathrm{D}^{m+1} L_k^{2m+1,r}(2\cdot -1) \qquad (k=0,\dots,r-1) \tag{3.4}$$

and $\mathbf{\Psi}^r_m := (\psi_k^{m,r})_{k=0}^{r-1}$. We shall show that the spline wavelets $\psi_k^{m,r}$ $(k = 0,\dots,r-1)$ and their integer translates form a Riesz basis of $W_0^{m,r}$.

Using the relations (2.2) and (3.2) we obtain for the vector $\hat{\mathbf{\Psi}}^r_m := (\hat{\psi}_k^{m,r})_{k=0}^{r-1}$ of Fourier transformed wavelets

$$\begin{aligned}
\hat{\mathbf{\Psi}}^r_m(u) &= [\mathrm{D}^{m+1} \mathbf{L}^r_{2m+1}(2\cdot -1)]^\wedge(u)\\
&= 1/2\,(iu/2)^{m+1}\, e^{-iu/2}\, \hat{\mathbf{L}}^r_{2m+1}(u/2)\\
&= 1/2\, e^{-iu/2}\, [\mathbf{H}^r_{2m+1}(e^{-iu/2})^{\mathrm{T}}]^{-1}\, \mathbf{D}^r_{m,1}(e^{-iu/2})\, \hat{\mathbf{N}}^r_m(u/2)
\end{aligned}$$

with $\mathbf{D}^r_{m,1}$ defined in (2.14). Thus, we have the two–scale relation

$$\hat{\mathbf{\Psi}}^r_m = \mathbf{Q}^r_m(e^{-i\cdot/2})\, \hat{\mathbf{N}}^r_m(\cdot/2) \tag{3.5}$$

with the *two–scale symbol of* $\mathbf{\Psi}_m^r$

$$\mathbf{Q}_m^r(z) := z/2\,[\mathbf{H}_{2m+1}^r(z)^{\mathrm{T}}]^{-1}\,\mathbf{D}_{m,1}^r(z) \quad (z \in \mathcal{T}). \tag{3.6}$$

Observe that the elements of the matrix $\mathbf{Q}_m^r$ belong to the Wiener class. The two–scale relation (3.5) implies that the functions $\psi_k^{m,r}$ $(k = 0, \ldots, r-1)$ lie in $V_1^{m,r}$. The functions $\psi_k^{m,r}$ belong to $W_0^{m,r}$ if and only if for $k, \nu = 0, \ldots, r-1$ and $l \in \mathbb{Z}$,

$$\langle N_k^{m,r}(\cdot - l), \psi_\nu^{m,r}\rangle := \int_{-\infty}^{\infty} N_k^{m,r}(x-l)\,\overline{\psi_\nu^{m,r}(x)}\,\mathrm{d}x = 0,$$

i.e., if and only if the condition

$$\mathbf{P}_m^r(z)\,\mathbf{\Phi}_m^r(z)\,\mathbf{Q}_m^r(z)^\star + \mathbf{P}_m^r(-z)\,\mathbf{\Phi}_m^r(-z)\,\mathbf{Q}_m^r(-z)^\star = \mathbf{0} \quad (z \in \mathcal{T}) \tag{3.7}$$

is satisfied (cf. [4]).

Theorem 3.2. *The functions* $\psi_k^{m,r}$ $(k = 0, \ldots, r-1)$ *belong to* $W_0^{m,r}$.

Proof: From the recursion relation (2.7) it follows with (2.8) and (2.14)

$$\mathbf{P}_m^r(z) = \frac{1}{2^{m+1}}\,\mathbf{D}_{m,0}^r(z^2)\,\mathbf{P}_{-1}^r\,\mathbf{D}_{m,0}^r(z)^{-1} \quad (z \in \mathcal{T}), \tag{3.8}$$

where for $z = 1$, (3.8) is understood according to (2.7). Using the relations (2.13) and (3.6) we obtain

$$\mathbf{P}_m^r(z)\,\mathbf{\Phi}_m^r(z)\,\mathbf{Q}_m^r(z)^\star = \frac{z}{2^{m+2}}\,\mathbf{D}_{m,0}^r(z^2)\,\mathbf{P}_{-1}^r\,\mathbf{D}_r,$$

where $\mathbf{P}_{-1}^r$ and $\mathbf{D}_r$ do not depend on z. Thus, the relation (3.7) is satisfied. ∎

We can even prove:

Theorem 3.3. *The functions* $\psi_k^{m,r}$ $(k = 0, \ldots, r-1)$ *form a Riesz basis of* W_0*, i.e., there exist Riesz bounds* $0 < \alpha \le \beta < \infty$ *such that*

$$\alpha \sum_{l=-\infty}^{\infty}\sum_{k=0}^{r-1} |c_l^k|^2 \le \|\sum_{l=-\infty}^{\infty}\sum_{k=0}^{r-1} c_l^k\,\psi_k^{m,r}(\cdot - l)\|_{L_2}^2 \le \beta \sum_{l=-\infty}^{\infty}\sum_{k=0}^{r-1} |c_l^k|^2$$

for any sequences $(c_l^k)_{l=-\infty}^{\infty} \in l_2$ $(k = 0, \ldots, r-1)$.

Proof: For $(c_l^k)_{l=-\infty}^{\infty} \in l_2$ $(k = 0, \ldots, r-1)$ let C_k denote their 2π-periodic symbols,

$$C_k := \sum_{l=-\infty}^{\infty} c_l^k\,e^{-iu\cdot} \quad (k = 0, \ldots, r-1).$$

Put $\mathbf{C} := (C_0, \ldots, C_{r-1})^{\mathrm{T}}$. Then by the Parseval identity we find

$$\| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_l^k \, \psi_k^{m,r}(\cdot - l) \|_{L_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathbf{C}(u)^{\mathrm{T}} \, \hat{\mathbf{\Psi}}_m^r(u)|^2 \, \mathrm{d}u$$
$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} \mathbf{C}(u)^{\mathrm{T}} \, \hat{\mathbf{\Psi}}_m^r(u + 2\pi l) \, \hat{\mathbf{\Psi}}_m^r(u + 2\pi l)^\star \, \overline{\mathbf{C}(u)} \, \mathrm{d}u.$$

Using the two–scale relation it follows

$$\| \sum_{l=-\infty}^{\infty} \sum_{k=0}^{r-1} c_l^k \, \psi_k^{m,r}(\cdot - l) \|_{L_2}^2$$
$$= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} \mathbf{C}(u)^{\mathrm{T}} \mathbf{Q}_m^r(e^{-i(u/2+\pi l)}) \, \hat{\mathbf{N}}_m^r(u/2 + \pi l) \, \hat{\mathbf{N}}_m^r(u/2 + \pi l)^\star$$
$$\cdot \, \mathbf{Q}_m^r(e^{-i(u/2+\pi l)})^\star \, \overline{\mathbf{C}(u)} \, \mathrm{d}u$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \mathbf{C}(u)^{\mathrm{T}} \Big(\mathbf{Q}_m^r(e^{-iu/2}) \, \mathbf{\Phi}_m^r(e^{-iu/2}) \, \mathbf{Q}_m^r(e^{-iu/2})^\star \Big) \, \overline{\mathbf{C}(u)} \, \mathrm{d}u$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \mathbf{C}(u)^{\mathrm{T}} \Big(\mathbf{Q}_m^r(-e^{-iu/2}) \, \mathbf{\Phi}_m^r(-e^{-iu/2}) \, \mathbf{Q}_m^r(-e^{-iu/2})^\star \Big) \, \overline{\mathbf{C}(u)} \, \mathrm{d}u. \tag{3.9}$$

Recall that $\mathbf{\Phi}_m^r(z)$ is Hermitian and positive definite for $z \in \mathcal{T}$. Thus, the matrix $\mathbf{Q}_m^r(z) \, \mathbf{\Phi}_m^r(z) \, \mathbf{Q}_m^r(z)^\star$ is Hermitian and positive definite for $z \in \mathcal{T}$, $z \neq 1$ and positive semidefinite for $z = 1$. It follows that for $\|\mathbf{C}\|_{L_2}^2 := \sum_{k=0}^{r-1} \|C_k\|_{L_2}^2 > 0$ the terms in (3.9) are nonnegative for all $z \in \mathcal{T}$ and at least one term in (3.9) is positive. By the definitions of $\mathbf{Q}_m^r$ and $\mathbf{\Phi}_m^r$ the terms in (3.9) are bounded for fixed m and r. ■

§4. The Spline Space $V_{1,0}^{2m+1,r}$

In view of the preceding result let us consider the subspace

$$V_{1,0}^{2m+1,r} := \{ s \in V_1^{2m+1,r} : \mathrm{D}^\nu s(n) = 0, \; \nu = 0, \ldots, r-1; \; n \in \mathbb{Z} \}$$

of cardinal splines of degree $2m+1$ and defect r with the knot sequence $2^{-1}\mathbb{Z}$. Then the functions

$$\Lambda_k^{2m+1,r} := L_k^{2m+1,r}(2 \cdot -1)$$

belong to $V_{1,0}^{2m+1,r}$ and we have:

Theorem 4.1. *For $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$ the functions*

$$\Lambda_k^{2m+1,r}(\cdot - l) \quad (k = 0, \ldots, r-1; \; l \in \mathbb{Z}) \tag{4.1}$$

form a Riesz basis of $V_{1,0}^{2m+1,r}$.

Proof: The cardinal Hermite fundamental splines $L_k^{2m+1,r}(2\cdot -l)$ $(k=0,\ldots, r-1;\ l\in\mathbb{Z})$ form a basis of $V_1^{2m+1,r}$, *i.e.*, an arbitrarily chosen element $G\in V_{1,0}^{2m+1,r}\subset V_1^{2m+1,r}$ can be uniquely represented in the form

$$G=\sum_{k=0}^{r-1}\sum_{l=-\infty}^{\infty} a_l^k\, L_k^{2m+1,r}(2\cdot -l).$$

The conditions $\mathrm{D}^\nu G(n)=0$ $(\nu=0,\ldots,r-1;\ n\in\mathbb{Z})$ imply that $a_{2l}^k=0$ $(k=0,\ldots,r-1;\ l\in\mathbb{Z})$, *i.e.*,

$$G=\sum_{k=0}^{r-1}\sum_{l=-\infty}^{\infty} b_l^k\,\Lambda_k^{2m+1,r}(\cdot-l)$$

with $b_l^k := a_{2l+1}^k$ $(k=0,\ldots,r-1;\ l\in\mathbb{Z})$. Thus, the functions in (4.1) form a basis of $V_{1,0}^{2m+1,r}$. To show the Riesz basis property we note that

$$\hat{\mathbf{\Lambda}}_{2m+1}^r := \left(\hat{\Lambda}_k^{2m+1,r}\right)_{k=0}^{r-1} = \mathbf{R}_{2m+1}^r(e^{-i\cdot/2})\,\hat{\mathbf{N}}_{2m+1}^r(\cdot/2)$$

with $\mathbf{R}_{2m+1}^r(z) := z\,(\mathbf{H}_{2m+1}^r(z)^{\mathrm{T}})^{-1}$ $(z\in\mathcal{T})$. Following the ideas in the proof of Theorem 3.3, we only have to consider the matrices

$$\mathbf{R}_{2m+1}^r(z)\,\mathbf{\Phi}_{2m+1}^r(z)\,\mathbf{R}_{2m+1}^r(z)^\star \quad (z\in\mathcal{T}),$$

where $\mathbf{\Phi}_{2m+1}^r(z)$ denotes the autocorrelation symbol for $\hat{\mathbf{N}}_{2m+1}^r$.
Since $\mathbf{H}_{2m+1}^r(z)$ is invertible and $\mathbf{\Phi}_{2m+1}^r(z)$ is positive definite for $z\in\mathcal{T}$, it follows that $\mathbf{R}_{2m+1}^r(z)\,\mathbf{\Phi}_{2m+1}^r(z)\,\mathbf{R}_{2m+1}^r(z)^\star$ is positive definite for $z\in\mathcal{T}$. Thus, the Riesz basis property is satisfied. ■

As a consequence of Theorems 4.1 and 3.3 we have the following result (cf. [2], p. 190 for $r=1$).

Theorem 4.2. *For $m\in\mathbb{N}_0$ and $r\in\mathbb{N}$ the $(m+1)$-th order differential operator D^{m+1} maps the spline space $V_{1,0}^{2m+1,r}$ one–to–one onto the wavelet space $W_0^{2m+1,r}$. Moreover, the Riesz basis $\{\Lambda_k^{2m+1,r}(\cdot-l);\ k=0,\ldots,r-1;\ l\in\mathbb{Z}\}$ of $V_{1,0}^{2m+1,r}$ corresponds to the Riesz basis $\{\psi_k^{m,r}(\cdot-l);\ k=0,\ldots, r-1;\ l\in\mathbb{Z}\}$ of $W_0^{m,r}$ via the relation $\psi_k^{m,r}=\mathrm{D}^{m+1}\,\Lambda_k^{2m+1,r}$ $(k=0,\ldots,r-1)$.*

§5. An Example

We want to apply the obtained formulas to the case $m = 3$, $r = 2$ of cubic spline wavelets with defect 2.

With $\hat{\mathbf{N}}^2_{-1} = (iu, 1)^{\mathrm{T}}$ and

$$\begin{aligned}\mathbf{D}^2_{3,0}(z) &= \mathbf{A}^2_3(z)\,\mathbf{A}^2_2(z)\,\mathbf{A}^2_1(z)\,\mathbf{A}^2_0(z)\\ &= 6\begin{pmatrix}1 & -1/2\\ -z & 1/2\end{pmatrix}\begin{pmatrix}1 & -1\\ -z & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -z & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1-z\end{pmatrix}\\ &= 3\begin{pmatrix}2+4z & -5+4z+z^2\\ -4z-2z^2 & 1+4z-5z^2\end{pmatrix}\end{aligned}$$

it follows from (2.2) that

$$\hat{\mathbf{N}}^2_3(u) = \frac{3}{(iu)^4}\begin{pmatrix}(2iu-5)+4(iu+1)e^{-iu}+e^{-2iu}\\ 1+4(-iu+1)e^{-iu}+(-2iu-5)e^{-2iu}\end{pmatrix}.$$

For the two–scale symbol satisfying $\hat{\mathbf{N}}^2_3 = \mathbf{P}^2_3(e^{-i\cdot/2})\,\hat{\mathbf{N}}^2_3(\cdot/2)$ we find with (2.8) and (3.8)

$$\begin{aligned}\mathbf{P}^2_3(z) &= \frac{1}{16}\,\mathbf{D}^2_{3,0}(z^2)\,\mathbf{P}^2_{-1}(z)\,\mathbf{D}^2_{3,0}(z)^{-1}\\ &= \frac{1}{16}\begin{pmatrix}2+6z+z^2 & 5+2z\\ 2z+5z^2 & 1+6z+2z^2\end{pmatrix}.\end{aligned}$$

The autocorrelation symbol reads

$$\mathbf{\Phi}^2_3(z) = \frac{1}{560}\begin{pmatrix}9z^{-1}+128+9z & 53z^{-1}+80+z\\ z^{-1}+80+53z & 9z^{-1}+128+9z\end{pmatrix}.$$

The Euler–Frobenius matrix $\mathbf{H}^2_7$ is given by

$$\mathbf{H}^2_7(z) = \frac{1}{432}\begin{pmatrix}37z+176z^2+3z^3 & 3z+176z^2+37z^3\\ 175z-224z^2-21z^3 & 21z+224z^2-175z^3\end{pmatrix},$$

such that (2.13) can simply be verified with

$$\begin{aligned}\mathbf{D}^2_{3,1}(z) &= \mathbf{A}^2_7(z)\,\mathbf{A}^2_6(z)\,\mathbf{A}^2_5(z)\,\mathbf{A}^2_4(z)\\ &= \frac{35}{9}\begin{pmatrix}6+26z+3z^2 & -17-18z\\ -18z-17z^2 & 3+26z+6z^2\end{pmatrix}.\end{aligned}$$

The matrix $\mathbf{H}^2_7(z)$ is invertible on the unit circle $z \in \mathcal{T}$ and we have

$$\mathbf{H}^2_7(z)^{-1} = \frac{12}{7z\,\Delta^2_7(z)}\begin{pmatrix}21+224z-175z^2 & -3-176z-37z^2\\ -175+224z+21z^2 & 37+176z+3z^2\end{pmatrix}$$

with

$$\Delta_7^2(z) = 1 - 72z + 262z^2 - 72z^3 + z^4.$$

Thus, the two–scale symbol $\mathbf{Q}_3^2$ of the wavelet vector $\mathbf{\Psi}_3^2$ is given by

$$\begin{aligned}\mathbf{Q}_3^2(z) &= z/2\,(\mathbf{H}_7^2(z)^{\mathrm{T}})^{-1}\,\mathbf{D}_{3,1}^2(z)\\ &= \frac{60}{\Delta_7^2(z)}\begin{pmatrix} 7(1+40z+30z^2 & 7(-7-64z+30z^2 \\ -64z^3-7z^4) & +40z^3+z^4) \\ & \\ -(1+100z+478z^2 & 9+252z+478z^2 \\ +252z^3+9z^4 & +100z^3+z^4) \end{pmatrix}.\end{aligned}$$

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